Table of Contents

Control System Engineering

ISBN 9788184314632

Published by :

Technical Publications Pune®
#1, Amit Residency, 412, Shaniwar Peth, Pune - 411 030, India.

Printer :

Alert DTPrinters
Sr.no. 10/3, Sinhagad Road,
Pune - 411 041

Chapter - 16 Control Components and Controllers (16 - 1) to (16 - 64)

1 Basics of Control System

1.1 Background

In recent years, concept of automatic control has achieved a very important position in advancement of modern science. Automatic control systems have played an important role in the advancement and improvement of engineering skills.

Practically, every activity in our day to day life is influenced by some sort of control system. Concept of control systems also plays an important role in the working of space vehicles, satellites, guided missiles etc. Such control systems are now integral part of the modern industrialization, industrial processes and home appliances. Control systems are found in number of practical applications like computerised control systems, transportation systems, power systems, temperature limiting systems, robotics etc.

Hence for an engineer it is absolutely necessary to get familiar with the analysis and designing methods of such control systems.

This chapter includes the concept of system and control system. Then it gives the classification of control systems. It includes the discussion of various types of control systems supported with number of real time applications.

1.2 Definitions

To understand the meaning of the word control system, first we will define the word system and then we will try to define the word control system.

System : *A system is a combination or an arrangement of different physical components which act together as an entire unit to achieve certain objective.*

Every physical object is actually a system. A classroom is a good example of physical system. A room along with the combination of benches, blackboard, fans, lighting arrangement etc. can be called a classroom which acts as an elementary system.

Another example of a system is a lamp. A lamp made up of glass, filament is a physical system. Similarly a kite made up of paper and sticks is an example of a physical system.

Similarly system can be of any type i.e. physical, ecological, biological etc.

In such system, output or part of the output is fedback to the input for comparison with the reference input applied to it.

Closed loop system can be represented as shown in the Fig. 1.12.

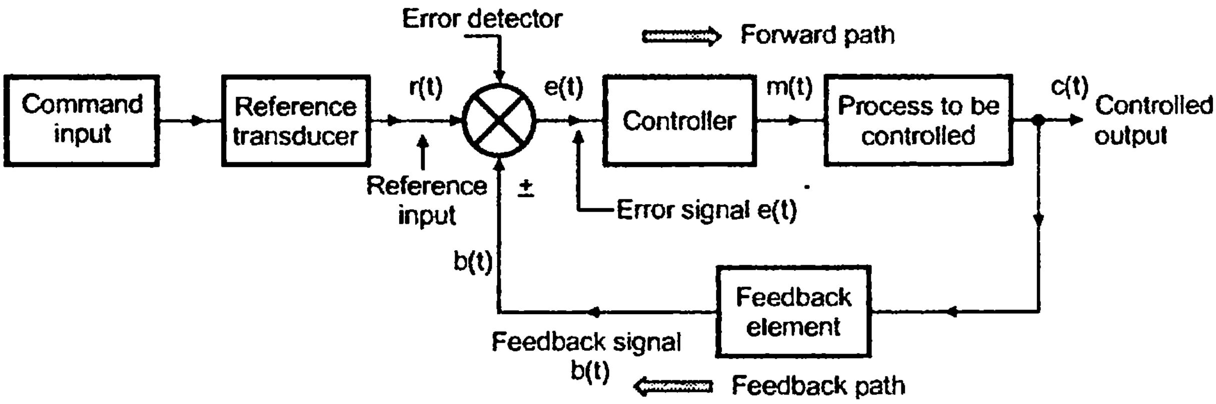


Fig. 1.12 Representation of closed loop control system

The various signals are,

r(t) = Reference input	e(t) = Error signal	
c(t) = Controlled output	m(t) = Manipulated signal	b(t) = Feedback signal

It is not possible in all the systems that available signal can be applied as input to the system. Depending upon nature of controller and plant it is required to reduce it or amplify it or to change its nature i.e. making it discrete from continuous type of signal etc. This changed input as per requirement is called **reference input** which is to be generated by using reference transducer. The main excitation to the system is called its **command input** which is then applied to the reference transducer to generate reference input.

Practically many electronic integrated circuits work on the d.c. voltage range of 5 to 10 V. The supply available is 230 V a.c. Hence the reference input voltage in the range of 5 to 10 V d.c. is obtained from the command input 230 V a.c. and proper rectifying unit.

The part of output, which is to be decided by feedback element is fed back to the reference input. The signal which is output of feedback element is called **'feedback signal'** b(t).

It is then compared with the reference input giving error signal $e(t) = r(t) \pm b(t)$

When feedback sign is positive, systems are called **positive feedback systems** and if it is negative systems are called **negative feedback systems.**

This error signal is then modified by controller and decides the proportional manipulated signal for the process to be controlled.

This manipulation is such that error will approach zero. This signal then actuates the actual system and produces an output. As output is controlled one, hence called **controlled output** c(t).

1.5.1 Advantages

The advantages of closed loop system are,

1) Accuracy of such system is always very high because controller modifies and manipulates the actuating signal such that error in the system will be zero.
2) Such system senses environmental changes, as well as internal disturbances and accordingly modifies the error.
3) In such system, there is reduced effect of nonlinearities and distortions.
4) Bandwidth of such system i.e. operating frequency zone for such system is very high.

1.5.2 Disadvantages

The disadvantages of closed loop system are,

1) Such systems are complicated and time consuming from design point of view and hence costlier.
2) Due to feedback, system tries to correct the error from time to time. Tendency to overcorrect the error may cause oscillations without bound in the system. Hence system has to be designed taking into consideration problems of instability due to feedback. The stability problems are severe and must be taken care of while designing the system.

1.5.3 Real Time Applications of Closed Loop System

1.5.3.1 Human Being

The best example is human being. If a person wants to reach for a book on the table, closed loop system can be represented as in the Fig. 1.13.

Position of the book is given as the reference. Feedback signal from eyes, compares the actual position of hands with reference position. Error signal is given to brain. Brain manipulates this error and gives signal to the hands. This process continues till the position of the hands get achieved appropriately.

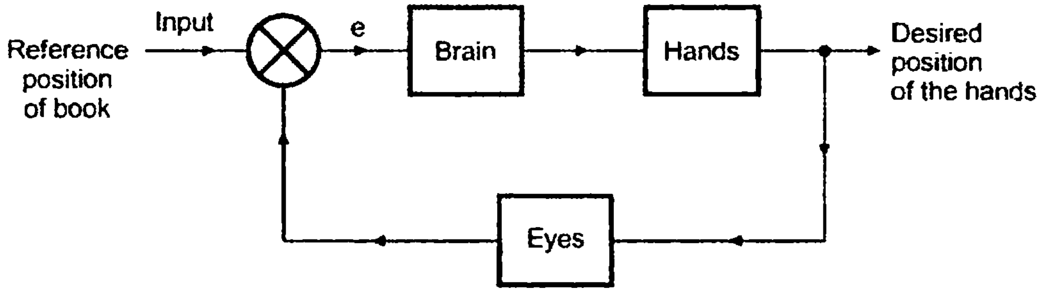


Fig. 1.13 Human being

1.5.3.2 Home Heating System

In this system, the heating system is operated by a valve. The actual temperature is sensed by a thermal sensor and compared with the desired temperature. The difference between the two, actuates the valve mechanism to change the temperature as per the requirement.

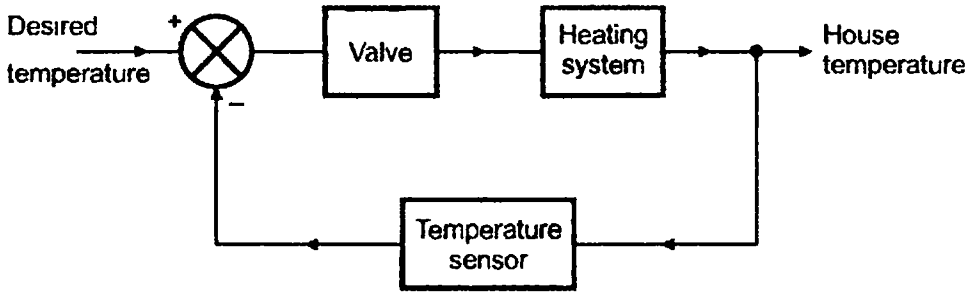


Fig. 1.14 Domestic heating system

1.5.3.3 Ship Stabilization System

In this system a roll sensor is used as a feedback element. The desired roll position is

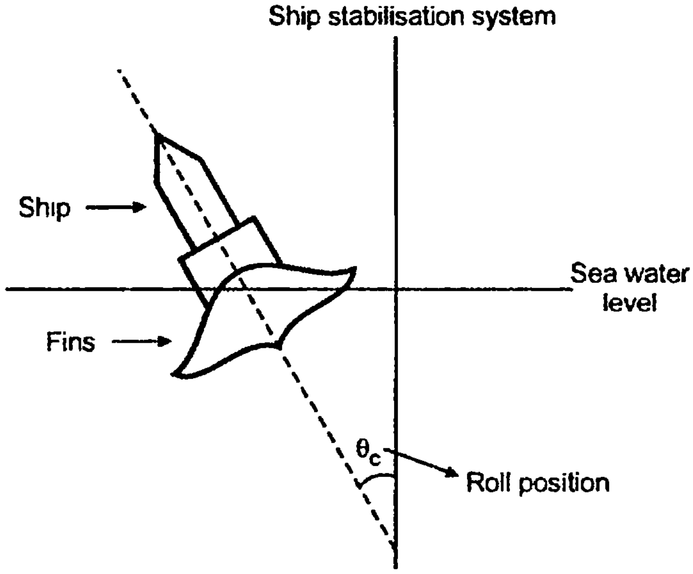


Fig. 1.15 Ship stabilization system

selected as θ_r while actual roll position is θ_c which is compared with θ_r to generate controlling signal. This activates fin actuator in proper way to stabilize the ship.

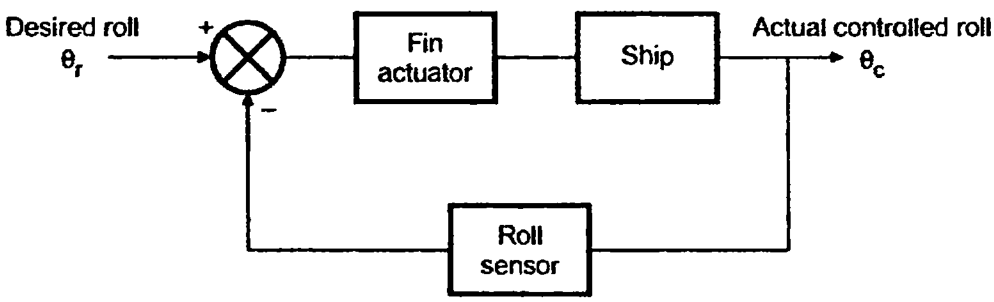


Fig. 1.16

1.5.3.4 Manual Speed Control System

A locomotive operator driving a train is a good example of a manual speed control system. The objective is to maintain the speed equal to the speed limits set. The entire system is shown in the block diagram in the Fig. 1.17.

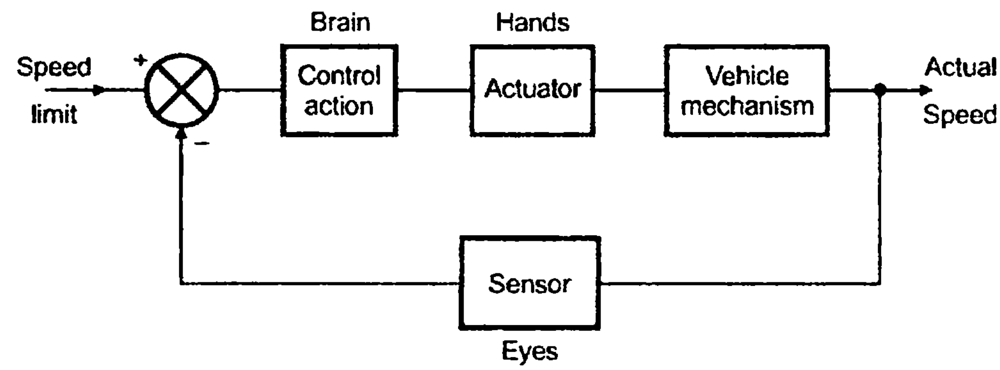


Fig. 1.17

1.5.3.5 D.C. Motor Speed Control

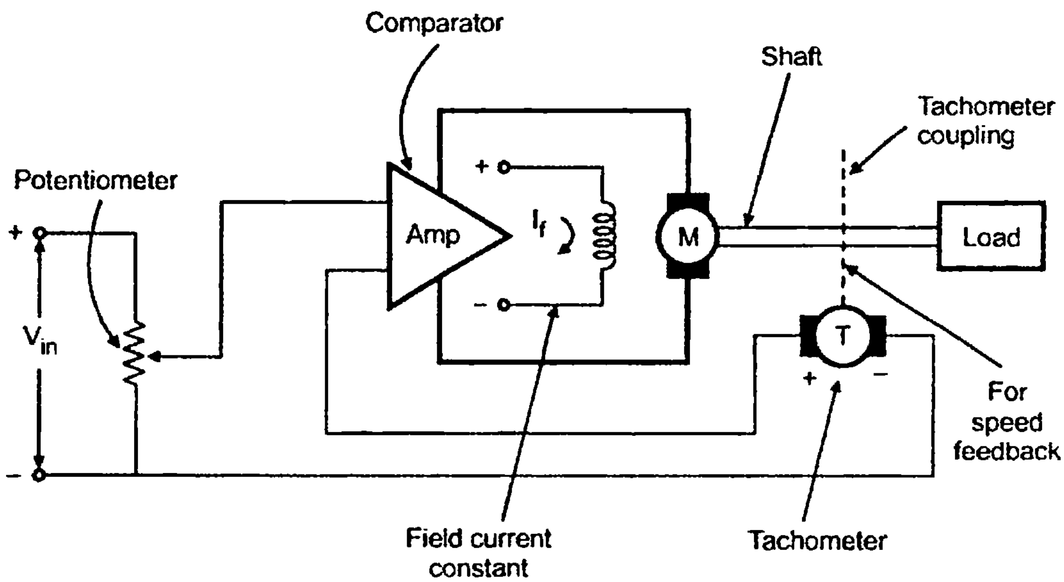


Fig. 1.18 Speed control system

The D.C. shunt motor is used where field current is kept constant and armature voltage is changed to obtain the desired speed. The feedback is taken by speed tachometer. This generates voltage proportional to speed which is compared with voltage required to the desired speed. This difference is used to change the input to controller which cumulatively changes the speed of the motor as required.

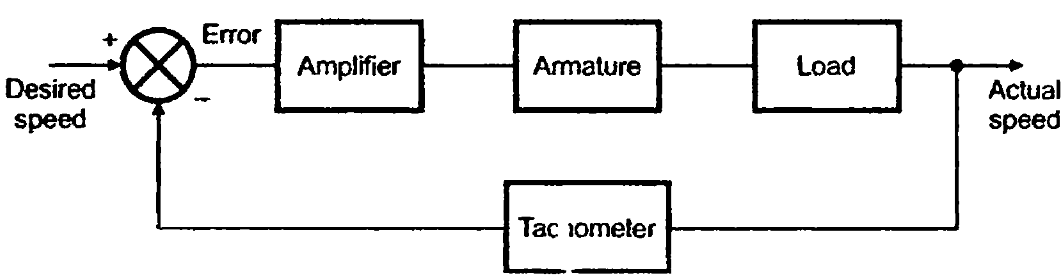


Fig. 1.19 Speed control system

1.5.3.6 Temperature Control System

The aim is to maintain hot water temperature constant. Water is coming with constant flow rate. Steam is coming from a valve. Pressure thermometer 'P' is used as a feedback element which sends a signal for comparison with the set point. This error actuates the valve which controls the rate of flow of steam, eventually controlling the temperature of the water.

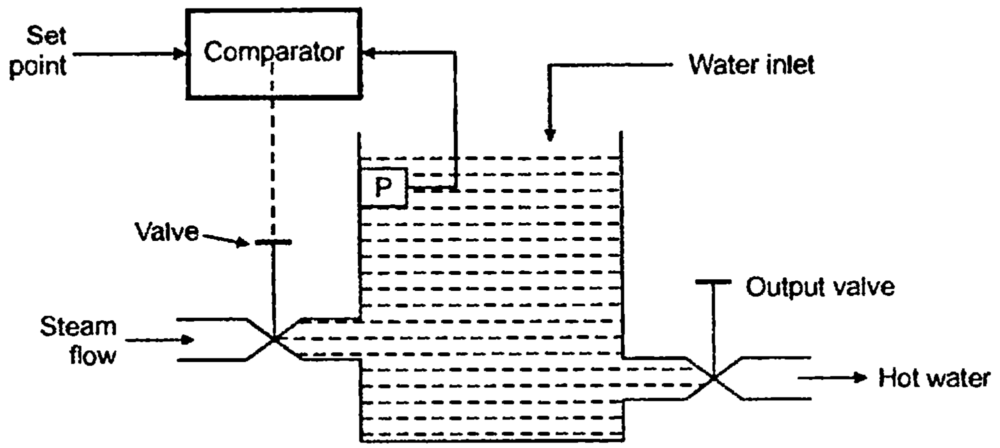


Fig. 1.20 Temperature control system

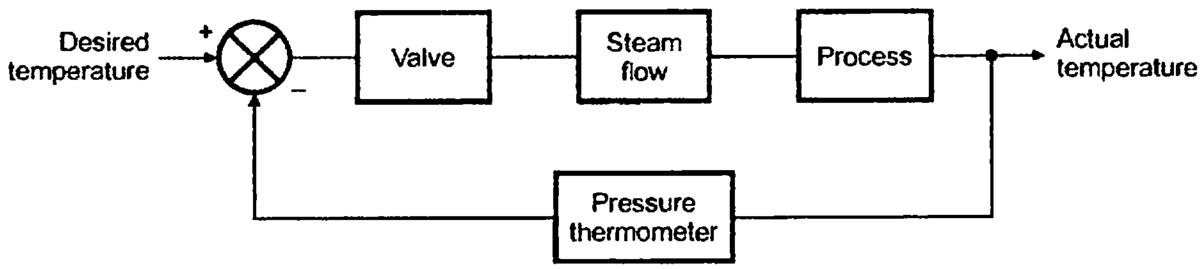


Fig. 1.21

1.5.3.7 Missile Launching System

This is sophisticated example of military applications of feedback control. The enemy plane is sighted by a radar which continuously tracks the path of the aeroplane. The launch computer calculates the firing angle interms of launch command, which when amplified drives the launcher. The launcher angular position is the feedback to the launch computer and the missile is triggered when error between the command signal and missile firing angle becomes zero. The system is shown in the Fig. 1.22.

1.5.3.8 Voltage Stabilizer

Supply voltage required for various single phase appliances must be constant and high fluctuations are generally not permitted. Voltage stabilizer is a device which accepts variable voltage and outputs a fixed voltage.

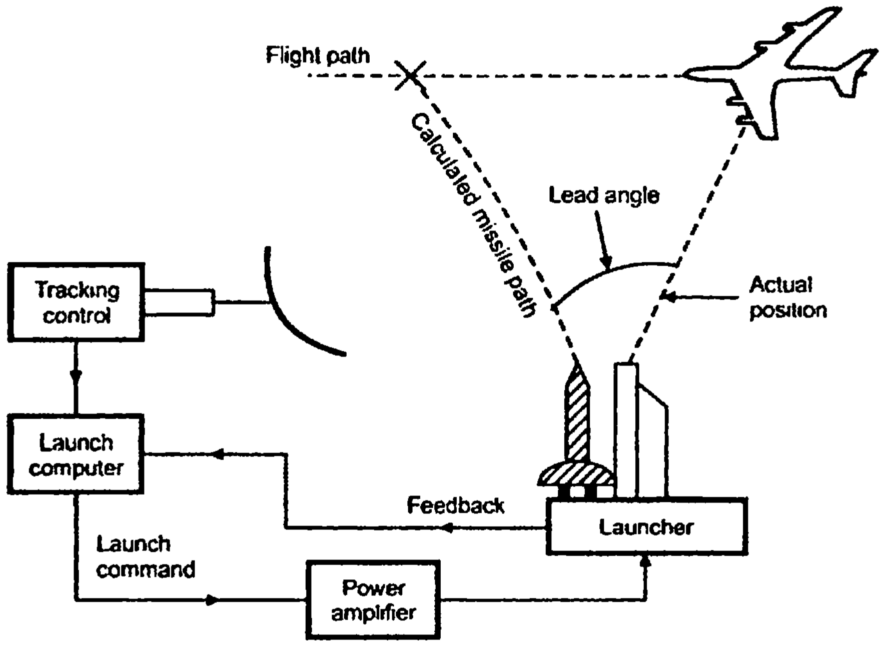


Fig. 1.22 Missile launching system

Principle of such stabilizer is based on controlling number of secondary turns as per requirement to increase or decrease the output voltage. The actual output voltage is sensed by a transformer and potential divider arrangement. The reference voltage is selected proportional to the desired output level. The actual output is compared with this to generate error which in turn is inputted to the controller. The controller takes the proper decision to increase or decrease the number of turns so as to adjust the output voltage. The scheme is shown in the Fig. 1.23.

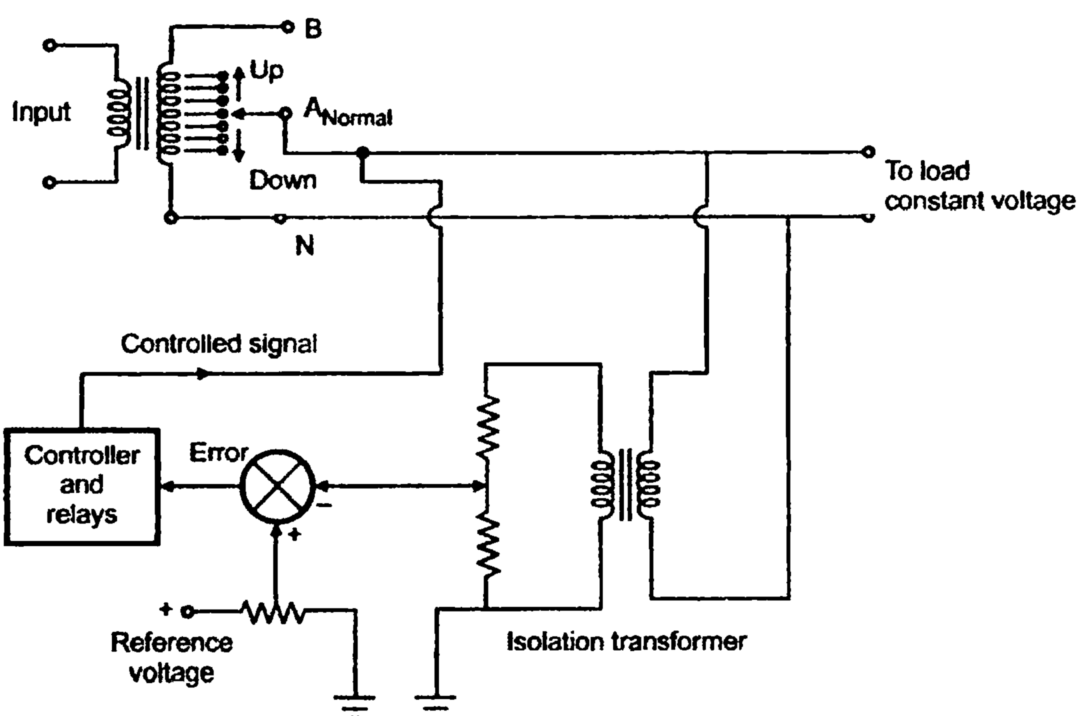


Fig. 1.23 Voltage stabilizer

The other examples of closed loop system are machine tool position control, positioning of radio and optical telescopes, auto pilots for aircrafts, inertial guidance system, automatic electric iron, railway reservation status display, sunseeker solar system, water level controllers, temperature control system. So in closed loop feedback control systems cause and effect relationship between input and output exists.

1.6 Comparison of Open Loop and Closed Loop Control System

Sr. No.	Open Loop	Closed Loop
1.	Any change in output has no effect on the input i.e. feedback does not exists.	Changes in output, affects the input which is possible by use of feedback.
2.	Output measurement is not required for operation of system.	Output measurement is necessary.
3.	Feedback element is absent.	Feedback element is present.
4.	Error detector is absent.	Error detector is necessary.
5.	It is inaccurate and unreliable.	Highly accurate and reliable.
6.	Highly sensitive to the disturbances.	Less sensitive to the disturbances.
7.	Highly sensitive to the environmental changes.	Less sensitive to the environmental changes.
8.	Bandwidth is small.	Bandwidth is large.
9.	Simple to construct and cheap.	Complicated to design and hence costly.
10.	Generally are stable in nature.	Stability is the major consideration while designing
11.	Highly affected by nonlinearities.	Reduced effect of nonlinearities.

1.7 Servomechanisms

Definition : *It is a feedback control system in which the controlled variable or the output is a mechanical position or its time derivatives such as velocity or acceleration.*

A simple example of servomechanism is a **position control system.** Consider a load which requires a constant position in its application. The position is sensed and converted to voltage using feedback potentiometer. It is compared with input potentiometer voltage to generate error signal. This is amplified and given to the controller. The controller in turn controls the voltage given to motor, due to which it changes its position.

The scheme is shown in the Fig. 1.24.

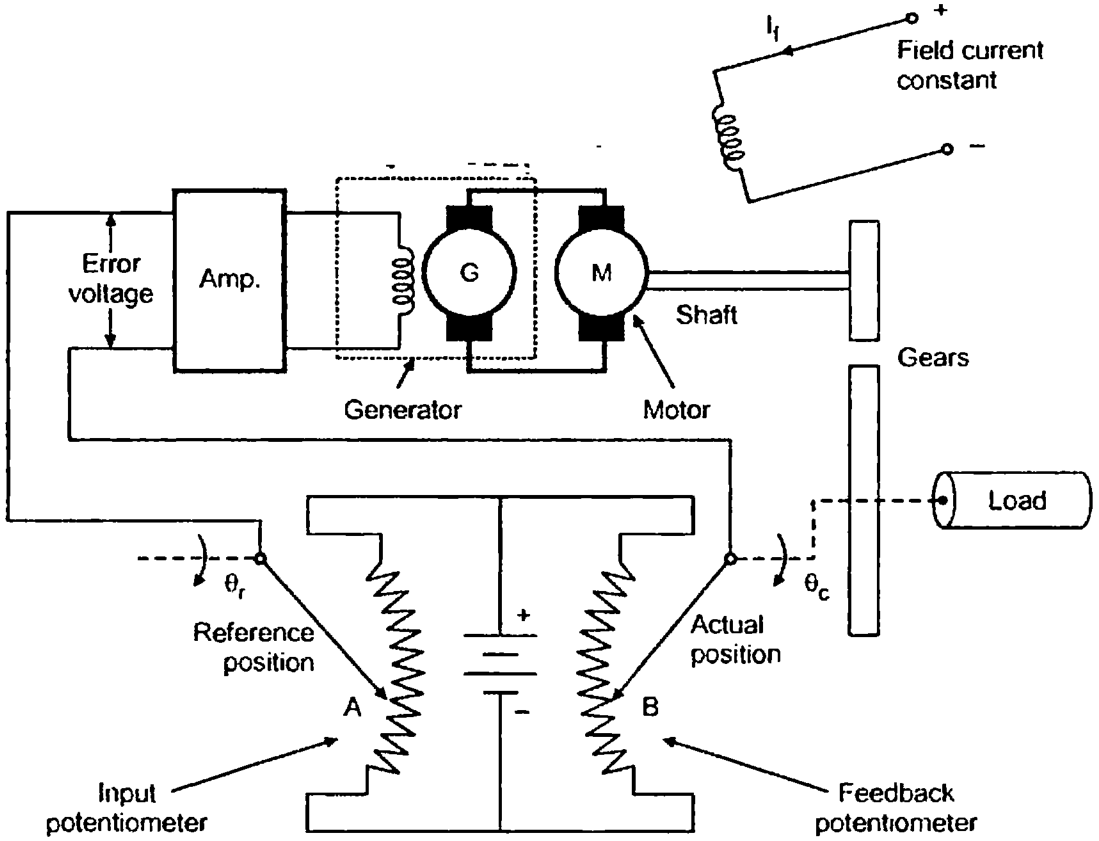


Fig. 1.24 Position control system

Few other examples of servomechanisms are,

1) Power steering apparatus for an automobile.

2) Machine tool position control.

3) Missile launchers.

4) Roll stabilization of ships.

1.8 Regulating Systems (Regulators)

Definition : *It is a feedback control system in which for a preset value of the reference input, the output is kept constant at its desired value.*

In such systems reference input remains constant for long periods. Most of the times the reference input or the desired output is either constant or slowly varying with time. In a regulator, the desired value of the controlled outputs is more or less fixed. Similarly the reference input is also fixed and called **set point.** Thus the regulator maintains a constant output for a fixed reference input. The problems due to disturbances are mainly rectified by the regulator. A simple example of such regulator system is servostabilizer. We have seen earlier that in voltage stabilizer position of tap on secondary is adjusted by using relay controls. But instead of fixed tap, the entire secondary can be smoothly tapped using a servomotor drive. The servomotor drives the shaft and controls the position of tap on secondary as per the controller signal. Due to the fluctuations in the main input if the load voltage changes, such effects are rectified by the regulator to keep load voltage constant.

The actual scheme is shown in the Fig. 1.25, while its block diagram representation is shown in the Fig. 1.26.

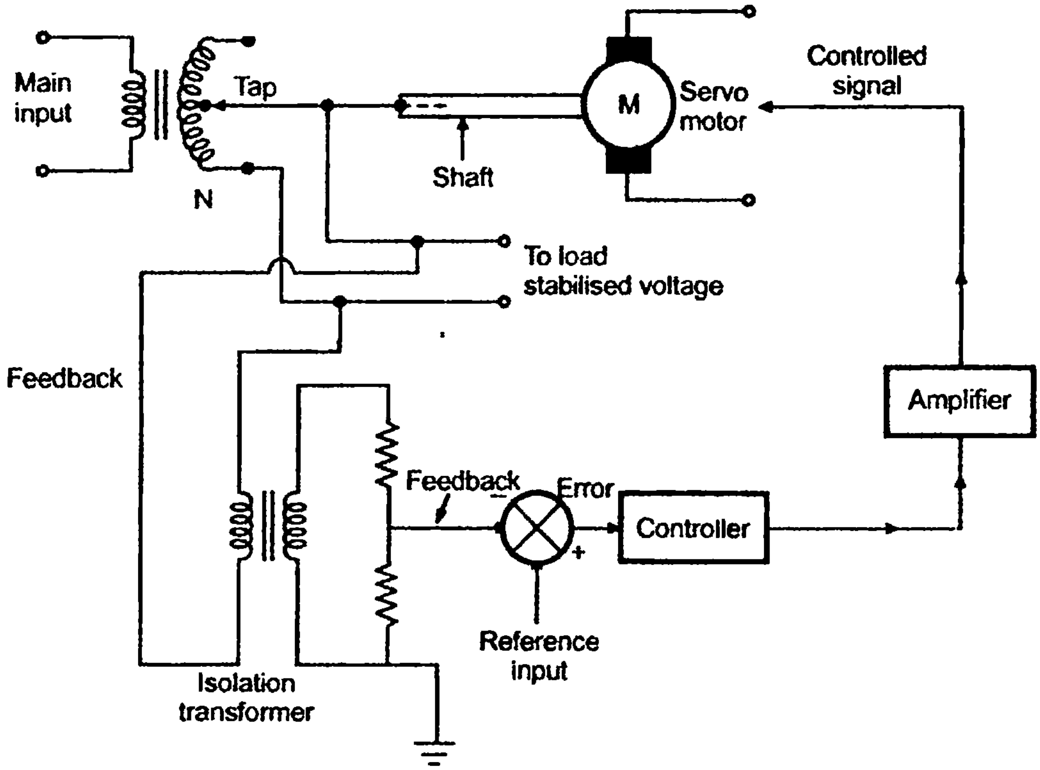


Fig. 1.25 Regulating system

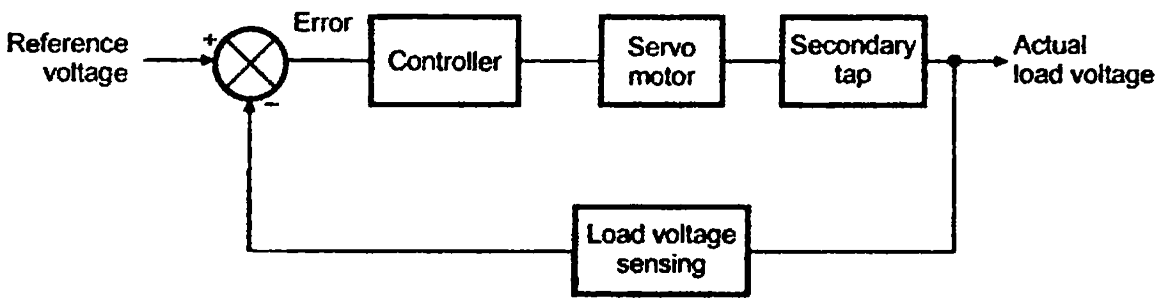


Fig. 1.26

Few other examples of regulating system are,

1) Temperature regulators.

2) Frequency controllers.

3) Speed governors.

1.9 Feedback and Feed Forward System

In the control systems considered uptill now, it is considered that the disturbance has affected the output adversely. Such an output is measured and compared with the reference input to generate an error. This error is fed to the controller which is successively operating the system to correct the output.

Thus such systems in which the effect of the disturbance must show up in the error before the controller can take proper corrective action are called feedback systems.

If the disturbance is measurable, then the signal can be added to the controller input to modify the actuating signal. Thus, a corrective action is initiated without waiting for the effect of the disturbance to show up in the output i.e. cumulatively in the error. Thus the undesirable effects of measurable disturbances by approximately compensating for them before they affect the output. This is much more advantageous as in normal feedback system the corrective action starts only after the output has been affected.

Key Point: *Such systems in which such corrective action is taken before disturbances affect the output are called feed forward system.*

A block diagram with feed forward concept is shown in the Fig. 1.27.

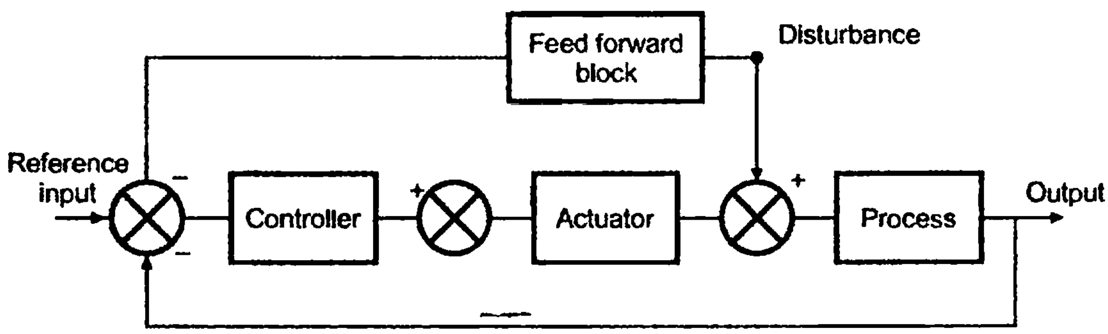


Fig. 1.27

The two difficulties associated with feed forward system are,

i) In some systems, the disturbance may not be measurable.

ii) The feed forward compensation is an open loop technique and if actuator transfer function is not known accurately, then such compensation cannot be achieved.

Control system : *To control means to regulate, to direct or to command. Hence a control system is an arrangement of different physical elements connected in such a manner so as to regulate, direct or command itself or some other system.*

For example, if in a classroom, professor is delivering his lecture, the combination becomes a control system as; he tries to regulate, direct or command the students in order to achieve the objective which is to impart good knowledge to the students. Similarly if lamp is switched ON or OFF using a switch, the entire system can be called a control system. The concept of physical system and a control system is shown in the Fig. 1.1 and Fig. 1.2.

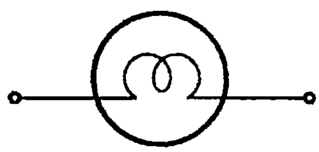


Fig. 1.1 Physical system

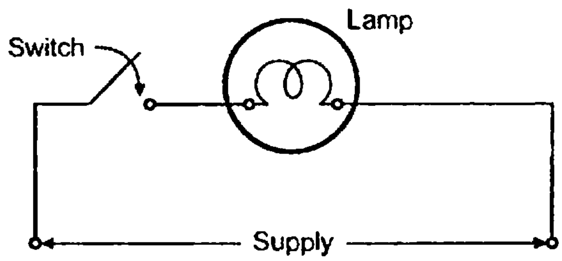


Fig. 1.2 Control system

When a child plays with the kite, he tries to control it with the help of string and entire system can be considered as a control system.

In short, a control system is in the broadest sense, an interconnection of the physical components to provide a desired function, involving some kind of controlling action in it.

Plant : *The portion of a system which is to be controlled or regulated is called the* ***plant or the Process.***

Controller : *The element of the system itself or external to the system which controls the plant or the process is called* ***controller.***

For each system, there must be an excitation and system accepts it as an input. And for analyzing the behaviour of system for such input, it is necessary to define the output of a system.

Input : *It is an applied signal or an excitation signal applied to a control system from an external energy source in order to produce a specified output.*

Output : *It is the particular signal of interest or the actual response obtained from a control system when input is applied to it.*

Disturbances : Disturbance is a signal which tends to adversely affect the value of the output of a system. If such a disturbance is generated within the system itself, it is called an **internal disturbance**. The disturbance generated outside the system acting as an extra input to the system in addition to its normal input, affecting the output adversely is called an **external disturbance**.

Control systems may have more than one input or output. From the information regarding the system, it is possible to well define all the inputs and outputs of the systems.

1.9.1 Real Time Application of Feed Forward System

In a particular process control industry, it is necessary to maintain the temperature of a molten metal constant before giving it to the next process. For this a general temperature control feedback system is used as shown in the Fig. 1.28.

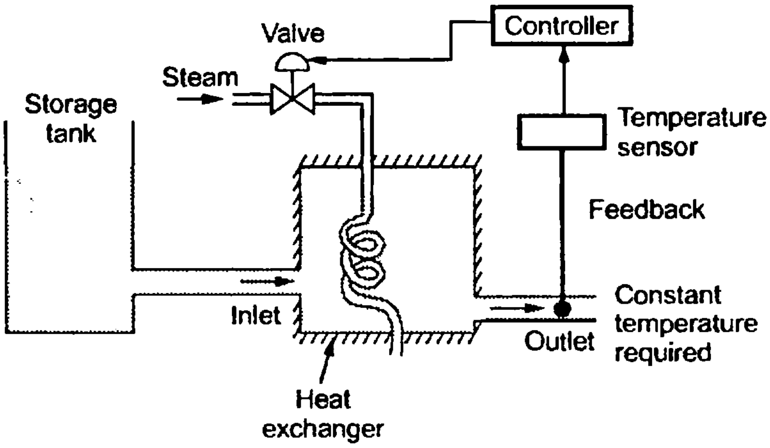


Fig. 1.28 Feedback system for temperature control

Practically if the rate of flow of metal from tank to heat exchanger gets disturbed, due to change in level then the temperature gets affected. But it takes a long time to see its effect at the output to take the corrective action.

Key Point: *In feedback system, the corrective action can not be taken unless and untill output gets disturbed.*

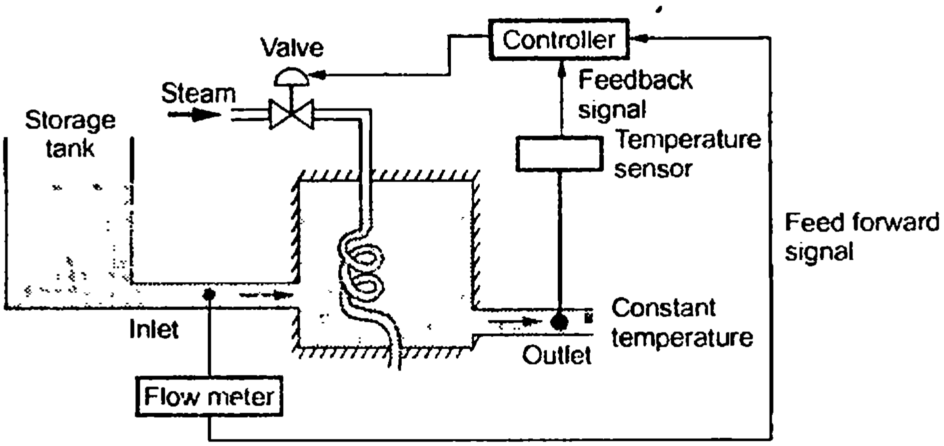
Due to the time lag, it is not possible to keep the output temperature constant within limits.

In such a case, feed forward system is used. The inlet flow rate is measured with the flowmeter. Immediately when there is a change in rate of flow, it is indicated to the controller through flowmeter before it is going to disturb the output. The controller takes the corrective action in advance by adjusting the steam flow. Thus the output temperature gets maintained constant within the limits.

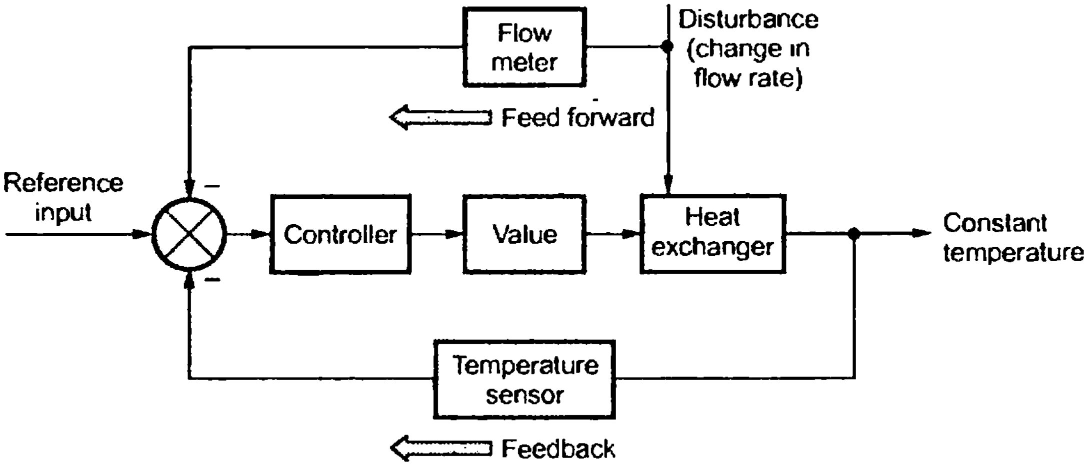
Key Point : *The feed forward compensation, compensates the effect of disturbance before it actually disturbs the output.*

The system is shown in the Fig. 1.29 (a) while its block diagram representation is shown in the Fig. 1.29 (b).

The feed forward minimizes the transient error due to measurable disturbances. While feedback compensates for unmeasurable disturbances and other effects. Thus it is advisable to incorporate both feedback and feed forward schemes in a system.



(a) Feed forward system



(b) Block diagram

Fig. 1.29 Practical example of feed forward system

1.10 Multivariable Control Systems

The control system in which there is only one output of the interest is called single variable system. But in many practical applications more than one variables are involved. A control system with multiple inputs and multiple outputs in called a **multivariable system.**

The block diagram representation of a multivariable control system is shown in the Fig. 1.30. The part of the system which is required to be controlled is called plant. The controller provides proper controlling action depending on the reference inputs. There are n reference inputs r_1, r_2, ... r_n.

There are n output variables $c_1(t)$, $c_2(t)$,.........$c_n(t)$. The values of these variables represent the performance of the plant. The control signals produced by the controller are applied to the plant. With the help of feedback elements the closed loop control of the plant is also possible. Due to the feedback, the controller takes into account the actual output values to decide the control signals.

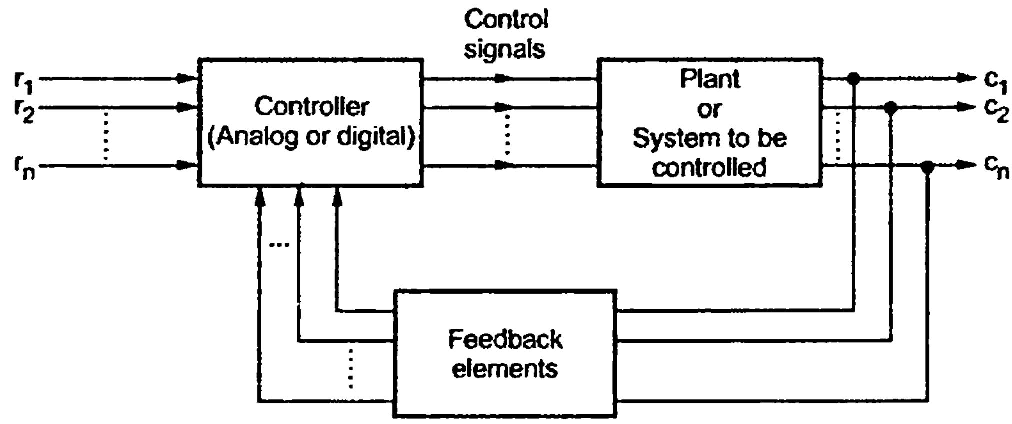


Fig. 1.30 Block diagram of multivariable control system

In case of multivariable systems, sometimes it is observed that a single input considerably affects more than one outputs. The system is said to be having strong **interactions** or **coupling**. This coupling is nothing but the disturbances for the separate systems. The interactions inherently present between inputs and outputs can be cancelled by designing a **decoupling controller**. Thus the resulting multivariable system is considered to have proper number of single input single output systems and the controller is designed for each system. The another way is to design a controller which will take care of all the inherent interactions present in the multivariable system.

In multivariable linear control system, each input is independently considered. Only one input and one output is considered and the total effect on any output because of all the inputs acting simultaneously is determined by addition of the outputs due to each input acting alone. Thus law of superposition is used to analyse multivariable linear control systems.

In many practical control systems, control is achieved by more than one input and the system may have many outputs. In chemical processes simultaneous control of pressure, temperature and concentration is required by commanding various inputs. Air crafts and space crafts are other examples where movement is controlled by various inputs. Power generators, atomic reactors and jet engines are some of other examples of multivariable systems.

Consider the block diagram of multivariable autopilot system shown in the Fig. 1.31.

The system shown in the Fig. 1.31 keeps a track of rocket vehicle in response to reference inputs given to it. The position, velocity and acceleration of the vehicle are fed to the digital controller using motion sensors. The controller takes appropriate decision and sends a controlling signal which will drive the actuator, which will move the engine. Thus there are three output variables which are to be observed and controlled and there are corresponding reference inputs hence the system is multivariable system.

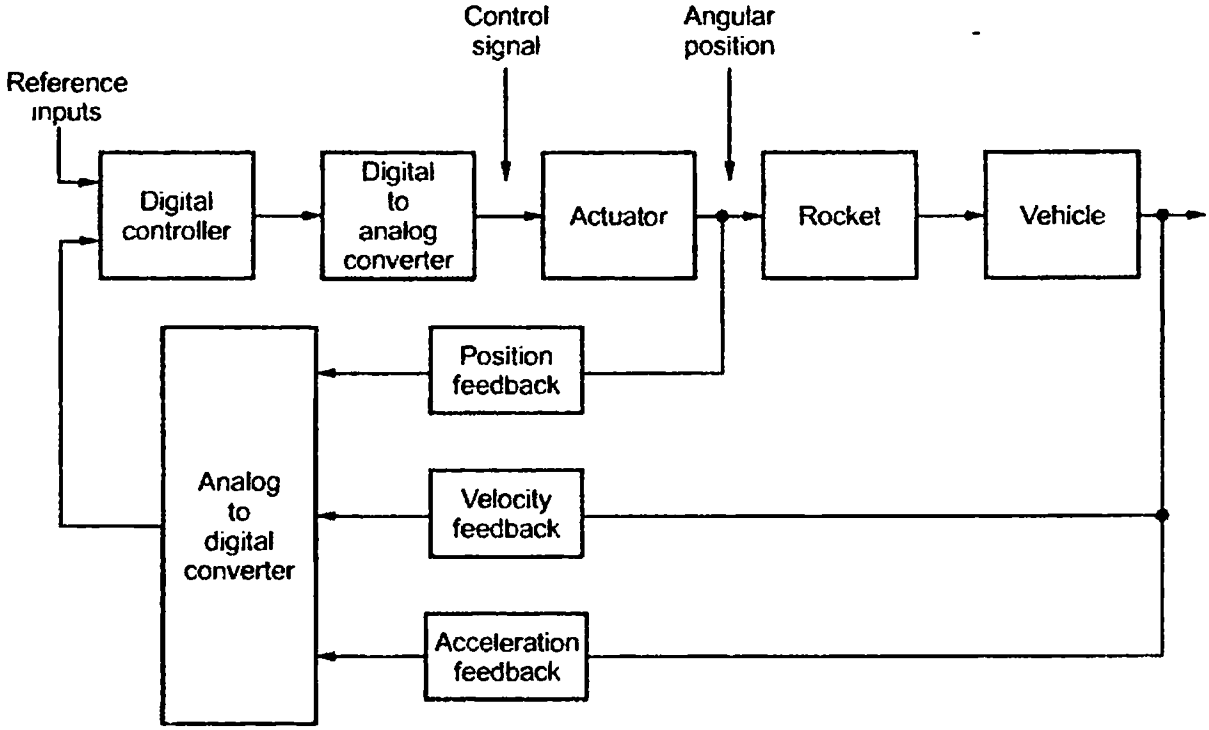


Fig. 1.31 Multivariable autopilot system

Review Questions

1. *Define the following terms*
   (i) System (ii) Control system (iii) Input (iv) Output (v) Disturbance.
2. *Explain how the control systems are classified*
3. *Define linear and nonlinear control systems.*
4. *What is time variant system? Give suitable example. How it is different than time invariant system?*
5. *Define open loop and closed loop system by giving suitable examples.*
6. *Differentiate between open loop and closed loop systems giving suitable examples.*
7. *With reference to feedback control system define the following terms*
   i) Command input (ii) Reference input (iii) Forward path (iv) Feedback path
8. *Explain the following terms giving suitable example*
   i) Servomechanism (ii) Regulator
9. *Distinguish between feedback control system and feed forward control system.*
10. *Differentiate between :*
    1. Linear and Nonlinear systems 2. Continuous and Discrete data systems
11. *Explain what is closed loop control system.*
12. *Write a note on multivariable control systems.*

□□□

The input variable is generally referred as the **Reference Input** and output is generally referred as the **Controlled Output**.

Fig. 1.3

Cause and effect relationship between input and output for a plant can be shown as in the Fig. 1.3.

1.3 Classification of Control Systems

Broadly control systems can be classified as,

1) **Natural Control Systems :** The biological systems, systems inside human being are of natural type.

Example 1 : The perspiration system inside the human being is a good example of natural control system. This system activates the secretion glands, secreating sweat and regulates the temperature of human body.

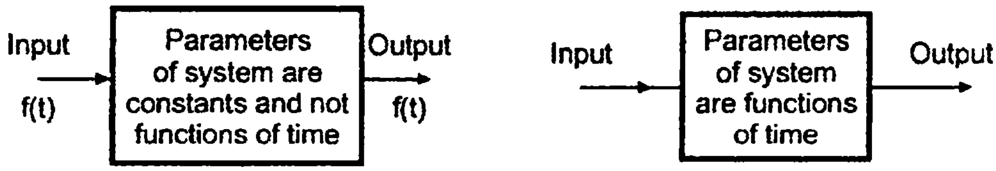
2) **Manmade Control Systems :** The various systems, we are using in our day to day life are designed and manufactured by human beings. Such systems like vehicles, switches, various controllers etc. are called manmade control systems.

Example 2 : An automobile system with gears, accelerator, braking system is a good example of manmade control system.

3) **Combinational Control Systems :** Combinational control system is one, having combination of natural and manmade together i.e. driver driving a vehicle. In such system, for successful operation of the system, it is necessary that natural systems of driver alongwith systems in vehicles which are manmade must be active.

   But for the engineering analysis, control systems can be classified in many different ways. Some of the classifications are given below.

4) **Time Varying and Time - Invariant Systems : Time varying control systems are those in which parameters of the systems are varying with time. It is not dependent on whether input and output are functions of time or not.** For example, space vehicle whose mass decreases with time, as it leaves earth. The mass is a parameter of space vehicle system. Similarly in case of a rocket, aerodynamic damping can change with time as the air density changes with the altitude. As against this **if even though the inputs and outputs are functions of time but the parameters of system are independent of time, which are not varying with time and are constants, then system is said to be time invariant system.** Different electrical networks consisting of the elements as resistances, inductances and capacitances are time invariant systems as the values of the elements of such system are constant and not the functions of time. The complexity of the control system design increases considerably if the control system is of the time varying type. This classification is shown in the Fig. 1.4.



(a) Time invariant system **(b) Time variant system**

Fig. 1.4

5) **Linear and Nonlinear Systems : A control system is said to be linear if it satisfies** following properties.

   a) The **principle of superposition** is applicable to the system. This means the response to several inputs can be obtained by considering one input at a time and then algebraically adding the individual results.

Mathematically principle of superposition is expressed by two properties,

   i) Additive property which says that for x and y belonging to the domain of the function f then we have,

$$f(x + y) = f(x) + f(y)$$

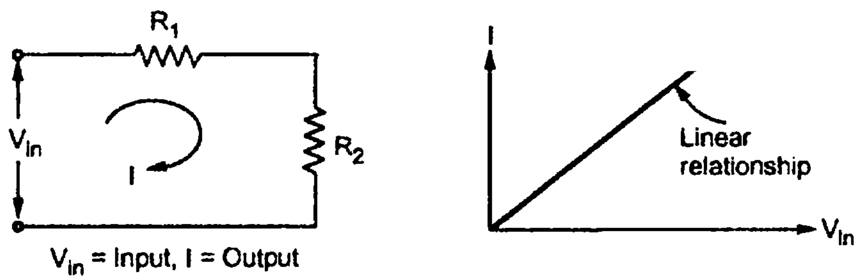
   ii) Homogeneous property which says that for any x belonging the domain of the function f and for any scalar constant α we have,

$$f(\alpha x) = \alpha f(x)$$

   b) The differential equation describing the system is linear having its coefficients as constants.

   c) Practically the output i.e. response varies linearly with the input i.e. forcing function for linear systems.

Real time example : A resistive network shown in the Fig. 1.5 (a) is a linear system. The Fig. 1.5 (b) shows the linear relationship existing between input and output.



(a) Linear system **(b) Response of system**

Fig. 1.5 Example of linear system

A control system is said to be nonlinear, if,

a. It does not satisfy the principle of superposition.

b. The equations describing the system are nonlinear in nature.

The function $f(x) = x^2$ is nonlinear because

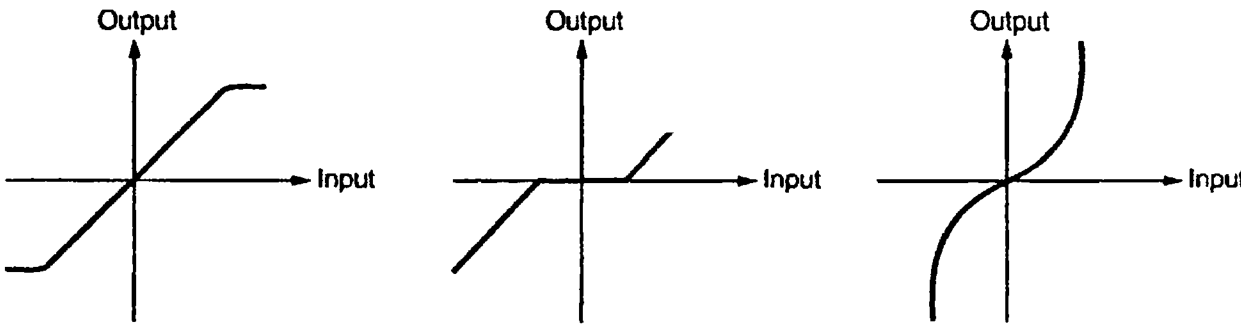
$$f(x_1 + x_2) = (x_1 + x_2)^2 \neq (x_1)^2 + (x_2)^2$$

and $$f(\alpha x) = (\alpha x)^2 \neq \alpha x^2 \quad \text{where } \alpha = \text{constant}$$

The equations of nonlinear system involves such nonlinear functions.

c. The output does not vary linearly for nonlinear systems.

The various nonlinearities practically present in the system are shown in the Fig. 1.6 (a), (b) and (c).

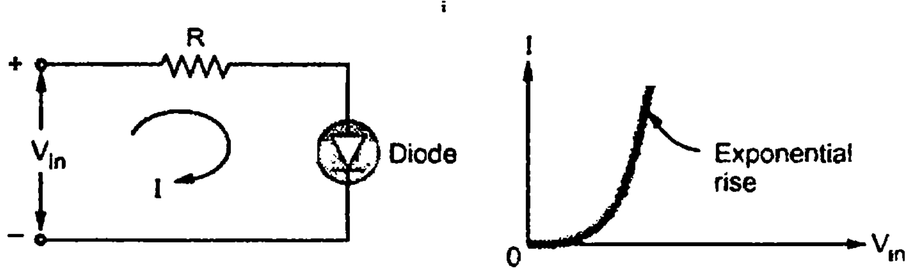


(a) Saturation (b) Dead zone (c) Exponential or square law

Fig. 1.6 Nonlinearities

The saturation means if input increases beyond certain limit, the output remains constant i.e. it does not remain linear. The flux and current relation i.e. B-H curve shows saturation in practice. In some big valves, though force increases upto certain value, the valve does not operate. So there is no response for certain time which is called dead zone.

The voltage-current equation of a diode is exponential and nonlinear thus diode circuit is an example of nonlinear system. This is shown in the Fig. 1.7.



(a) System (b) Response

Fig. 1.7 Example of nonlinear system

It can be seen that as long as V_{in} increases upto certain value, current remains almost zero. This is a dead zone and thereafter voltage-current are exponentially related to each other which is a nonlinear function.

Key Point : ***In practice it is difficult to find perfectly linear system. Most of the physical systems are nonlinear to certain extent.***

But if the presence of certain nonlinearity is negligible and not affecting the system response badly, keeping response within its linear limits then the nonlinearity can be neglected and for practical purpose the system can be treated to be linear.

Procedures for finding the solutions of nonlinear system problems are complicated and time consuming. Because of this difficulty most of the nonlinear systems are treated as linear systems for the limited range of operation with some assumptions and approximations. The number of linear methods, then can be applied for analysis of such linear systems.

6) **Continuous Time and Discrete Time Control Systems : In a continuous time control system all system variables are the functions of a continuous time variable 't'.** The speed control of a d.c. motor using a tachogenerator feedback is an example of continuous data system. At any time 't' they are dependent on time. **In discrete time systems one or more system variables are known only at certain discrete intervals of time.** They are not continuously dependent on the time. Microprocessor or computer based systems use such discrete time signals. The reasons for using such signals in digital controllers are,

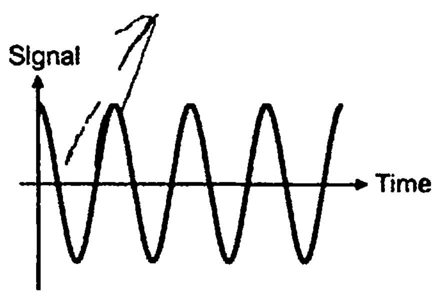
1) Such signals are less sensitive to noise.

2) Time sharing of one equipment with other channels is possible.

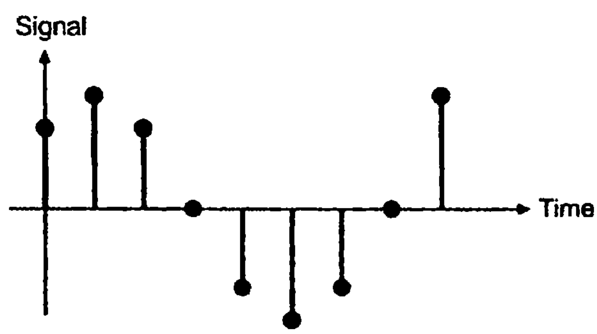
3) Advantageous from point of view of size, speed, memory, flexibility etc.

The systems using such digital controllers or sampled signals are called sampled data systems.

Continuous time system uses the signals as shown in the Fig. 1.8 (a) which are continuous with time while discrete system uses the signals as shown in the Fig. 1.8 (b).



(a) Continuous signal



(b) Discrete signal

Fig. 1.8

7) **Deterministic and Stochastic Control Systems :** A control system is said to be deterministic when its response to input as well as behaviour to external disturbances is predictable and repeatable. If such response is unpredictable, system is said to be stochastic in nature.

8) **Lumped Parameter and Distributed Parameter Control Systems :** Control system that can be described by **ordinary differential equations** is called lumped parameter control system. For example, electrical networks with different parameters as resistance, inductance, etc. are lumped parameter systems. Control systems that can be described by **partial differential equations** are called distributed parameter control systems. For example, transmission line having its parameters resistance and inductance totally distributed along it. Hence description of transmission line characteristics is always by use of partial differential equations. The lumped parameters are **physically separable** and can be shown to be located at a particular point while representing the system. The distributed parameters **can not be physically separated** and hence can not be represented at a particular place.

9) **Single Input Single Output (SISO) and Multiple Input Multiple Output (MIMO) Systems :** A system having only one input and one output is called single input single output system. For example, a position control system has only one input (desired position) and one output (actual output position). Some systems may have multiple type of inputs and multiple outputs, these are called multiple input multiple output systems.

10) **Open Loop and Closed Loop Systems :** This is another important classification. The features of both these types are discussed in detail in coming sections.

1.4 Open Loop System

Definition : A system in which output is dependent on input but controlling action or input is totally independent of the output or changes in output of the system, is called an Open Loop System.

In a broad manner it can be represented as in Fig. 1.9.

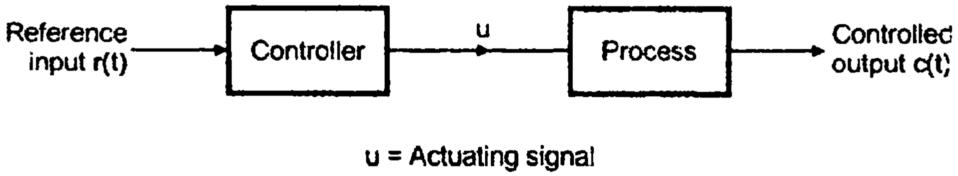


Fig. 1.9 Open loop control system

Reference input [r(t)] is applied to the controller which generates the actuating signal (u) required to control the process which is to be controlled. Process is giving out the necessary desired controlled output c(t).

1.4.1 Advantages

The advantages of open loop control system are,

1) Such systems are simple in construction.
2) Very much convenient when output is difficult to measure.
3) Such systems are easy from maintenance point of view.
4) Generally these are not troubled with the problems of stability.
5) Such systems are simple to design and hence economical.

1.4.2 Disadvantages

The disadvantages of open loop control system are,

1) Such systems are inaccurate and unreliable because accuracy of such systems are totally dependent on the accurate precalibration of the controller.
2) Such systems give inaccurate results if there are variations in the external environment i.e. such systems cannot sense environmental changes.
3) Similarly they cannot sense internal disturbances in the system, after the controller stage.
4) To maintain the quality and accuracy, recalibration of the controller is necessary from time to time.

To overcome all the above disadvantages, generally in practice closed loop systems are used.

The good example of an open loop system is an electric switch. This is open loop because output is light and switch is controller of lamp. Any change in light has no effect on the ON-OFF position of the switch, i.e. its controlling action.

Similarly automatic washing machine. Here output is degree of cleanliness of clothes. But any change in this output will not affect the controlling action or will not decide the operation time or will not decide the amount of detergent which is to be used. Some other examples are traffic signal, automatic toaster system etc.

1.4.3 Real Time Applications of an Open Loop System

The various illustrations of an open loop system are discussed below,

1.4.3.1 Sprinkler used to Water a Lawn

The system is adjusted to water a given area by opening the water valve and observing the resulting pattern. When the pattern is considered satisfactory, the system is "calibrated" and no further valve adjustment is made.

1.4.3.2 Stepper Motor Positioning System

The actual position in such system is usually not monitored. The motor controller commands a certain number of steps by the motor to drive the output to a previously determined location.

1.4.3.3 Automatic Toaster System

In this system, the quality of toast depends upon the time for which the toast is heated. Depending on the time setting, bread is simply heated in this system. The toast quality is to be judged by the user and has no effect on the inputs.

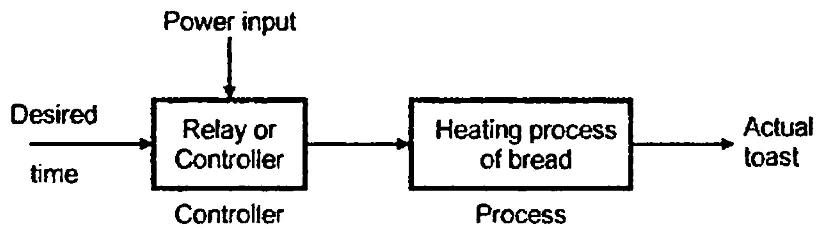


Fig. 1.10

1.4.3.4 Traffic Light Controller

A traffic flow control system used on roads is time dependent. The traffic on the road becomes mobile or stationary depending on the duration and sequence of lamp glow. The sequence and duration are controlled by relays which are predetermined and not dependent on the rush on the road.

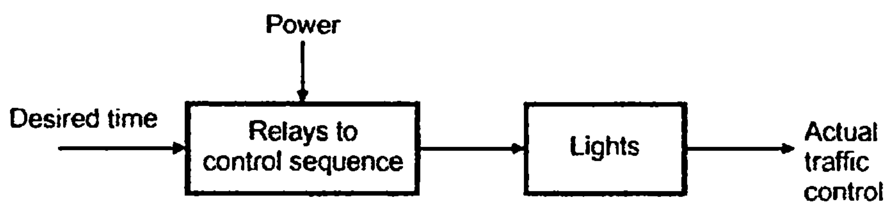


Fig. 1.11

1.4.3.5 Automatic Door Opening and Closing System

In this system, photo sensitive devices are used. When a person interrupts a light, photo device generates actuating signal which opens the door for specific time. When person passes through the door, light becomes continuous closing the door. The opening and closing of the door is the output which has nothing to do with the inputs, hence an open loop system.

The room heater, fan regulator, automatic coffee server, electric lift, theatre lamp dimmer, automatic dryer are examples of open loop system.

1.5 Closed Loop System

Definition : *A system in which the controlling action or input is somehow dependent on the output or changes in output is called closed loop system.*

To have dependence of input on the output, such system uses the feedback property.

Feedback : *Feedback is a property of the system by which it permits the output to be compared with the reference input to generate the error signal based on which the appropriate controlling action can be decided.*

2 Basics of Laplace Transform

2.1 Background

The various methods used to solve the engineering problems are based on the replacement of functions of time by certain frequency dependent variables. This makes the computation job very easy. The known example of such method is the use of Fourier series to solve certain electrical problems.

The transformation technique relating the time functions to frequency dependent functions of a complex variable is called the **Laplace transformation technique.** Such transformation is very useful in solving linear differential equations. The transfer function of a system, which is heart of the control system analysis is based on the Laplace transform. This chapter gives the definition of Laplace transform, some commonly used functions and Laplace transform pairs and useful properties of Laplace and inverse Laplace transform. Some examples are also included demonstrating the superiority of Laplace approach over the conventional approach.

2.2 Definition of Laplace Transform

The Laplace transform is defined as below :

Let f(t) be a real function of a real variable t defined for t > 0, then

$$F(s) = L\{f(t)\} = \int_0^\infty f(t).e^{-st}\, dt$$

Where F(s) is called Laplace transform of f(t). And the variable 's' which appears in F(s) is frequency dependent complex variable. It is given by,

$$s = \sigma + j\omega$$

Where σ = Real part of complex variable 's'.

ω = Imaginary part of complex variable 's'.

The time function f(t) is obtained back from the Laplace transform by a process called Inverse Laplace transform and denoted as L^{-1}. Thus,

Example 2.3 : *Find the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s+2)}{s(s+3)(s+4)}$$

Solution : The degree of N(s) is less than D(s). Hence F(s) can be expressed as,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{(s+3)} + \frac{K_3}{(s+4)}$$

Where $K_1 = s \,.\, F(s)\Big|_{s=0} = s \,.\, \dfrac{(s+2)}{s(s+3)(s+4)}\Big|_{s=0} = \dfrac{2}{3\times 4} = \dfrac{1}{6}$

$$K_2 = (s+3)\,.\,F(s)\Big|_{s=-3} = (s+3)\,.\,\frac{(s+2)}{s(s+3)(s+4)}\Big|_{s=-3} = \frac{(-3+2)}{(-3)(-3+4)} = \frac{1}{3}$$

$$K_3 = (s+4)\,.\,F(s)\Big|_{s=-4} = (s+4)\,.\,\frac{(s+2)}{s(s+3)(s+4)}\Big|_{s=-4} = \frac{(-4+2)}{(-4)(-4+3)} = \frac{1}{2}$$

$$\therefore \quad F(s) = \frac{1/6}{s} + \frac{1/3}{(s+3)} - \frac{1/2}{(s+4)}$$

Taking inverse Laplace transform,

$$\therefore \quad f(t) = \frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-4t}$$

2.4.2 Multiple Roots

The given function is of the form,

$$F(s) = \frac{N(s)}{(s-a)^n\, D'(s)}$$

Here there is multiple root of the order 'n' existing at s = a. The method of writing the partial fraction expansion for such multiple roots is,

$$F(s) = \frac{K_0}{(s-a)^n} + \frac{K_1}{(s-a)^{n-1}} + \frac{K_2}{(s-a)^{n-2}} + \ldots + \frac{K_{n-1}}{(s-a)} + \frac{N'(s)}{D'(s)}$$

Where $\dfrac{N'(s)}{D'(s)}$ represents remaining terms of the expansion of F(s).

Key Point: *Thus a separate coefficient is assumed for each power of repetative root, starting from its highest power n to 1.*

Here find the L.C.M. of the entire right hand side and express numerator interms of $K_0, K_1, K_2, \ldots$ The numerator N(s) on left hand side is known. Compare the coefficients of all powers of s in the numerator of both sides which will give simultaneous equations interms of $K_0, K_1, K_2, \ldots$ Solving these equations we can obtain the coefficients $K_0, K_1, K_2, \ldots$

For ease of solving simultaneous equations, we can find out the coefficient K_0 by the same method as discussed for simple roots.

$$K_0 = (s-a)^n \cdot F(s)\Big|_{s=a}$$

Similarly coefficients for simple roots present if any, can also be calculated by the method discussed earlier, for ease of solving simultaneous equations.

While finding Laplace inverse transform of expanded F(s) refer to standard transform pairs,

$$L[t^n] = \frac{n!}{s^{n+1}} \text{ and}$$

$$L[e^{-at}\, t^n] = \frac{n!}{(s+a)^{n+1}} \text{ and } L[e^{+at}\, t^n] = \frac{n!}{(s-a)^{n+1}}$$

Example 2.4 : *Obtain the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s-2)}{s(s+1)^3}$$

Solution : The given F(s) can be expressed as,

$$F(s) = \frac{K_0}{(s+1)^3} + \frac{K_1}{(s+1)^2} + \frac{K_2}{(s+1)} + \frac{K_3}{s}$$

Finding L.C.M. of right hand side,

$$\frac{(s-2)}{s(s+1)^3} = \frac{K_0(s) + K_1(s+1)s + K_2(s+1)^2 s + K_3(s+1)^3}{s(s+1)^3}$$

$$\therefore (s-2) = K_0 s + K_1 s^2 + K_1 s + K_2 s^3 + 2K_2 s^2 + K_2 s + K_3 s^3 + 3K_3 s^2 + 3K_3 s + K_3$$

Comparing coefficients of various powers of s on both sides,

For s^3, $K_2 + K_3 = 0$ ... (1)

For s^2, $K_1 + 2K_2 + 3K_3 = 0$ ... (2)

For s^1, $K_0 + K_1 + K_2 + 3K_3 = 1$ ... (3)

For s^0, $K_3 = -2$ ... (4)

As $K_3 = -2$

from (1), $K_2 = 2$

$\therefore$ from (2), $K_1 = 2$

∴ from (3), $K_0 = 3$

$$\therefore \quad F(s) = \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2}{s}$$

Now $$L[e^{-at}\, t^n] = \frac{n!}{(s+a)^{n+1}}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{e^{-at}\, t^n}{n!}$$

$$\therefore \quad F(s) = 3 \cdot \frac{1}{(s+1)^3} + 2 \cdot \frac{1}{(s+1)^2} + 2 \cdot \frac{1}{(s+1)} - 2 \cdot \frac{1}{s}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = \frac{3}{2!} e^{-t} \cdot t^2 + \frac{2}{1!} e^{-t} \cdot t + 2e^{-t} - 2$$

$$\therefore \quad f(t) = \frac{3}{2} t^2 e^{-t} + 2 t e^{-t} + 2 e^{-t} - 2$$

2.4.3 Complex Conjugate Roots

If there exists a quadratic term in D(s) of F(s) whose roots are complex conjugates then the F(s) is expressed with a first order polynomial in s in the numerator as,

$$F(s) = \frac{As+B}{\left(s^2+\alpha s+\beta\right)} + \frac{N'(s)}{D'(s)}$$

Where $\left(s^2+\alpha s+\beta\right)$ is the quadratic whose roots are complex conjugates while $\frac{N'(s)}{D'(s)}$ represents remaining terms of the expansion. The A and B are partial fraction coefficients.

The method of finding the coefficients in such a case is same as discussed earlier for the multiple roots. Once A and B are known then use the following method for calculating inverse Laplace transform.

Consider $$F_1(s) = \frac{As+B}{s^2+\alpha s+\beta} \qquad \text{A and B are known}$$

Now complete the square in the denominator by calculating last term as,

$$L.T. = \frac{(M.T.)^2}{4\,(F.T.)}$$

Where L.T = Last term

M.T = Middle term

F.T = First term

$$\therefore \quad \text{L.T.} = \frac{\alpha^2}{4}$$

$$\therefore \quad F_1(s) = \frac{As+B}{s^2+\alpha s+\frac{\alpha^2}{4}+\beta-\frac{\alpha^2}{4}} = \frac{As+B}{\left(s+\frac{\alpha}{2}\right)^2+\omega^2}$$

Where $\omega = \sqrt{\beta-\frac{\alpha^2}{4}}$

Now adjust the numerator As + B in such a way that it is of the form,

$$L[e^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2} \quad \text{or} \quad L[e^{-at}\cos\omega t] = \frac{(s+a)}{(s+a)^2+\omega^2}$$

Key Point: *Thus inverse Laplace transform of F(s) having complex conjugate roots of D(s), always contains sine, cosine or damped sine or damped cosine functions.*

Example 2.5 : *Find the inverse Laplace transform of*

$$F(s) = \frac{s^2+3}{(s^2+2s+5)(s+2)}$$

Solution : The given F(s) can be written as,

$$F(s) = \frac{As+B}{s^2+2s+5}+\frac{C}{s+2}$$

As s^2+2s+5 has complex conjugate roots. To find A, B and C find L.C.M. of right hand side,

$$\therefore \quad F(s) = \frac{(s+2)(As+B)+C(s^2+2s+5)}{(s^2+2s+5)(s+2)}$$

$$\therefore \quad \frac{s^2+3}{(s^2+2s+5)(s+2)} = \frac{As^2+2As+Bs+2B+Cs^2+2sC+5C}{(s^2+2s+5)(s+2)}$$

Comparing the coefficients of various powers of s, of the numerators of both sides

$$\therefore \quad s^2+3 = s^2(A+C)+s(2A+B+2C)+(2B+5C)$$

$$\therefore \quad A+C = 1 \quad \quad ...(1)$$

$$\therefore \quad 2A+B+2C = 0 \quad \quad ...(2)$$

$$\therefore \quad 2B+5C = 3 \quad \quad ...(3)$$

To solve the equations quickly, the coefficient C corresponding to the simple, real root can be obtained as,

$$\therefore \quad C = F(s).(s+2)\Big|_{s=-2} = \frac{(s^2+3)(s+2)}{(s^2+2s+5)(s+2)}\Bigg|_{s=-2} = \frac{(4+3)}{(4-4+5)} = \frac{7}{5}$$

Substituting in (1) and (2),

$$A = -\frac{2}{5}$$

and

$$B = -2$$

$$\therefore \quad F(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5}+\frac{\frac{7}{5}}{(s+2)}$$

Consider

$$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5}$$

Completing square in the denominator,

$$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+1+5-1} = \frac{-\frac{2}{5}s-2}{(s+1)^2+(2)^2} = -\frac{2}{5}\left[\frac{s+5}{(s+1)^2+(2)^2}\right]$$

$$= -\frac{2}{5}\left[\frac{s+1+4}{(s+1)^2+(2)^2}\right] \qquad \text{split 4 as } 2\times 2$$

$$= -\frac{2}{5}\left[\frac{s+1}{(s+1)^2+(2)^2}+2\times\frac{2}{(s+1)^2+(2)^2}\right]$$

$$\therefore \quad F(s) = -\frac{2}{5}\left\{\frac{(s+1)}{(s+1)^2+(2)^2}+2\times\frac{2}{(s+1)^2+(2)^2}\right\}+\frac{\frac{7}{5}}{(s+2)}$$

As

$$L^{-1}\left[\frac{(s+a)}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\cos\omega t\right] \quad \text{and}$$

$$L^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\sin\omega t\right]$$

Hence taking inverse Laplace transform of F(s),

$$f(t) = -\frac{2}{5}\left[e^{-t}\cos 2t+2\,e^{-t}\sin 2t\right]+\frac{7}{5}\,e^{-2t}$$

2.5 Use of Laplace Transform in Control System

The control systems can be classified as electrical, mechanical, hydraulic, thermal and so on. All systems can be described by integrodifferential equations of various orders. While the output of such systems for any input can be obtained by solving such integrodifferential equations. Mathematically, it is very difficult to solve such equations in time domain. The Laplace transform of such integrodifferential equations converts them into simple algebraic equations. All the complicated computations then can be easily performed in s domain as the equations to be handled are algebraic in nature. Such transformed equations are known as equations in frequency domain.

Then by eliminating unwanted variable, the required variable in s domain can be obtained. Then by using technique of Laplace inverse, time domain function for the required variable can be obtained. Hence making the computations easy by converting the integrodifferential equations into algebraic is the main essence of the Laplace transform.

Example 2.6 : *Obtain the expression for y(t) which is satisfying the differential equation*

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8\, y(t) = 16e^{-t}.$$ *Neglect initial conditions.*

Solution : Taking Laplace transform of both sides of the given differential equation and neglecting initial condition terms in Laplace transform of $\frac{d^2y(t)}{dt^2}$ and $\frac{dy(t)}{dt}$ we get,

$$s^2Y(s) + 6sY(s) + 8Y(s) = \frac{16}{s+1}$$

$$\therefore \quad (s^2+6s+8)\,Y(s) = \frac{16}{(s+1)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s^2+6s+8)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s+2)(s+4)}$$

$$\therefore \quad Y(s) = \frac{a_1}{s+1} + \frac{a_2}{s+2} + \frac{a_3}{s+4}$$

$$\therefore \quad Y(s) = \frac{5.33}{s+1} - \frac{8}{s+2} + \frac{2.66}{s+4}$$

Taking inverse Laplace transform of Y(s),

$$y(t) = 5.33e^{-t} - 8e^{-2t} + 2.66e^{-4t}$$

This is the required solution of differential equation.

2.6 Special Case of Inverse Laplace Transform

Let us see now if the order of P(s) and Q(s) of the function F(s) is same. In such case P(s) must be divided by Q(s), to obtain the seperation of F(s) as a constant term which is result of the division and the remainder polynomial P'(s) having order less than Q(s).

So $$F(s) = \frac{P(s)}{Q(s)}$$ ... order of P(s) and Q(s) same

$$= K+\frac{P'(s)}{Q(s)} \quad \text{.... after dividing P(s) by Q(s)}$$

Now Laplace inverse of constant term is impulse function. Refer last pair in the Table 2.2

$\therefore$ | $L^{-1}\{K\} = K\delta(t)$ where $\delta(t)$ = unit impulse. |

While $P'(s) / Q(s)$ can now be expressed to obtain partial fraction expansion, to get its inverse very easily.

Note : The same method is to be applied F(s) with order of numerator polynomial P(s) is greater than denominator polynomial Q(s)

Example 2.7 : *Find the Laplace inverse of* $F(s) = \frac{s^3+18s^2+3s+5}{s^3+8s^2+17s+10}$.

Solution : Divide P(s) by Q(s).

$$\begin{array}{r|l} s^3+8s^2+17s+10 & s^3+18s^2+3s+5 \;(1 \rightarrow K \\ & s^3+8s^2+17s+10 \\ & - \quad - \quad - \quad - \\ \hline & 10s^2-14s-5 \quad \rightarrow P'(s) \end{array}$$

$$\therefore \quad F(s) = 1+\frac{10s^2-14s-5}{s^3+8s^2+17s+10}$$

$$= 1+\frac{10s^2-14s-5}{(s+2)\,(s+1)\,(s+5)} = 1+\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{s+5}$$

$$\therefore \quad A = \left.\frac{10s^2-14s-5}{(s+1)\,(s+5)}\right|_{s=-2} = -21$$

$$B = \left.\frac{10s^2-14s-5}{(s+2)\,(s+5)}\right|_{s=-1} = 4.75$$

$$C = \left.\frac{10s^2-14s-5}{(s+1)\,(s+2)}\right|_{s=-5} = 26.25$$

$$\therefore \quad F(s) = 1-\frac{21}{s+2}+\frac{4.75}{s+1}+\frac{26.25}{s+5}$$

$$\therefore \quad f(t) = L^{-1}\{F(s)\} = \delta(t)-21\,e^{-2t}+4.75\,e^{-t}+26.25\,e^{-5t}$$

Where $L^{-1}\{1\} = \delta(t)$ = Unit impulse function.

Examples with Solutions

Example 2.8 : *In the circuit given, the values of R and C are 1 MΩ and 1 μF respectively. Obtain the expression for the current flowing in the circuit if it is supplied with an input of step voltage of 1 V at t = 0.*

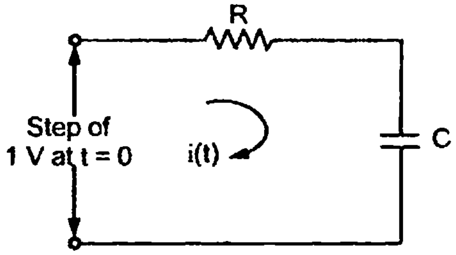


Fig. 2.1

Solution : Let us write down the equation for the circuit given, assuming supply voltage as v(t).

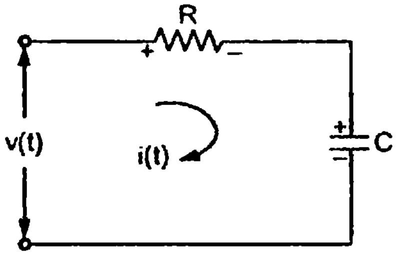


Fig. 2.1(a)

Applying Kirchhoff's voltage law we get,

$$v(t) = i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

Where voltage across capacitor is $\frac{1}{C}\int i(t)\,dt$

While v(t) = Step of 1 V at t = 0 as shown in the Fig. 2.1(b).

Taking Laplace transform of both sides of above equation,

$$V(s) = I(s)\,R + \frac{1}{C}\left[\frac{I(s)}{s}\right]$$

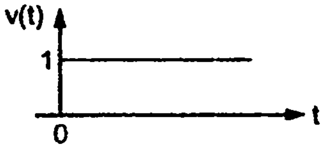


Fig. 2.1(b)

Neglecting initial conditions hence neglecting the term of initial condition in Laplace transform of $\int i(t)\,dt$.

Now $V(s) = \frac{1}{s}$ as v(t) is step of 1 V ... Refer pair 1 in Table 2.2

$$\therefore \quad \frac{1}{s} = I(s)\,R + \frac{1}{sC}\,I(s)$$

$$\therefore \quad \frac{1}{s} = I(s)\left[R + \frac{1}{sC}\right]$$

$$\therefore \quad \frac{1}{s} = \frac{I(s)\,[1+sRC]}{sC}$$

$$\therefore \quad I(s) = \frac{C}{1+sRC} = \frac{C}{RC\left[s+\frac{1}{RC}\right]}$$

$$\therefore \quad I(s) = \frac{1}{R}\left[\frac{1}{s+\frac{1}{RC}}\right]$$

Taking Laplace inverse, referring Table 2.1

$$i(t) = \frac{1}{R}\,e^{-\frac{1}{RC}t}$$

Substituting values of R and C,

$$i(t) = \frac{1}{1\times 10^{+6}}\,e^{-\frac{t}{1\times 10^{6}\times 1\times 10^{-6}}}$$

$$\therefore \quad i(t) = 1\times 10^{-6}\,e^{-t}\ \text{A}$$

Key Point: *It can be observed in this example that solving the equation in time domain for the current involves calculation of complementary function, then particular integral and arbitrary constants, independently. In Laplace approach we get the answers of all in a single step. Hence Laplace approach proves to be superior over a normal approach.*

Example 2.9 : *A series circuit consisting of resistance R and an inductance of L henry is connected to a supply of v(t) volts. Find the expression of the current in s domain. Also calculate the value of current at t = 0.5 msec with R = $1\times10^3\ \Omega$, L = 25 mH and supply is a step voltage of 50 V. Neglect initial condition.*

Solution : The circuit is shown in the Fig. 2.2.

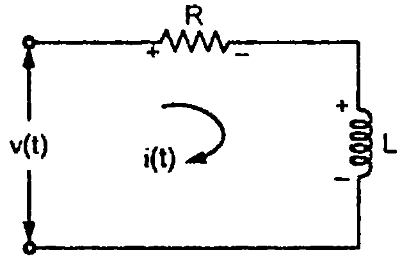


Fig. 2.2

Applying KVL we get,

$$v(t) = i(t)\,R + L\frac{di(t)}{dt}$$

Taking Laplace and neglecting initial conditions of current we get,

$$V(s) = I(s)R + LsI(s)$$

$$V(s) = I(s)\,[R + sL]$$

$$I(s) = \frac{V(s)}{R+sL}$$

Now $R = 1\times10^3\,\Omega$ $L = 25\times10^{-3}H$ and $V(s) = \frac{50}{s}$

$$\therefore \quad I(s) = \frac{50}{s[1\times 10^3 + s\times 25\times 10^{-3}]}$$

$$\therefore \quad I(s) = \frac{50}{25\times 10^{-3}\ s\times [s+4\times 10^4]} \quad \text{...Taking } 25\times10^{-3} \text{ outside}$$

$$\therefore \quad I(s) = \frac{2000}{s\,[s+4\times 10^4]} = \frac{A}{s} + \frac{B}{[s+4\times 10^4]} \quad \text{... Partial fractions}$$

$\therefore$ A = 0.05 and B = − 0.05

$$\therefore \quad I(s) = \frac{0.05}{s} - \frac{0.05}{s+4\times 10^4}$$

Taking inverse Laplace transform,

$$i(t) = 0.05 - 0.05\ e^{-4\times10^4 t}\ A$$

$$\therefore \quad \mathbf{i(t) = 0.05\left(1-e^{-4\times10^4 t}\right) A}$$

At $t = 0.5$ msec $= 0.5\times 10^{-3}$ sec

$$i(t) = 0.05\left(1-e^{-4\times10^4 \times 0.5\times10^{-3}}\right) A$$

$$\therefore \quad \mathbf{i(t) = 0.05\ A.}$$

Example 2.10 : *Calculate Laplace inverse of* $F(s) = \frac{20}{s(s^2+2s+5)}$

Solution : $F(s) = \frac{20}{s(s^2+2s+5)}$

Now quadratic s^2+2s+5 has complex conjugate roots.

Therefore partial fraction expansion of F(s) is,

$$F(s) = \frac{20}{s(s^2+2s+5)} = \frac{a_1}{s} + \frac{a_2\,s+a_3}{s^2+2s+5}$$

$$\therefore \quad 20 = a_1\,(s^2+2s+5)\ s +\ s(a_2\,s+a_3)$$

$$\therefore \quad 20 = s^2\,(a_1+a_2)+s(2a_1+a_3)+5a_1$$

$$\therefore \quad 5a_1 = 20 \qquad \text{equating constants}$$

$$2a_1+a_3 = 0 \qquad \text{equating coefficients of s}$$

$$a_1+a_2 = 0 \qquad \text{equating coefficients of } s^2$$

$$L^{-1}[F(s)] = L^{-1}\{L(f(t))\} = f(t)$$

The time function f(t) and its Laplace transform F(s) is called **transform pair.**

Example 2.1 : *Find the Laplace transform of e^{-at} and 1 for $t \geq 0$.*

Solution : (i) $f(t) = e^{-at}$

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{-at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt = \left[-\frac{1}{(s+a)} \cdot e^{-(s+a)t}\right]_0^\infty$$

$$= 0 - \left(\frac{-1}{s+a}\right) = \frac{1}{s+a}$$

$$\therefore \quad L\{e^{-at}\} = \frac{1}{s+a} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

(ii) $f(t) = 1$

$$\therefore \quad F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\therefore \quad L\{1\} = \frac{1}{s} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s}\right\} = 1$$

2.3 Properties of Laplace Transform

Number of important properties of the Laplace transform are discussed in this section. The table of Laplace transform pairs is developed using these properties.

2.3.1 Linearity

The transform of a finite sum of time functions is the sum of the Laplace transforms of the individual functions.

So if $F_1(s)$, $F_2(s)$, ..., $F_n(s)$ are the Laplace transforms of the time functions $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ respectively then,

$$L\{f_1(t) + f_2(t) + \ldots\ldots + f_n(t)\} = F_1(s) + F_2(s) + \ldots\ldots + F_n(s)$$

The property can be further extended if the time functions are multiplied by the constants i.e.

$$L\{a_1 f_1(t) + a_2 f_2(t) + \ldots\ldots + a_n f_n(t)\} = a_1 F_1(s) + a_2 F_2(s) + \ldots\ldots + a_n F_n(s)$$

Where a_1, a_2,............, a_n are constants.

$$a_1 + a_2 = 0 \qquad \text{equating coefficients of } s^2$$

$$\therefore \quad a_1 = 4 \qquad a_2 = -4 \qquad a_3 = -8$$

$$\therefore \quad F(s) = \frac{4}{s} + \frac{-4s-8}{s^2+2s+5} = \frac{4}{s} - 4\left[\frac{s+2}{s^2+2s+5}\right]$$

Completeing square in the denominator

$$F(s) = \frac{4}{s} - 4\left[\frac{s+2}{s^2+2s+1+5-1}\right] = \frac{4}{s} - 4\left[\frac{s+2}{(s+1)^2+(2)^2}\right]$$

Now the denominator is in the form $(s+a)^2+\omega^2$. Let us adjust numerator to get the expression in standard form.

$$\therefore \quad F(s) = \frac{4}{s} - 4\left[\frac{s+1+1}{(s+1)^2+(2)^2}\right]$$

$$= \frac{4}{s} - 4\left[\frac{s+2}{(s+1)^2+(2)^2} + \frac{1}{(s+1)^2+(2)^2}\right]$$

Now first term is in the form $\frac{(s+a)}{(s+a)^2+\omega^2}$ which is Laplace of $e^{-at}\cos\omega t$.

While the second term needs adjustment of constant to get in the form $\frac{\omega}{(s+a)^2+\omega^2}$ which is Laplace of $e^{-at}\sin\omega t$.

$$\therefore \quad F(s) = \frac{4}{s} - 4\left[\frac{(s+1)}{(s+1)^2+(2)^2} + \frac{1}{2}\cdot\frac{2}{(s+1)^2+(2)^2}\right]$$

Taking inverse Laplace transform we get.

$$f(t) = 4 - 4\left[e^{-t}\cos 2t + 0.5\, e^{-t}\sin 2t\right]$$

$$\therefore \quad \mathbf{f(t) = 4 - e^{-t}\,(4\cos 2t + 2\sin 2t)}$$

□□□

2.3.2 Scaling Theorem

If K is a constant then the Laplace transform of K f(t) is given as K times the Laplace transform of f(t).

$$L\{K f(t)\} = K F(s) \qquad \text{... K is constant}$$

2.3.3 Real Differentiation (Differentiation in Time Domain)

Let F(s) be the Laplace transform of f(t). Then,

$$L\left\{\frac{d f(t)}{dt}\right\} = s F(s) - f(0^-)$$

Where $f(0^-)$ indicates value of f(t) at $t = 0^-$ i.e. just before the instant t = 0.

The theorem can be extended for n^{th} order derivative as,

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - f^{(n-1)}(0^-)$$

Where $f^{(n-1)}(0^-)$ is the value of $(n-1)^{th}$ derivative of f(t) at $t = 0^-$.

i.e for n = 2, $L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - s f(0^-) - f'(0^-)$

for n = 3, $L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0^-) - s f'(0^-) - f''(0^-)$ and so on.

This property is most useful as it transforms differential time domain equations to simple algebraic equations, along with the initial conditions, if any.

2.3.4 Real Integration

If F(s) is the Laplace transform of f(t) then,

$$L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

This property can be extended for multiple integrals as,

$$L\left\{\int_0^{t_1}\int_0^{t_2} \ldots\ldots\ldots\ldots \int_0^{t_n} f(t)dt_1, dt_2, \ldots\ldots dt_n\right\} = \frac{F(s)}{s^n}$$

2.3.5 Differentiation by s

If F(s) is the Laplace transform of f(t) then the differentiation by s in the complex frequency domain corresponds to the multiplication by t in the time domain.

$$L\{t\,f(t)\} = \frac{-dF(s)}{ds}$$

Thus,
$$L\{t\} = L\{t \times 1\} = -\frac{d}{ds}[L\{1\}] = -\frac{d}{ds}\left[\frac{1}{s}\right] = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$$

$$L\{t^2\} = L\{t \times t\} = -\frac{d}{ds}[L\{t\}] = -\frac{d}{ds}\left[\frac{1}{s^2}\right] = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$$

$$\therefore \quad L\{t^n\} = \frac{n!}{s^{n+1}}$$

2.3.6 Complex Translation

If F(s) is the Laplace transform of f(t) then by the complex translation property,

$$F(s-a) = L\{e^{at} f(t)\}$$ and $$F(s+a) = L\{e^{-at} f(t)\}$$

$$F(s \mp a) = F(s)\Big|_{s = s \mp a}$$

Where F(s) is the Laplace transform of f(t).

2.3.7 Real Translation (Shifting Theorem)

This theorem is useful to obtain the Laplace transform of the shifted or delayed function of time.

If F(s) is the Laplace transform of f(t) then the Laplace transform of the function **delayed by time T is,**

$$L\{f(t-T)\} = e^{-Ts} F(s)$$

2.3.8 Initial Value Theorem

The Laplace transform is very useful to find the initial value of the time function f(t). Thus if F(s) is the Laplace transform of f(t) then,

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s\,F(s)$$

The only restriction is that f(t) must be continuous or at the most, a step discontinuity at t = 0.

2.3.9 Final Value Theorem

Similar to the initial value, the Laplace transform is also useful to find the final value of the time function f(t). Thus if F(s) is the Laplace transform of f(t) then the final value theorem states that,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

The only restriction is that the roots of the denominator polynomial of F(s) i.e. poles of F(s) have negative or zero real parts.

Example 2.2 : *Find the Laplace transform of sin ωt.*

Solution : The sin ωt can be expressed using Euler's equation as,

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\therefore \quad L\{\sin \omega t\} = L\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\}$$

$$= L\left\{\frac{e^{j\omega t}}{2j}\right\} - L\left\{\frac{e^{-j\omega t}}{2j}\right\} \quad \text{... using Linearity property}$$

Now $L\{e^{at}\} = \frac{1}{s-a}$ and $L\{e^{-at}\} = \frac{1}{s+a}$

$$\therefore \quad L\{\sin \omega t\} = \frac{1}{2j}\left[\frac{1}{s-j\omega}\right] - \frac{1}{2j}\left[\frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{s+j\omega-s+j\omega}{(s-j\omega)(s+j\omega)}\right] = \frac{1}{2j}\left[\frac{2j\omega}{s^2-(j\omega)^2}\right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

$$\therefore \quad L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

Table of Laplace Transforms :

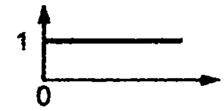
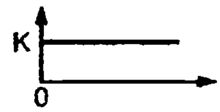
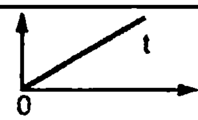
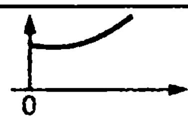
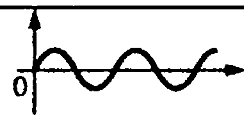
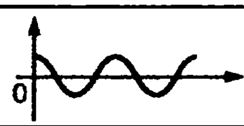
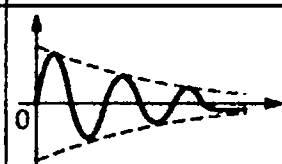
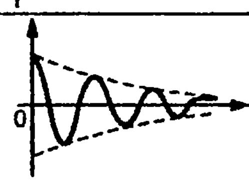
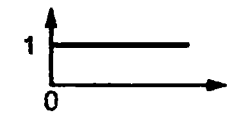
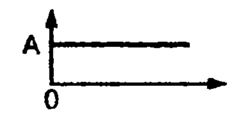
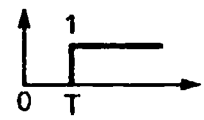
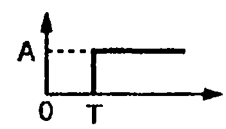
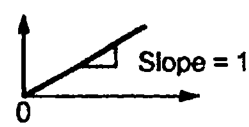
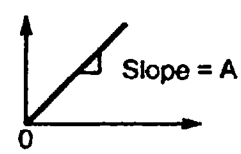
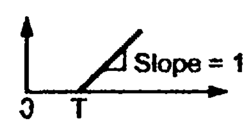
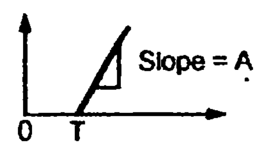
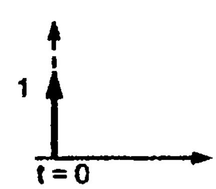
f(t)	F(s)	Waveform
1	$\frac{1}{s}$	
Constant K	$\frac{K}{s}$	
K f(t), K is constant	K F(s)	
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	
e^{-at}	$\frac{1}{s+a}$	
e^{at}	$\frac{1}{s-a}$	
$e^{-at}\ t^n$	$\frac{n!}{(s+a)^{n+1}}$	
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$	

Table 2.1 Standard Laplace transform pair

Function f(t)	Laplace Transform F(s)	Waveforms
Unit step = u(t)	$\frac{1}{s}$	
A u(t)	$\frac{A}{s}$	
Delayed unit step = u(t − T)	$\frac{e^{-Ts}}{s}$	
A u(t − T)	$\frac{A\,e^{-Ts}}{s}$	
Unit ramp = r(t) = t u(t)	$\frac{1}{s^2}$	
A t u(t)	$\frac{A}{s^2}$	
Delayed unit ramp = r(t − T) = (t − T) u(t − T)	$\frac{e^{-Ts}}{s^2}$	
A (t − T) u(t − T)	$\frac{A\,e^{-Ts}}{s^2}$	
Unit impulse = δ (t)	1	

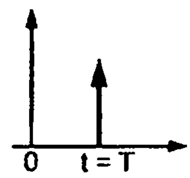
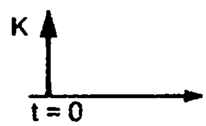
Delayed unit impulse = $\delta(t-T)$	e^{-Ts}	
Impulse of strength K i.e. $K\,\delta(t)$	K	

Table 2.2 Laplace transforms of standard time functions

2.4 Inverse Laplace Transform

As mentioned earlier, inverse Laplace transform is calculated by partial fraction method rather than complex integration evaluation. Let F(s) is the Laplace transform of f(t) then the inverse Laplace transform is denoted as,

$$F(t) = L^{-1}[F(s)]$$

The F(s), in partial fraction method, is written in the form as,

$$F(s) = \frac{N(s)}{D(s)}$$

Where $N(s)$ = Numerator polynomial in s

and $D(s)$ = Denominator polynomial in s

Key Point : ***The given function F(s) can be expressed in partial fraction form only when degree of N(s) is less than D(s).***

Hence if degree of N(s) is equal or higher than D(s) then mathematically divide N(s) by D(s) to express F(s) in quotient and remainder form as,

$$F(s) = Q + F_1(s)$$

$$= Q + \frac{N'(s)}{D'(s)}$$

Where Q = Quotient obtained by dividing N(s) by D(s)

and $F_1(s) = \frac{N'(s)}{D'(s)}$ = Remainder

Now in the remainder, degree of $N'(s)$ is less than $D'(s)$ and hence $F_1(s)$ can be expressed in the partial fraction form. Once F(s) is expanded interms of partial fractions, inverse Laplace transform can be easily obtained by adjusting the terms and referring to the table of standard Laplace transform pairs (Table 2.1).

The roots of denominator polynomial D(s) play an important role in expanding the given F(s) into partial fractions. There are three types of roots of D(s). The method of finding partial fractions for each type is different. Let us discuss these three cases of roots of D(s).

2.4.1 Simple and Real Roots

The roots of D(s) are simple and real. Hence the function F(s) can be expressed as,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-a)(s-b)(s-c)\,...}$$

Where a, b, c ... are the simple and real roots of D(s). The degree of N(s) should be always less than D(s). This can be further expressed as,

$$F(s) = \frac{N(s)}{(s-a)(s-b)(s-c)\,...} = \frac{K_1}{(s-a)} + \frac{K_2}{(s-b)} + \frac{K_3}{(s-c)} + ...$$

Where K_1, K_2, K_3, ... are called **partial fraction coefficients.** The values of K_1, K_2, K_3 ... can be obtained as,

$$K_1 = (s-a)\ F(s)\Big|_{s=a}$$

$$K_2 = (s-b)\,.\,F(s)\Big|_{s=b}$$

$$K_3 = (s-c)\,.\,F(s)\Big|_{s=c}$$

In general, $K_n = (s-s_n)\,.\,F(s)\Big|_{s=s_n}$ and so on.

Where s_n = n^{th} root of D(s)

$$L\{e^{\pm at}\} = \frac{1}{(s \mp a)}$$

is standard Laplace transform pair. Hence once F(s) is expressed in terms of partial fractions, with coefficients K_1, K_2 ... K_n, the inverse Laplace transform can be easily obtained.

$$\therefore \quad f(t) = L^{-1}\left[F(s)\right] = K_1\,e^{at} + K_2\,e^{bt} + K_3\,e^{ct} + ...$$

3 Transfer Function and Impulse Response

3.1 Background

The mathematical indication of cause and effect relationship existing between input and output means to decide the transfer function of the given system. It is commonly used to characterize the input-output relationship of the system.

Transfer function explains mathematical function of the parameters of system, performing on the applied input in order to produce the required output. Laplace transform plays an important role in making mathematical analysis easy. Laplace transform and its use in control system analysis is thoroughly discussed in Chapter-2. In this chapter concept of transfer function, transfer function models and impulse response models of the systems are discussed.

3.2 Concept of Transfer Function

In any system, first the system parameters are designed and their values are selected as per the requirement. The input is selected next, to see the performance of the designed system. This is shown in the Fig. 3.1.

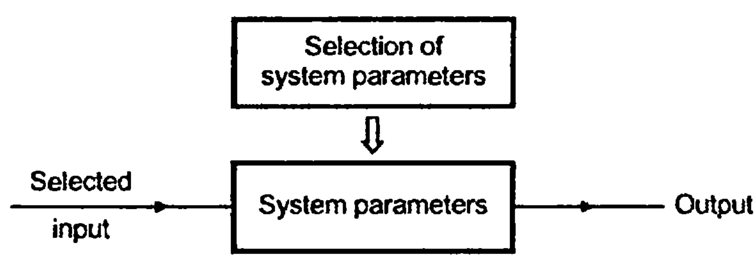


Fig. 3.1

Now performance of the system can be expressed in terms of its output as,

Output = Effect of system parameters on the selected input

∴ Output = Input × Effect of system parameters,

$$\therefore \quad \boxed{\text{Effect of system parameters} = \frac{\text{Output}}{\text{Input}}}$$

This effect of system parameters, role of system parameters in the performance of system can be expressed as ratio of output to input. Mathematically such a function

$$\therefore \quad C(s) = \frac{0.4}{s} - \frac{0.4}{s+5}$$

Taking Laplace inverse of this equation,

$$c(t) = 0.4 - 0.4\, e^{-5t} \qquad \text{... Output}$$

3.5 Some Important Terminologies Related to the T.F.

As transfer function is a ratio of Laplace of output to Laplace of input and hence can be expressed as a ratio of polynomials in 's'.

$$\text{T.F.} = \frac{P(s)}{Q(s)}$$

This can be further expressed as,

$$= \frac{a_0 s^m + a_1 s^{m-1} + a_2 s^{m-2} + ... + a_m}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + ... + b_n}$$

The numerator and denominator can be factorised to get the factorised form of the transfer function as,

$$\therefore \quad \boxed{\text{T.F.} = \frac{K(s-s_a)(s-s_b)......(s-s_m)}{(s-s_1)(s-s_2)......(s-s_n)}}$$

where K is called **system gain factor.** Now if in the transfer function, values of 's' are substituted as s_1 , s_2 , s_3 ...... s_n in the denominator then value of T.F. will become infinity.

3.5.1 Poles of a Transfer Function

Definition : *The values of 's', which make the T.F. infinite after substitution in the denominator of a T.F. are called* ***'Poles'*** *of that T.F.*

So values s_1 , s_2 , s_3 ...... s_n are called poles of the T.F.

These poles are nothing but the roots of the equation obtained by equating denominator of a T.F. to zero.

For example, let the transfer function of a system be,

$$T(s) = \frac{2(s+2)}{s(s+4)}$$

The equation obtained by equating denominator to zero is,

$$s(s+4) = 0$$

$$\therefore \quad s = 0 \quad \text{and} \quad s = -4$$

If these values are used in the denominator, the value of transfer function becomes infinity. Hence poles of this transfer function are s = 0 and – 4.

$$2(s+1)^2 (s+2)(s^2+2s+2) = 0$$

i.e. Simple zero at $s = -2$

Repeated zero at $s = -1$ (twice)

Complex conjugate zeros at $s = -1 \pm j1$.

The zeros are indicated by small circle or zero 'o' in the s-plane.

3.5.4 Pole-Zero Plot

> **Definition :** *Plot obtained by locating all poles and zeros of a T.F. in s-plane is called pole-zero plot of a system.*

3.5.5 Order of a Transfer Function

> **Definition :** *The highest power of 's' present in the characteristic equation i.e. in the denominator polynomial of a closed loop transfer function of a system is called 'Order' of a system.*

3.5.6 D.C. Gain

The value of the transfer function obtained for s = 0 i.e. zero frequency is called the d.c. gain of the system.

$$\text{D.C. Gain} = T(s)\big|_{s=0}$$

Key Point *: It is not possible to indicate the value of d.c. gain on pole-zero plot as it is a constant value. It is required to be separately specified, alongwith the pole-zero plot.*

The d.c. gain decides the resultant constant of the system transfer function.

For example, consider example 3.1 discussed earlier. The system T.F. is $\frac{1}{1+sRC}$.

So $1 + sRC = 0$ is its characteristic equation and system is first order system.

Then $s = -1/RC$ is a pole of that system and T.F. has no zeros.

The corresponding pole-zero plot can be shown as in the Fig. 3.9.

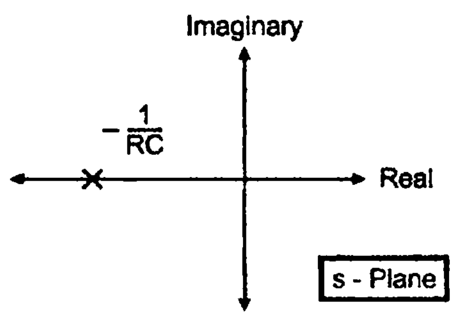


Fig. 3.9

Similarly for example 3.2, the T.F. calculated is,

$$\text{T.F.} = \frac{1}{s^2LC + sRC + 1} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

- The characteristic equation is,

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = 0$$

So system is 2nd order and the two poles are, $-\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$

T.F. has no zeros.

Now if values of R, L and C selected are such that both poles are real, unequal and negative, the corresponding pole-zero plot can be shown as in the Fig. 3.10.

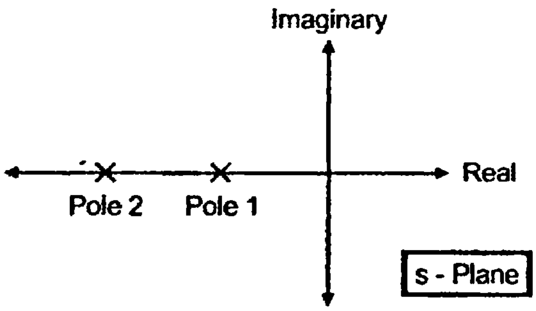


Fig. 3.10

For a system having T.F. as,

$$\frac{C(s)}{R(s)} = \frac{(s+2)}{s\left[s^2+2s+2\right]\left[s^2+7s+12\right]}$$

The characteristic equation is,

$$s(s^2+2s+2)(s^2+7s+12) = 0 \quad \text{i.e.} \quad s(s^2+2s+2)\ (s+3)(s+4) = 0$$

i.e. System is 5th order and there are 5 poles. Poles are $0, -1 \pm j, -3, -4$ while zero is located at '−2'.

The corresponding pole-zero plot can be drawn as shown in the Fig. 3.11.

After getting familiar with introductory remarks about control system, now it is necessary to see how overall systems are represented and the methods to represent the given system, based on the transfer function approach.

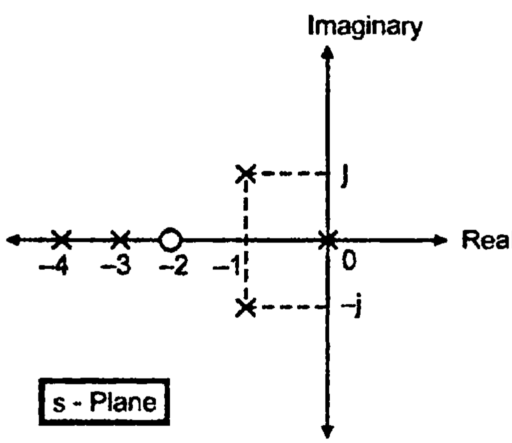


Fig. 3.11

3 · Laplace Transform of Electrical Network

In the use of Laplace in electrical systems, it is always easy to redraw the system by finding Laplace transform of the given network. Electrical network mostly consists of the parameters R, L and C. The various expressions related to these parameters in time domain and Laplace domain are given in the table below.

Element	Time domain expression for voltage	Laplace domain expression for voltage	Laplace domain behaviour
Resistance R	$i(t) \times R$	$I(s)R$	R
Inductance L	$L \frac{d\,i(t)}{dt}$	$sLI(s)$	sL
Capacitance C	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{sC} I(s)$	$\frac{1}{sC}$

Table 3.1

From the table it can be seen that after taking Laplace transform of time domain equations, **neglecting the initial conditions,** the resistance R behaves as R, the inductance behaves as sL, while the capacitance behaves as $\frac{1}{sC}$ and all time domain functions get converted to Laplace domain like i(t) to I(s), v(t) to V(s) and so on.

By using these transformations, the parameters can be replaced by their Laplace transform to get Laplace transform of the entire network. Once this is obtained, simple algebraic equations relating Laplace of various voltages and currents can be directly obtained. This eliminates the step of writing the integrodifferential equations and taking Laplace of them.

e.g. Consider a network shown below,

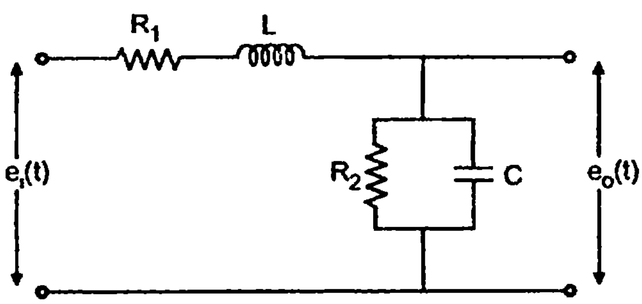


Fig. 3.12

The Laplace of the above network can be obtained by following replacements.

$R_1 \rightarrow R_1$ $\quad L \rightarrow sL$

$R_2 \rightarrow R_2$ $\quad C \rightarrow \frac{1}{sC}$

$e_i(t) \rightarrow E_i(s)$ $\quad e_o(t) \rightarrow E_o(s)$

The other variables then can be introduced which will be directly Laplace variables to obtain the Laplace domain equations directly. Such Laplace of network is shown in the Fig. 3.13.

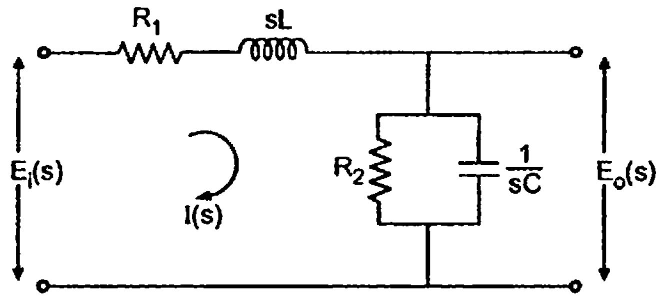


Fig. 3.13

Examples with Solutions

Example 3.6 : *The transfer function of a system is given by,*

$$T(s) = \frac{K(s+6)}{s\,(s+2)\,(s+5)\,(s^2+7s+12)}$$

Determine i) Poles ii) Zeros iii) Characteristic equation and iv) Pole-zero plot in s-plane

Solution :

i) Poles are the roots of the equation obtained by equating denominator to zero i.e. roots of,

$$s\,(s+2)\,(s+5)\,(s^2+7s+12) = 0$$

i.e. $s(s+2)(s+5)(s+3)(s+4) = 0$

So there are 5 poles located at $s = 0, -2, -5, -3$ and -4

ii) Zeros are the roots of the equation obtained by equating numerator to zero i.e. roots of $K(s+6) = 0$

i.e. $s = -6$

There is only one zero.

iii) Characteristic equation is one, whose roots are the poles of the transfer function. So it is,

$$s\,(s+2)\,(s+5)\,(s^2+7s+12) = 0$$

i.e. $s\,(s^2+7s+10)\,(s^2+7s+12) = 0$

i.e $s^5+14s^4+71s^3+154s^2+120s = 0$

iv) Pole-zero plot

This is shown in the Fig. 3.14

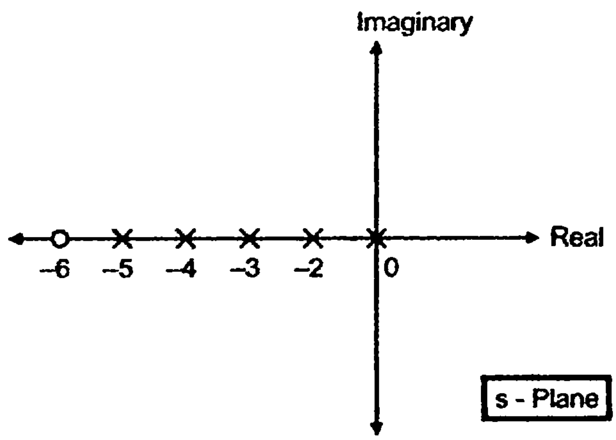


Fig. 3.14

Example 3.7: *The unit impulse response of a system is given by $T(t) = e^{-t}(1-\cos 2t)$ Determine its transfer function.*

Solution : Laplace transform of the impulse response is the transfer function.

$$T(s) = L\{T(t)\}$$

$$= L\{e^{-t}(1-\cos 2t)\} = L\{e^{-t}\} - L\{e^{-t}\cos 2t\}$$

$$= \frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+(2)^2} = \frac{1}{s+1} - \frac{(s+1)}{(s^2+2s+5)}$$

$$\therefore \quad \boxed{T(s) = \frac{4}{(s+1)(s^2+2s+5)}}$$

Example 3.8 : *Obtain the transfer function of the lead network shown in the Fig. 3.15.*

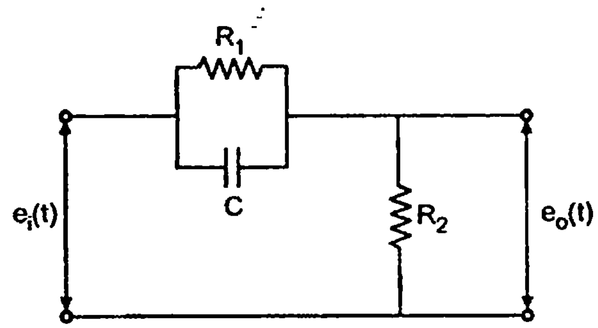


Fig. 3.15

Solution : Take Laplace transform of the network, as shown in the Fig. 3.16.

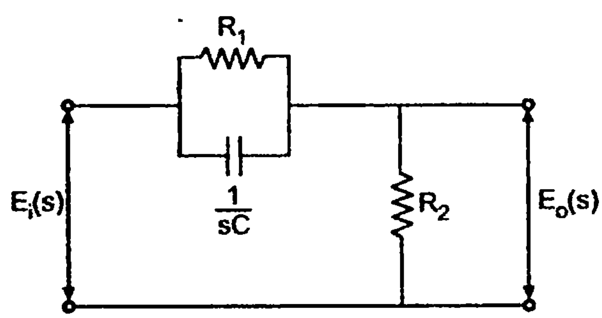


Fig. 3.16

The parallel combination of R_1 and $\frac{1}{sC}$ gives impedance of,

$$Z = \frac{Z_1 Z_2}{Z_1+Z_2} = \frac{R_1 \times \frac{1}{sC}}{R_1 + \frac{1}{sC}} = \frac{R_1}{1+sR_1C}$$

Once the parameters are replaced by corresponding Laplace domain conversions, all the parameters R, sL and 1/sC behave as impedances. Hence their series and parallel combinations can be easily obtained by the algebraic calculations.

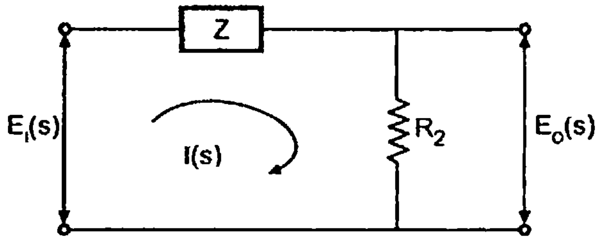


Fig. 3.17

Applying KVL to the circuit,

$$E_i(s) = Z\, I(s) + I(s)\, R_2 \quad \text{... (1)}$$

$$E_o(s) = I(s)\, R_2 \quad \text{... (2)}$$

$$\therefore \quad I(s) = \frac{E_o(s)}{R_2} \quad \text{from (2)}$$

Substituting in (1) we get,

$$E_i(s) = I(s)\,[Z + R_2] = \frac{E_o(s)}{R_2}[Z + R_2]$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{R_2}{Z + R_2}$$

Substituting Z, T. F. $= \dfrac{R_2}{\dfrac{R_1}{1 + sR_1C} + R_2} = \dfrac{R_2(1 + sR_1C)}{R_1 + R_2(1 + sR_1C)}$

$$= \frac{sR_1R_2C + R_2}{R_1 + sR_1R_2C + R_2} = \frac{s + \alpha}{s + \beta}$$

where $\alpha = \dfrac{1}{R_1C}, \quad \beta = \dfrac{(R_1 + R_2)}{R_1R_2C}$

This circuit is also called **lead compensator.**

Example 3.9 : *Find the transfer function of the given circuit.* **(M.U.: Dec.-2003)**

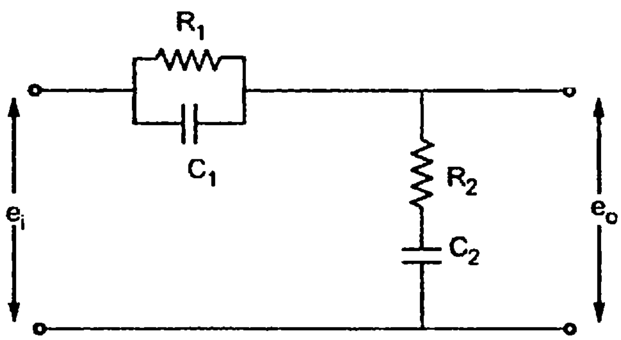


Fig. 3.18

Solution : Draw the Laplace domain network of the given system.

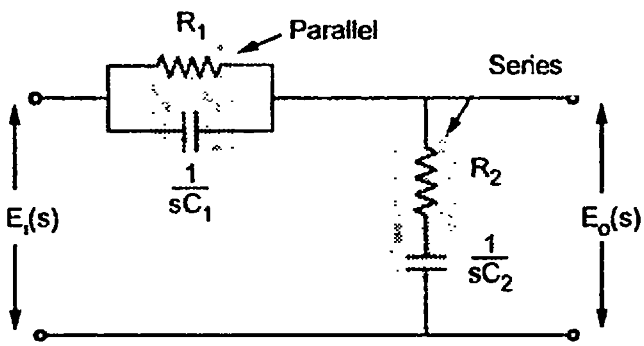


Fig. 3.19

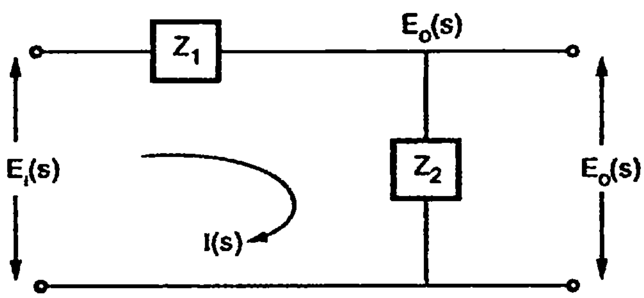


Fig. 3.20

$$Z_1 = R_1 \parallel \frac{1}{sC_1} = \frac{R_1 \times \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}}$$

$$= \frac{R_1}{1+sR_1C_1}$$

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{1+sR_2C_2}{sC_2}$$

$$I(s) = \frac{E_i(s) - E_o(s)}{Z_1} \quad \text{... (1)}$$

$$E_o(s) = I(s)\ Z_2 \quad \text{... (2)}$$

$$\therefore E_o(s) = \frac{E_i(s) - E_o(s)}{Z_1} \times Z_2$$

$$\therefore E_o(s)\left[1 + \frac{Z_2}{Z_1}\right] = \frac{Z_2}{Z_1} E_i(s)$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

$$= \frac{\frac{1+sR_2C_2}{sC_2}}{\frac{R_1}{1+sR_1C_1} + \frac{(1+sR_2C_2)}{sC_2}}$$

$$\therefore \boxed{\frac{E_o(s)}{E_i(s)} = \frac{(1+sR_2C_2)(1+sR_1C_1)}{s^2R_1R_2C_1C_2 + s\,[R_1C_1 + R_2C_2 + R_1C_2] + 1}}$$

Example 3.10 : *Find the transfer function* $\frac{C(s)}{R(s)}$ *of a system having differential equation given below.*

$$2\frac{d^2 c(t)}{dt^2} + 2\frac{d\,c(t)}{dt} + c(t) = r(t) + 2r(t-1)$$

Solution : Taking Laplace transform of the given equation and assuming all initial conditions zero we get,

$$2s^2C(s) + 2s\,C(s) + C(s) = R(s) + 2e^{-s}R(s)$$

Laplace transform of delayed function is,

$$L\{f(t-T)\} = e^{-sT}F(s) \qquad \text{(Refer Table 2.2)}$$

$$\therefore \quad L\{r(t-1)\} = e^{-s.1}R(s)$$

Combining terms of C(s) and R(s) we get,

$$(2s^2 + 2s + 1)\,C(s) = R(s)\,(1 + 2e^{-s})$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{1 + 2e^{-s}}{2s^2 + 2s + 1}}$$

Example 3.11 : *Find* $V_o(s) / V_i(s)$*. Assume gain of buffer amplifier as 1.*

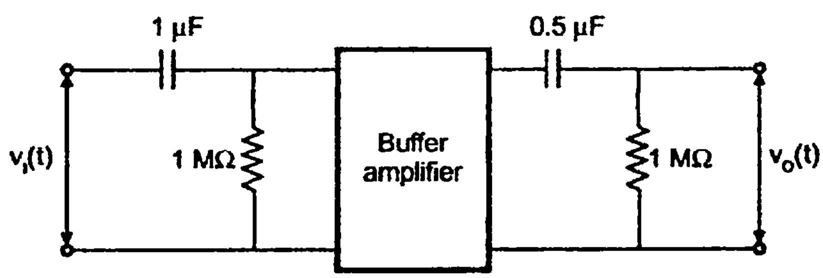


Fig. 3.21

Solution : Taking Laplace transform of the network,

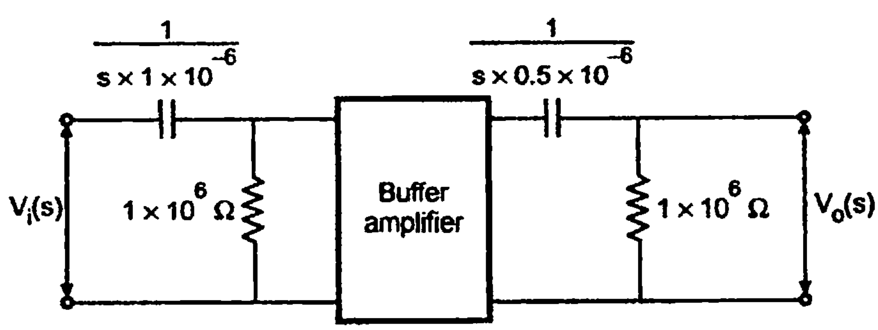


Fig. 3.22

explaining the effect of system parameters on input to produce output is called **transfer function.** Due to the own characteristics of the system parameters, the input gets transferred into output, once applied to the system. This is the concept of transfer function. The exact definition of the transfer function is given in the next section.

3.3 Transfer Function

3.3.1 Definition

Mathematically it is defined as the ratio of Laplace transform of output (response) of the system to the Laplace transform of input (excitation or driving function), under the assumption that ***all initial conditions are zero.***

Symbolically system can be represented as shown in the Fig. 3.2(a). While the transfer function of system can be shown as in the Fig 3.2(b).

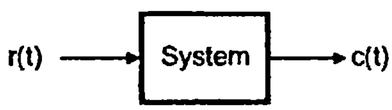


Fig. 3.2(a)

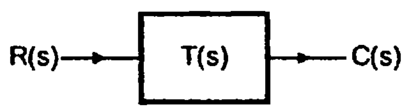


Fig. 3.2(b)

Transfer function of this system is $\frac{C(s)}{R(s)}$, where C(s) is Laplace of c(t) and R(s) is Laplace of r(t).

If T(s) is the transfer function of the system then,

$$T(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{C(s)}{R(s)}$$

Example 3.1 : *For a system shown in the Fig. 3.3, calculate its transfer function where $v_o(t)$ is output and $v_i(t)$ is input to the system.*

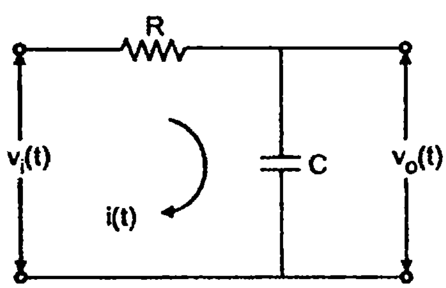


Fig. 3.3

Solution : We can write for this system, equations by applying KVL as,

$$v_i(t) = R \cdot i(t) + \frac{1}{C}\int i(t)\,dt \quad \text{... (1)}$$

and

$$v_o(t) = \frac{1}{C}\int i(t)\,dt \quad \text{... (2)}$$

Let us divide the network into two parts,

Part 1) :

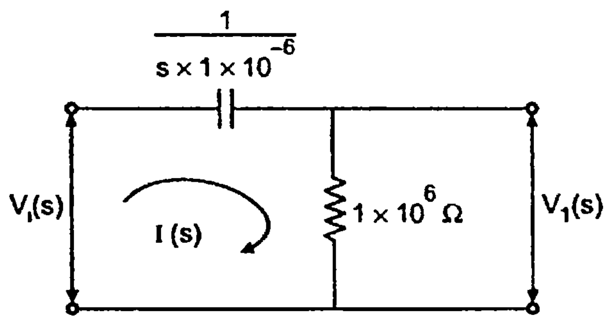


Fig. 3.23

Applying KVL,

$$V_i(s) = \frac{1}{s \times 1 \times 10^{-6}} I(s) + 1 \times 10^6 I(s) \quad \text{... (1)}$$

$$V_1(s) = 1 \times 10^6 I(s) \quad \text{... (2)}$$

$$\therefore \quad I(s) = \frac{V_1(s)}{1 \times 10^6}$$

Substituting in (1) $V_i(s) = \left[\frac{10^6}{s} + 10^6\right] I(s) = \left[\frac{10^6 + s\,10^6}{s}\right]\left[\frac{V_1(s)}{10^6}\right]$

$$\therefore \quad \frac{V_1(s)}{V_i(s)} = \frac{s}{s+1}$$

Part 2) :

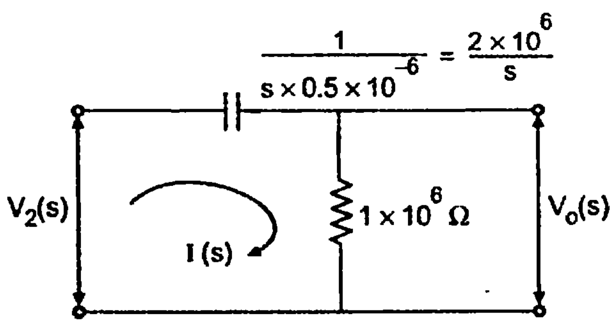


Fig. 3.24

$$\therefore \quad V_2(s) = I(s)\left[\frac{2 \times 10^6}{s} + 1 \times 10^6\right] \quad \text{... (1)}$$

$$V_o(s) = I(s)\, 1 \times 10^6 \quad \text{... (2)}$$

$$\therefore \quad I(s) = \frac{V_o(s)}{10^6}$$

Substituting in (1)

$$V_2(s) = \frac{V_o(s)}{10^6}\left[\frac{2+s}{s}\right]10^6$$

$$\therefore \quad \frac{V_o(s)}{V_2(s)} = \frac{s}{s+2}$$

Now gain of buffer amplifier is 1 (unity)

$$\therefore \quad V_1(s) = V_2(s)$$

$$\therefore \quad \left(\frac{s}{s+1}\right)V_i(s) = \frac{(s+2)}{s}V_o(s)$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{s^2}{(s+1)(s+2)} \quad \text{...T.F.}}$$

Example 3.12 : *Determine the transfer function if the d.c. gain is equal to 10 for the system whose pole-zero plot is shown below.*

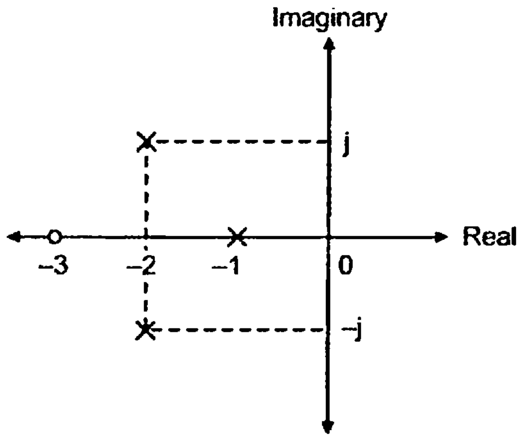


Fig. 3.25

Solution : From pole-zero plot given, the transfer function has 3 poles at s = –1, –2+j and –2–j. And it has one zero at s = –3.

$$\therefore \quad T(s) = \frac{K(s+3)}{(s+1)(s+2+j)(s+2-j)} = \frac{K(s+3)}{(s+1)[(s+2)^2-(j)^2]}$$

$$= \frac{K(s+3)}{(s+1)[s^2+4s+5]}$$

Now d.c. gain is value of T(s) at s = 0 which is given as 10.

$$\therefore \quad \text{d.c. gain} = T(s)|_{\text{at } s=0}$$

$$\therefore \quad 10 = \frac{K \times 3}{1 \times 5}$$

$$\therefore \quad K = \frac{50}{3} = 16.667$$

$$\therefore \quad \boxed{T(s) = \frac{16.667\,(s+3)}{(s+1)\,(s^2+4s+5)}}$$

This is the required transfer function.

Example 3.13 : *If the system transfer function is*

$$\frac{Y(s)}{X(s)} = \frac{s+4}{s^2+2s+5}$$

Obtain the differential equation representing the system.

Solution : $$\frac{Y(s)}{X(s)} = \frac{s+4}{s^2+2s+5}$$

$$\therefore \quad (s^2+2s+5)\ Y(s) = (s+4)\ X(s)$$

$$\therefore \quad s^2\ Y(s) + 2s\ Y(s) + 5Y(s) = 5X(s) + 4X(s)$$

Replacing variable s by $\frac{d}{dt}$ and Y(s) by y(t) and X(s) by x(t) we get,

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 5y(t) = \frac{d}{dt}x(t) + 4x(t)$$

$$\therefore \quad \boxed{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5 = \frac{dx}{dt} + 4x} \quad \text{... Differential equation}$$

Example 3.14 : *For the two port network shown in the Fig. 3.26, obtain the transfer functions*

i) $\frac{V_2(s)}{V_1(s)}$ *and ii)* $\frac{V_1(s)}{I_1(s)}$

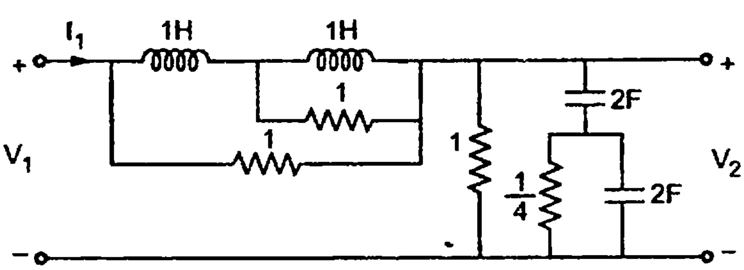


Fig. 3.26

Solution : Draw the Laplace transform of the given network as shown by replacing L by sL, C by $\frac{1}{sC}$ and all time variables by the Laplace variables,

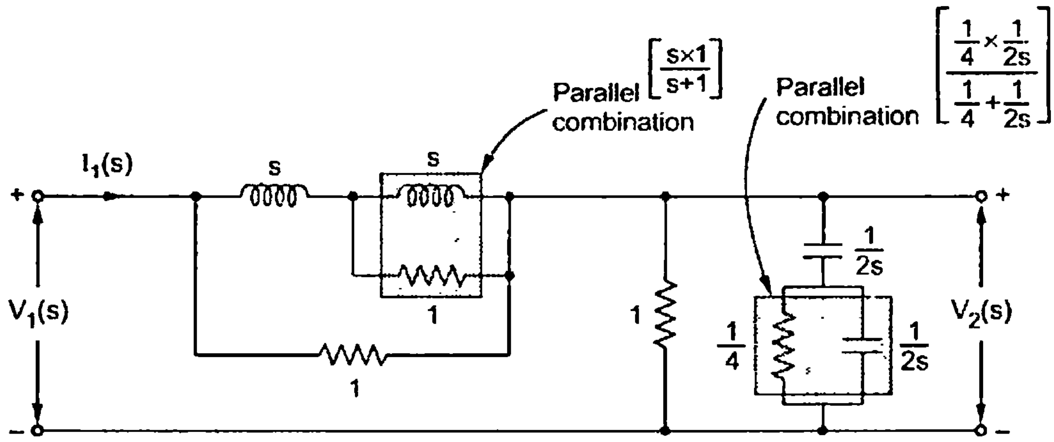


Fig. 3.26(a)

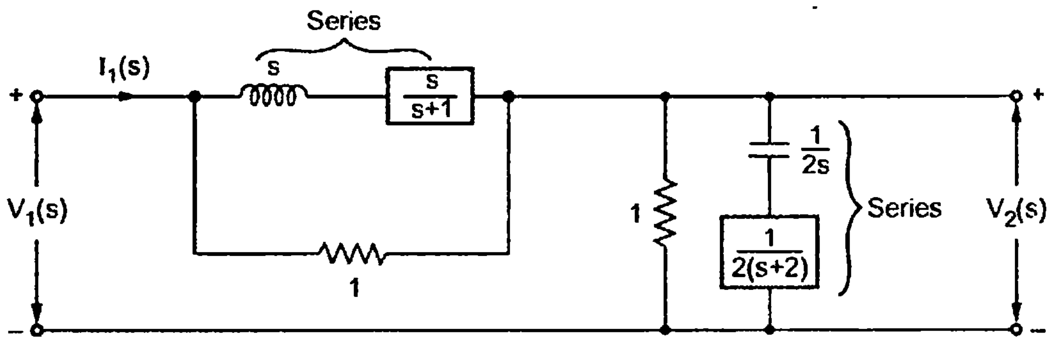


Fig. 3.26(b)

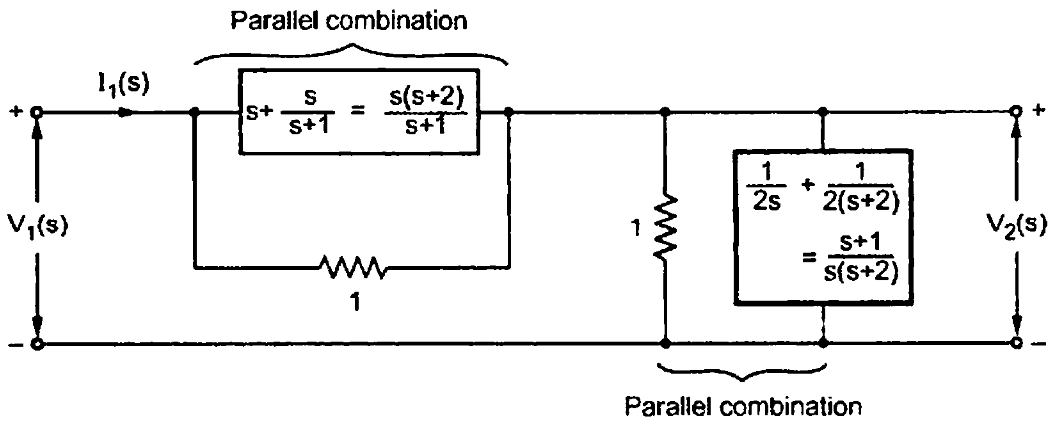


Fig. 3.26(c)

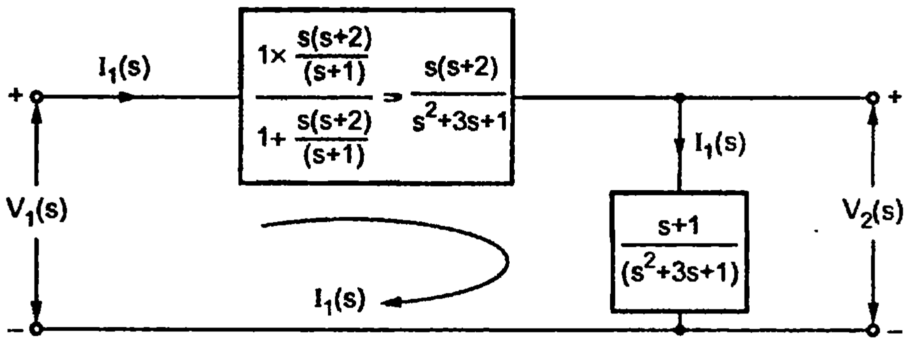


Fig. 3.26(d)

$$I_1(s) = \frac{V_1(s)}{\frac{s(s+2)}{s^2+3s+1}+\frac{s+1}{s^2+3s+1}} = \frac{V_1(s)}{\left[\frac{s^2+3s+1}{s^2+3s+1}\right]} = V_1(s)$$

$\therefore$

$$\boxed{\frac{V_1(s)}{I_1(s)} = 1}$$

$\therefore$

$$V_2(s) = I_1(s)\times\frac{(s+1)}{(s^2+3s+1)} = V_1(s)\times\frac{(s+1)}{(s^2+3s+1)}$$

$\therefore$

$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{s+1}{s^2+3s+1}}$$

Example 3.15 : *What will be the transfer function of a system whose unit step response is a unit impulse function ?* **(M.U. : Dec.-2003)**

Solution : The output c(t) = δ(t) = unit impulse function

and the input r(t) = 1 = unit step function

$\therefore$

$$\text{T.F.} = \frac{C(s)}{R(s)} = \frac{L\{\delta(t)\}}{L\{1\}} = \frac{1}{\frac{1}{s}} = s$$

Review Questions

1) *Define the transfer function of a system.*
2) *Explain the significance of a transfer function stating its advantages and features.*
3) *What are the limitations of transfer function approach?*
4) *How transfer function is related to unit impulse response of a system?*
5) *Define and explain the following terms related to the transfer function of a system.*
   (i) Poles (ii) Zeros (iii) Characteristic equation (iv) Pole-zero plot (v) Order.
6) *The unit impulse response of a system is e^{-7t}. Find its transfer function.* **Ans. :** $\frac{1}{s+7}$

7) *A certain system is described by a differential equation*

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 11y(t) = 5x(t)$$

where y(t) is the output and the x(t) is the input. Obtain the transfer function of the system.

Ans.: $\frac{Y(s)}{X(s)} = \frac{5}{s^2 + 3s + 11}$

8) *A certain system has its transfer function as*

$$\frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + s + 1}$$

Obtain its differential equation.

Ans.: $\frac{d^2c(t)}{dt^2} + \frac{dc(t)}{dt} + c(t) = 2\frac{dr(t)}{dt} + r(t)$

9) *If a system equation is given as*

$$3\frac{dc(t)}{dt} + 2c(t) = r(t - T)$$

Where c(t) is output and r(t) is input shifted by T seconds. Obtain its transfer function.

Ans.: $\frac{e^{sT}}{3s + 2}$

10) *A system when excited by unit step type of input gives following response*

$c(t) = 1 - 2e^{-t} + 4e^{-3t}$

Obtain its transfer function C(s)/R(s)

Ans.: $\frac{3s^2 + 2s + 3}{(s + 1)(s + 3)}$

11) *Derive the transfer function of the system shown in Fig. 3.27. The amplifier gain is K.*

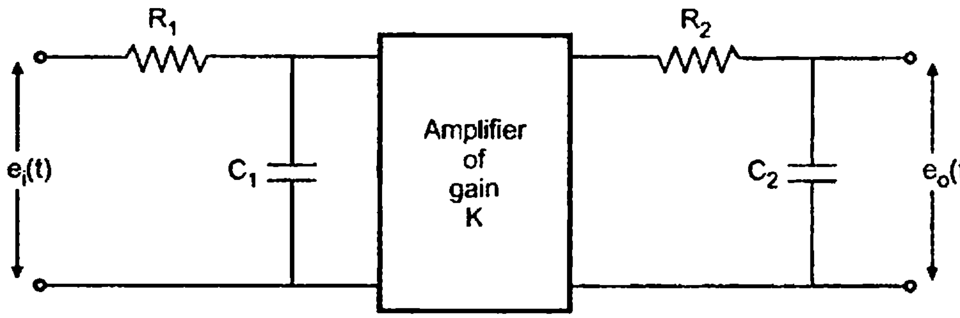


Fig. 3.27

Ans.: $\frac{E_o(s)}{E_i(s)} = \frac{K}{(1 + sR_1C_1)(1 + sR_2C_2)}$

12) *The transfer function of a system is given by*

$$T(s) = \frac{10(s + 8)}{s(s + 4)(s^2 + 6s + 25)}$$

Obtain its (i) Poles (ii) Zeros (iii) Order.

Sketch its pole-zero plot.

Ans.: Poles at 0, –4, –3 ± j4, Zero at –8, Order 4

13) Obtain the transfer function of the network shown in the Fig. 3.28.

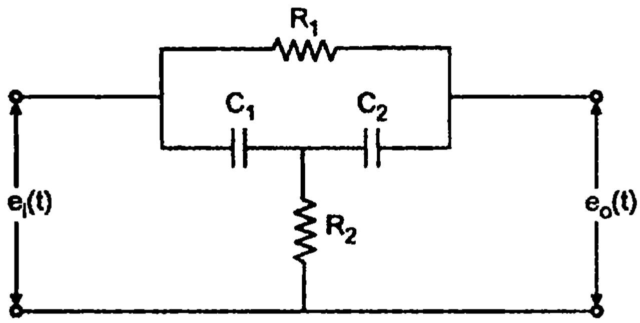


Fig. 3.28

Ans. : $\dfrac{E_o(s)}{E_i(s)} = \dfrac{R_1 R_2 C_1 C_2 s^2 + R_2(C_1 + C_2)\, s + 1}{R_1 R_2 C_1 C_2 s^2 + (R_1 C_2 + R_2 C_1 + R_2 C_2) s + 1}$

14) Obtain the transfer function of the network shown in the Fig. 3.29.

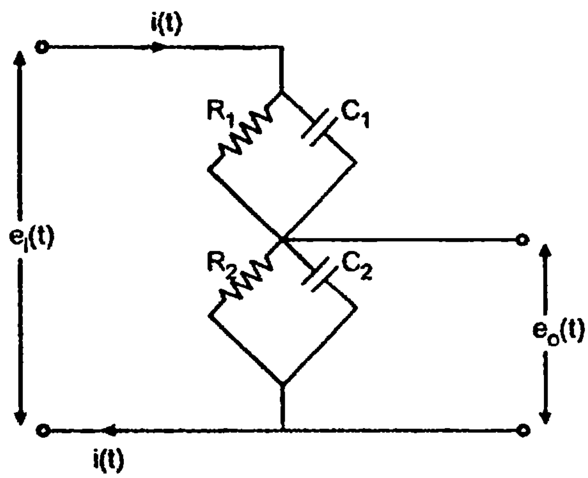


Fig. 3.29

Ans. : $\dfrac{E_o(s)}{E_i(s)} = \dfrac{K(1 + T_1 s)}{(1 + T_2 s)}$ where

$K = \dfrac{R_2}{R_1 + R_2}$

$T_1 = R_1 C_1$, and $T_2 = \dfrac{R_1 R_2 C_1 + R_1 R_2 C_2}{R_1 + R_2}$

□□□

We are interested in $\frac{V_o(s)}{V_i(s)}$ where $V_o(s)$ is Laplace of $v_o(t)$ and $V_i(s)$ is Laplace of $v_i(t)$ and initial conditions are to be neglected.

So taking Laplace of above two equations and assuming initial conditions zero we can write,

$$V_i(s) = RI(s) + \frac{1}{sC} I(s) \quad ... (3)$$

$$V_o(s) = \frac{1}{sC} I(s) \quad ... (4)$$

$$\therefore \quad I(s) = sCV_o(s)$$

Substituting in equation (3),

$$V_i(s) = sCV_o(s)\left[R + \frac{1}{sC}\right]$$

$$\therefore \quad V_i(s) = sCR\,V_o(s) + V_o(s) = V_o(s)[1 + sCR]$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sCR}}$$

We can represent above system as in the Fig. 3.4, which is called transfer function model of the system.

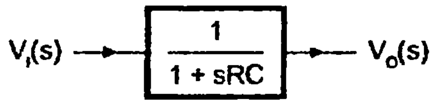


Fig. 3.4

Example 3.2 : *Find out the T.F. of the given network.*

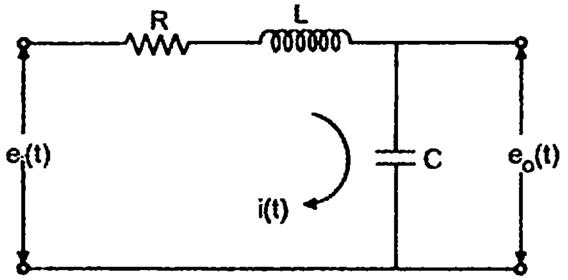


Fig. 3.5

Solution : Applying we get the equations as,

$$e_i(t) = iR + L\frac{di}{dt} + \frac{1}{C}\int idt \quad ... (1)$$

$$\text{Input} = e_i(t) \quad ; \quad \text{Output} = e_o(t)$$

Laplace transform of $\int F(t)\,dt = \frac{F(s)}{s}$, ... neglecting initial conditions

and Laplace transform of $\frac{df(t)}{dt} = sF(s)$ ... neglecting initial conditions

Take Laplace transform,

$$\therefore \quad E_i(s) = I(s)\left[R + sL + \frac{1}{sC}\right]$$

$$\frac{I(s)}{E_i(s)} = \frac{1}{\left[R + sL + \frac{1}{sC}\right]} \qquad \text{... (2)}$$

Now $$e_o(t) = \frac{1}{C}\int i\,dt \qquad \text{... (3)}$$

$$\therefore \quad E_o(s) = \frac{1}{sC} I(s)$$

$$\therefore \quad I(s) = sCE_o(s) \qquad \text{... (4)}$$

Substituting value of I(s) in equation (2),

$$\therefore \quad \frac{sCE_o(s)}{E_i(s)} = \frac{1}{\left[R + sL + \frac{1}{sC}\right]}$$

$$\therefore \quad \boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{sC\left[R + sL + \frac{1}{sC}\right]} = \frac{1}{RsC + s^2 LC + 1}}$$

So we can represent the system as in the Fig. 3.6.

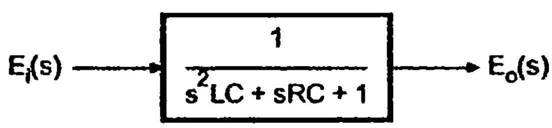


Fig. 3.6 Transfer function model

3.3.2 Advantages and Features of Transfer Function

The various features of the transfer function are,

i) It gives mathematical models of all system components and hence of the overall system. Individual analysis of various components is also possible by the transfer function approach.

ii) As it uses a Laplace approach, it converts time domain equations to simple algebraic equations.

iii) It suggests operational method of expressing equations which relate output to input.

iv) The transfer function is expressed only as a function of the complex variable 's'. It is not a function of the real variable, time or any other variable that is used as the independent variable.

v) It is the property and characteristics of the system itself. Its value is dependent on the parameters of the system and independent of the values of inputs. In the example 3.1, if the output i.e. focus of interest is selected as voltage across resistance R rather than the voltage across capacitor C, the transfer function will be different. So transfer function is to be obtained for a pair of input and output and then it remains constant for any selection of input as long as output variable is same. It helps in calculating the output for any type of input applied to the system.

vi) Once transfer function is known, output response for any type of reference input can be calculated.

vii) It helps in determining the important information about the system i.e. poles, zeros, characteristic equation etc.

viii) It helps in the stability analysis of the system.

ix) The system differential equation can be easily obtained by replacing variable 's' by d/dt.

x) Finding inverse, the required variable can be easily expressed in the time domain. This is much more easy than to analyse the entire system in the time domain.

3.3.3 Disadvantages

The few limitations of the transfer function approach are,

i) Only applicable to linear time invariant systems.

ii) It does not provide any information concerning the physical structure of the system. From transfer function, physical nature of the system whether it is electrical, mechanical, thermal or hydraulic, cannot be judged.

iii) Effects arising due to initial conditions are totally neglected. Hence initial conditions loose their importance.

3.3.4 Procedure to Determine the Transfer Function of a Control System

The procedure used in Ex. 3.1 and Ex. 3.2 can be generalised as below :

1) Write down the time domain equations for the system by introducing different variables in the system.

2) Take the Laplace transform of the system equations assuming all initial conditions to be zero.
3) Identify system input and output variables.
4) Eliminating introduced variables, get the resultant equation in terms of input and output variables.
5) Take the ratio of Laplace transform of output variable to Laplace transform of input variable to get the transfer function model of the system.

Example 3.3 : *Find out the T.F. of the given network.*

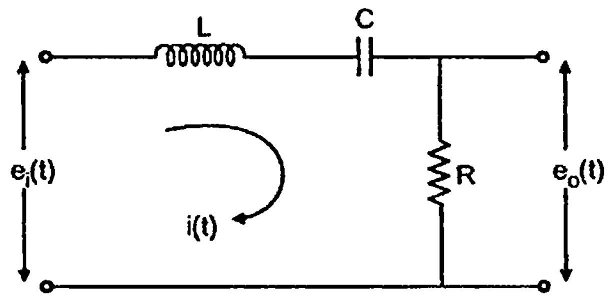


Fig. 3.7

Solution : Applying KVL we can write,

$$e_i(t) = L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt + i(t)R \quad \text{... (1)}$$

while $$e_o(t) = i(t)R \quad \text{... (2)}$$

where $e_i(t)$ = input and

$e_o(t)$ = output

Taking Laplace of equations (1) and (2), neglecting the initial conditions.

$$E_i(s) = sLI(s) + \frac{1}{C}\frac{I(s)}{s} + RI(s) \quad \text{... (3)}$$

$$E_o(s) = I(s)R \quad \text{... (4)}$$

∴ $$E_i(s) = I(s)\left[sL + \frac{1}{sC} + R\right] \text{ from (3)}$$

Substituting $I(s) = \frac{E_o(s)}{R}$ from (4) in the above equation we get,

$$E_i(s) = \frac{E_o(s)}{R}\left[sL + \frac{1}{sC} + R\right]$$

$$\therefore \quad E_i(s) = \frac{E_o(s)}{R} \times \left[\frac{s^2LC + 1 + sCR}{sC}\right]$$

$$\therefore \quad \boxed{\frac{E_o(s)}{E_i(s)} = \frac{sRC}{s^2\,LC + sRC + 1}}$$

This is the required transfer function.

Key Point: *The network in Ex. 3.2 and Ex. 3.3 is same but as focus of interest i.e. output is changed, the transfer function is changed. For a fixed output, transfer function is constant and independent of any type of input applied to the system. If the output variable is changed, the T.F. also changes accordingly.*

3.4 Impulse Response and Transfer Function

The impulse function is defined as,

$$f(t) = A \quad \text{for } t = 0$$
$$= 0 \quad \text{for } t \neq 0$$

A **unit impulse function** $\delta(t)$ can be considered a narrow pulse (of any shape) occuring at zero time such that area under the pulse is unity and the time for which the pulse occurs tends to zero. In the limit $t \to 0$, the pulse reduces to a unit impulse $\delta(t)$. Consider a narrow rectangular pulse of width A and height $1/A$ units, so that the area under the pulse = 1, as shown in the Fig. 3.8(a).

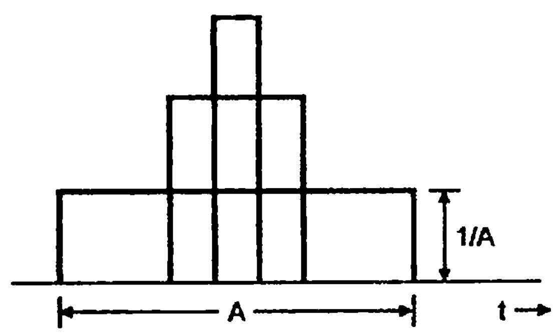
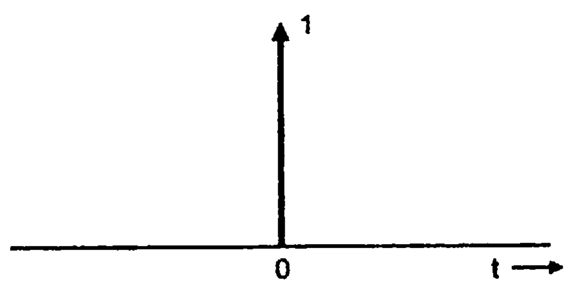


Fig. 3.8 (a)

Now if we go on reducing width A and maintain the area as unity then the height $1/A$ will go on increasing. Ultimately when $A \to 0$, $1/A \to \infty$ and it results the pulse of infinite magnitude. It may then be called an **impulse of magnitude unity** and it is denoted by $\delta(t)$. It is not possible to draw an impulse function on paper, hence it is represented by a vertical arrow at t = 0 as shown in the Fig. 3.8(b).

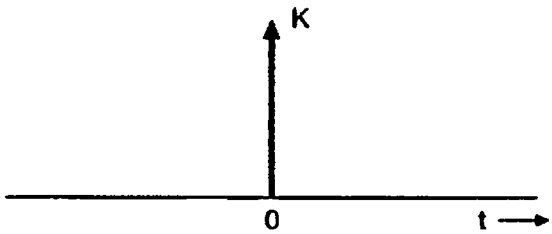


Symbol of δ(t)

Fig. 3.8 (b)

So mathematically unit impulse is defined as,

$$\boxed{\begin{aligned} \delta(t) &= 1, \quad t = 0 \\ &= 0, \quad t \neq 0 \end{aligned}}$$



Symbol of Kδ(t)

Fig. 3.8 (c)

If in the above example the area under the narrow pulse is maintained at K units while the period of pulse is reduced, it is called to be an impulse of magnitude 'K' and is denoted by K $\delta(t)$, as shown in the Fig.3.8(c).

An important property of impulse function is that if it is multiplied by any function and integrated then the result is the value of the function at t = 0.

Thus $\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = \int_{0^-}^{t} f(t)\,\delta(t)\,dt = \int_{0^-}^{0^+} f(t)\,\delta(t)\,dt = f(t)\big|_{t=0}.$

This is called '**sampling**' property of impulse. Hence if we define Laplace transform of $\delta(t)$ as,

$$L[\delta(t)] = \int_{0}^{\infty} \delta(t)\, e^{-st} dt \quad \text{... by definition}$$

$$= e^{-st}\big|_{t=0} \quad \text{... by sampling property.}$$

$$= e^{-0} = 1$$

$\therefore$

$$L\{\delta(t)\} = 1$$

Thus Laplace transform of impulse function δ(t) = 1

Now $$T(s) = \frac{C(s)}{R(s)}$$

$\therefore$ $$C(s) = R(s) \cdot T(s)$$

So response C(s) can be determined for any input once T(s) is determined.

Key Point: *The equation [C(s) = R(s) · T(s)] is applicable only in Laplace domain and cannot be used in time domain. The equation [c(t) = r(t) · t(t)] is **not at all valid** in time domain.*

Now consider that input be unit impulse i.e.

$$r(t) = \delta(t) = \text{unit impulse input}$$

$\therefore$ $$R(s) = L\{\delta(t)\} = 1$$

Substituting in above,

$$C(s) = 1 \cdot T(s) = T(s)$$

Now $$c(t) = L^{-1}\{C(s)\} = L^{-1}\{T(s)\} = T(t)$$

Thus we can say that for impulse input, impulse response C(s) equals the transfer function T(s). So impulse response is c(t) = T(t) as C(s) = T(s) hence we can conclude that,

> **Key Point :** *Laplace transform of impulse response of a linear time invariant system is its transfer function with all the initial conditions assumed to be zero.*

Example 3.4 : *The unit impulse response of a certain system is found to be* e^{-4t}. *Determine its transfer function.*

Solution : Laplace of unit impulse response is the transfer function.

$$\therefore \quad L\{e^{-4t}\} = T(s)$$

$$\therefore \quad \boxed{T(s) = \frac{1}{s+4}}$$

Example 3.5 : *The Laplace inverse of the transfer function in time domain of a certain system is* e^{-5t} *while its input is* $r(t) = 2$. *Determine its output* $c(t)$.

Solution : Let T(s) be the transfer function.

$$L^{-1}[T(s)] = T(t) = e^{-5t} \quad \text{... given}$$

$$r(t) = 2$$

But $$c(t) \neq r(t) \times T(t),$$

It is mentioned earlier that $\frac{c(t)}{r(t)} = T(t)$ is not at all valid in time domain, so

$$c(t) \neq 2e^{-5t}$$

Hence the equation valid according to the definition of transfer function must be used,

$$T(s) = \frac{C(s)}{R(s)}$$

so $$T(s) = L\{T(t)\} = L\{e^{-5t}\} = \frac{1}{s+5}$$

$$R(s) = \frac{2}{s} \quad \text{... as } r(t) = 2$$

$$\therefore \quad \frac{1}{s+5} = \frac{C(s)}{\left(\frac{2}{s}\right)}$$

$$\therefore \quad C(s) = \frac{2}{s(s+5)} = \frac{a_1}{s} + \frac{a_2}{s+5}$$

4

Mathematical Modeling of Systems

4.1 What is Mathematical Model ?

To study and examine a control system, it is necessary to have some type of equivalent representation of the system. Such a representation can be obtained from the mathematical equations, governing the behaviour of the system. Most of such mathematical equations are differential equations whether the system may be electrical, mechanical, thermal, hydraulic etc.

Key Point : *The set of mathematical equations, describing the dynamic characteristics of a system is called* ***mathematical model*** *of the system.*

Obtaining the mathematical model is the first step in analysing a given system. In the mathematical model, the various operations in the system are represented by the mathematical equations.

Most of the control systems contain mechanical or electrical or both types of elements and components. To analyse such systems, it is necessary to convert such systems into mathematical models based on transfer function approach. From mathematical angle of view, models of mechanical and electrical components are exactly analogous to each other. Not only this, but we can show that for given mechanical system there is always an analogous electrical network exists and vice versa. The mathematical equations describing both the systems are exactly same in nature.

As we are well familiar with the behaviour of electrical networks and methods of writing equations for it, it will be better if we can draw equivalent electrical networks for given mechanical systems. This will help us in writing system equations in simplified manner and with more detailed understanding.

This chapter explains the concept of analogous networks, method of writing differential equations for various physical systems and derives the transfer functions of various commonly used control systems.

Torque equation of side 1 is,

$$T = J_1 \frac{d^2 \theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + T_1(t) \quad \text{... (1)}$$

Torque equation of side 2 is,

$$T_2 = J_2 \frac{d^2 \theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + T_L(t) \quad \text{... (2)}$$

Now $\frac{T_1}{T_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1}$

$\therefore$ $T_2 = \frac{N_2}{N_1} T_1$

Substituting in equation (2)

$$\therefore \quad \frac{N_2}{N_1} T_1 = J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_L$$

$$\therefore \quad T_1 = \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L \quad \text{... (3)}$$

Substituting value of T_1 in equation (1)

$$T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt_2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L$$

Substituting $\theta_2 = \frac{N_1}{N_2} \theta_1$

$$\therefore T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{N_1}{N_2} \frac{d^2\theta_1}{dt_2} + \frac{N_1}{N_2} \cdot B_2 \frac{N_1}{N_2} \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$$\therefore T = \left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 \right] \frac{d^2\theta_1}{dt^2} + \left[B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2 \right] \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$\therefore$ J_{1e} = Equivalent inertia referred to primary side

$$\therefore \quad \boxed{J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2}$$

and B_{1e} = Equivalent friction referred to primary side

$$\therefore \quad \boxed{B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2}$$

$$\therefore \quad \boxed{T = J_{1e} \frac{d^2\theta_1}{dt_2} + B_{1e} \frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right) T_L}$$

Similarly the equation can be written referred to load side also, where applied torque gets transferred to load as $\left(\frac{N_2}{N_1} T\right)$.

$$\left(\frac{N_2}{N_1}\right) T = J_{2e} \frac{d^2 \theta_2}{dt^2} + B_{2e} \frac{d\theta_2}{dt} + T_L$$

where $J_{2e} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1$ and $B_{2e} = B_2 + \left(\frac{N_2}{N_1}\right)^2 B_1$

4.6.2. Belt or Chain Drives

Belt and chain drives perform same function as that of gear train. Assuming that there is no slippage between belt and pulleys we can write,

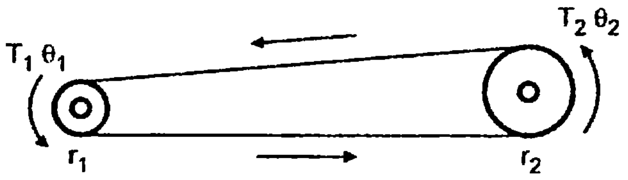


Fig. 4.16

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$$ for such drive

4.6.3 Levers

The lever system is shown in the Fig. 4.17. This transmits translational motion and forces, similar to gear trains.

By law of moment,

$$f_1 l_1 = f_2 l_2$$

By work done $f_1 x_1 = f_2 x_2$

Hence $$\frac{f_1}{f_2} = \frac{l_2}{l_1} = \frac{x_2}{x_1}$$

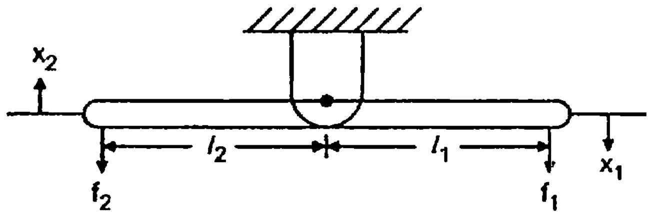


Fig. 4.17

4.7 Electrical Systems

Similar to the mechanical systems, very commonly used systems are of electrical type. The behaviour of such systems is governed by Ohm's law. The dominant elements of an electrical system are,

i) Resistor ii) Inductor iii) Capacitor

i) **Resistor :** Consider a resistance carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = IR$$

Suppose it carries a current $(I_1 - I_2)$ then for the polarity of the voltage drop shown its equation is,

$$V = (I_1 - I_2) R$$

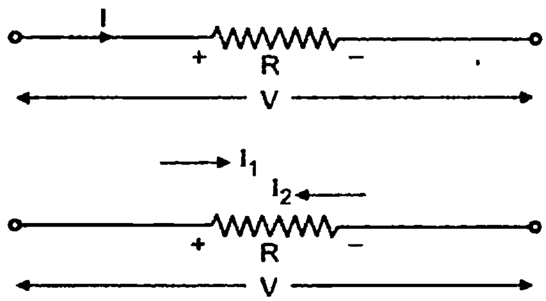


Fig. 4.18

ii) **Inductor :** Consider an inductor carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = L\frac{dI}{dt} \quad \text{or}$$

$$I = \frac{1}{L}\int V\,dt$$

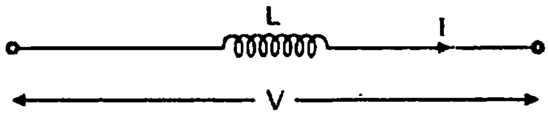


Fig. 4.19

If it carries a current $(I_1 - I_2)$ then for the polarity shown its voltage equation is ,

$$V = L\frac{d(I_1 - I_2)}{dt}$$

or

$$(I_1 - I_2) = \frac{1}{L}\int V\,dt$$

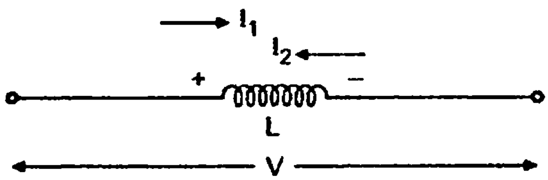


Fig. 4.20

iii) **Capacitor :** Consider a capacitor carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = \frac{1}{C}\int I\,dt$$

or

$$I = C\frac{dV}{dt}$$

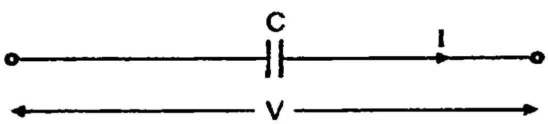


Fig. 4.21

If it carries a current $(I_1 - I_2)$ then for the polarity shown its voltage equation is,

$$V = \frac{1}{C}\int (I_1 - I_2)\,dt$$

or

$$(I_1 - I_2) = C\frac{dV}{dt}$$

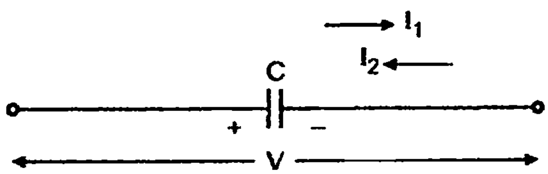


Fig. 4.22

4.8 Analogous Systems

In between electrical and mechanical systems there exists a fixed analogy and their exists a similarity between their equilibrium equations. Due to this, it is possible to draw an electrical system which will behave exactly similar to the given mechanical system, this is called electrical analogous of given mechanical system and vice versa. It is always advantageous to obtain **electrical analogous** of the given mechanical system as we are well familiar with the methods of analysing electrical network than mechanical systems.

There are two methods of obtaining electrical analogous networks, namely

1) Force - Voltage Analogy i.e. Direct Analogy.

2) Force - Current Analogy i.e. Inverse Analogy.

4.8.1 Mechanical System

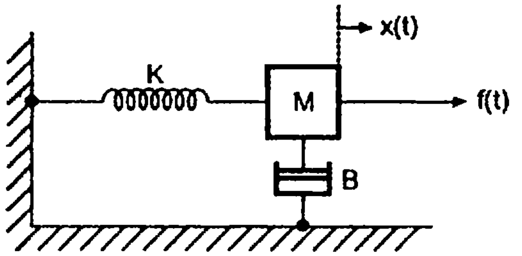


Fig. 4.23

Consider simple mechanical system as shown in the Fig. 4.23.

Due to the applied force, mass M will displace by an amount x(t) in the direction of the force f(t) as shown in the Fig. 4.23.

According to Newton's law of motion, applied force will cause displacement x(t) in spring, acceleration to mass M against frictional force having constant B

$$\therefore \quad f(t) = Ma + Bv + K\,x(t)$$

Where, a = Acceleration, v = Velocity

$$\therefore \quad f(t) = M\frac{d^2\,x(t)}{dt^2} + B\frac{d\,x(t)}{dt} + K\,x(t)$$

Taking Laplace,

$$F(s) = Ms^2\,X(s) + Bs\,X(s) + K\,X(s)$$

This is equilibrium equation for the given system.

Now we will try to derive analogous electrical network.

4.8.2 Force Voltage Analogy (Loop Analysis)

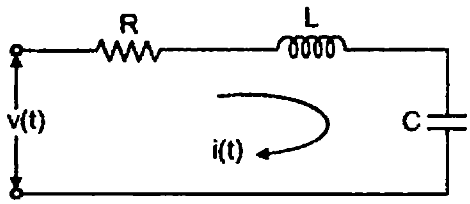


Fig. 4.24

In this method, to the force in mechanical system, voltage is assumed to be analogous one. Accordingly we will try to derive other analogous terms. Consider electric network as shown in the Fig. 4.24.

The equation according to Kirchhoff's law can be written as,

$$v(t) = i(t) R + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t)dt$$

Taking Laplace,

$$V(s) = I(s) R + Ls I(s) + \frac{I(s)}{sC}$$

But we cannot compare F(s) and V(s) unless we bring them into same form. For this we will use current as rate of flow of charge.

$$\therefore \quad i(t) = \frac{dq}{dt}$$

i.e. $$I(s) = s Q(s) \quad \text{or} \quad Q(s) = \frac{I(s)}{s}$$

Replacing in above equation,

$$V(s) = L s^2 Q(s) + Rs Q(s) + \frac{1}{C} Q(s)$$

Comparing equations for F(s) and V(s) it is clear that,

i) Inductance `L' is analogous to mass M

ii) Resistance `R' is analogous to friction B.

iii) Reciprocal of capacitor i.e. 1/C is analogous to spring of constant K.

Translational	Rotational	Electrical
Force	Torque T	Voltage V
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant K Nm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x} = \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Current $I = \frac{dq}{dt}$

Table 4.2 Tabular form of force-voltage analogy

4.8.3 Force Current Analogy (Node Analysis)

In this method, current is treated as analogous quantity to force in the mechanical system. So force shown is replaced by a current source in the system shown in the Fig. 4.25.

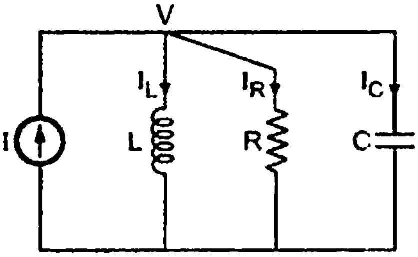


Fig. 4.25

The equation according to Kirchhoff's current law for above system is,

$$I = I_L + I_R + I_C$$

Let node voltage be V,

$$\therefore \quad I = \frac{1}{L}\int V\,dt + \frac{V}{R} + C\frac{dV}{dt}$$

Taking Laplace,

$$I(s) = \frac{V(s)}{sL} + \frac{V(s)}{R} + sC\,V(s)$$

But to get this equation in the similar form as that of F(s) we will use,

$$v(t) = \frac{d\phi}{dt} \qquad \text{where } \phi = \text{flux}$$

$$\therefore \quad V(s) = s\,\phi(s) \quad \text{i.e.} \quad \phi(s) = \frac{V(s)}{s}$$

Substituting in equation for I(s)

$$\therefore \quad I(s) = Cs^2\,\phi(s) + \frac{1}{R}\,s\,\phi(s) + \frac{1}{L}\,\phi(s)$$

Comparing equations for F(s) and I(s) it is clear that,

i) Capacitor `C' is analogous to mass M.

ii) Reciprocal of resistance $\frac{1}{R}$ is analogous to frictional constant B.

iii) Reciprocal of inductance $\frac{1}{L}$ is analogous to spring constant K.

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'e' $= \frac{d\phi}{dt}$

Table 4.3 Tabular form of force-current analogy

Key Point : *The elements which are in series in F - V analogy, get connected in parallel in F – I analogous network and which are in parallel in F - V analogy, get connected in series in F – I analogous network.*

4.9 Steps to Solve Problems on Analogous Systems

Step 1 : Identify all the displacements due to the applied force. The elements spring and friction between two moving surfaces cause change in displacement.

Step 2 : Draw the equivalent mechanical system based on node basis. The elements under same displacement will get connected in parallel under that node. Each displacement is represented by separate node. Element causing change in displacement (either friction or spring) is always between the two nodes.

Step 3 : Write the equilibrium equations. At each node algebraic sum of all the forces acting at the node is zero.

Step 4 : In F-V analogy, use following replacements and rewrite equations,

$$F \to V, \quad M \to L, \quad B \to R, \quad K \to 1/C, \quad x \to q, \quad \dot{x} \to i \text{ (current)}$$

Step 5 : Simulate the equations using loop method. Number of displacements equal to number of loop currents.

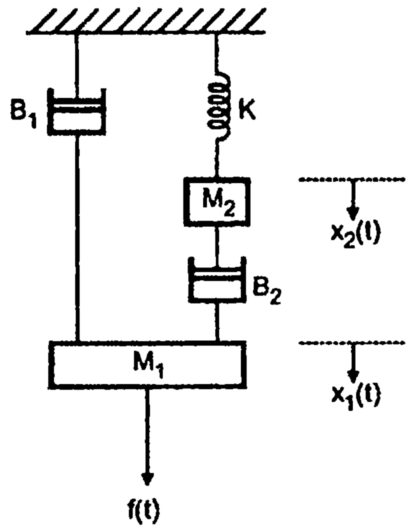
Step 6 : In F-I analogy, use following replacements and rewrite equations,

$$F \to I, \quad M \to C, \quad B \to 1/R, \quad K \to 1/L, \quad x \to \phi, \quad \dot{x} = e(\text{e.m.f.})$$

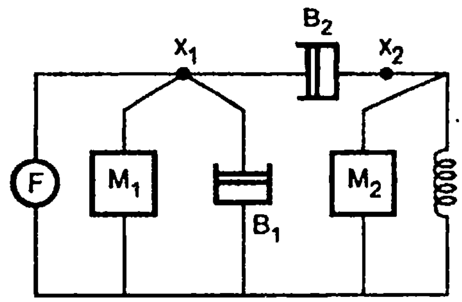
Step 7 : Simulate the equations using node basis. Number of displacements equal to number of node voltages. Infact the system will be exactly same as equivalent mechanical system obtained in step 2 with appropriate replacements.

Example 4.1 : *Draw the equivalent mechanical system of the given system. Hence write the set of equilibrium equations for it and obtain electrical analogous circuits using,*

i) F-V Analogy and ii) F-I Analogy



Solution : The displacement of M_1 is $x_1(t)$ and as B_1 is between M_1 and fixed support hence it is also under the influence of $x_1(t)$. While B_2 changes the displacement from $x_1(t)$ to $x_2(t)$ as it is between two moving points. And M_2 and K are under the displacement $x_2(t)$ as K is between mass and fixed support.



Equivalent system

$$\Sigma F = 0$$

At node 1, $F = M_1 s^2 X_1 + B_1 s X_1 + B_2 s(X_1 - X_2)$ ... (1)

At node 2, $0 = M_2 s^2 X_2 + K X_2 + B_2 s(X_2 - X_1)$ ... (2)

Now (i) **F - V Analogy** $M \to L \quad B \to R \quad K \to 1/C \quad x \to q$

$\therefore$ $V(s) = L_1 s^2 q_1 + R_1 s q_1 + R_2 s(q_1 - q_2)$ ... (3)

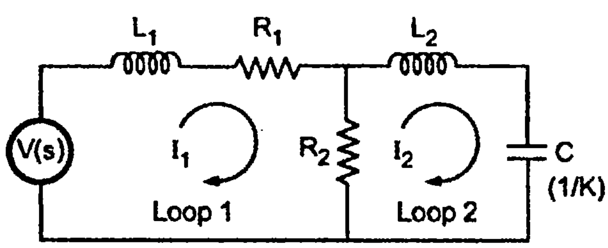
$0 = L_2 s^2 q_2 + (1/C) q_2 + R_2 s(q_2 - q_1)$ ... (4)

Replacing $\frac{I(s)}{s} = Q(s)$ i.e. $I(s) = s\, Q(s)$

$$V(s) = L_1 s I_1(s) + R_1 I_1(s) + R_2 [I_1(s) - I_2(s)] \to \text{Loop 1}$$

$$0 = L_2 s I_2(s) + \frac{1}{sC} I_2(s) + R_2 [I_2(s) - I_1(s)] \to \text{Loop 2}$$

Hence,



Number of loop currents equal to number of displacements.

(ii) F - I Analogy

$F \to I \quad M \to C \quad B \to 1/R \quad K \to 1/L \quad x \to \phi$

$$I(s) = C_1 s^2 \phi_1 + \frac{1}{R_1} s\, \phi_1 + \frac{1}{R_2} s\, (\phi_1 - \phi_2) \quad \text{... (1)}$$

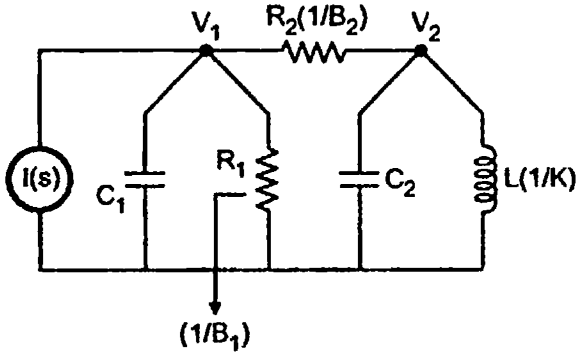
$$0 = \frac{1}{R_2} s(\phi_2 - \phi_1) + C_2 s^2 \phi_2 + \frac{1}{L} \cdot \phi_2 \quad \text{... (2)}$$

Replacing $s\, \phi(s) = V(s)$

$$I(s) = C_1\, s\, V_1(s) + \frac{V_1(s)}{R_1} + \frac{1}{R_2}\,[V_1(s) - V_2(s)] \qquad \text{...Node 1}$$

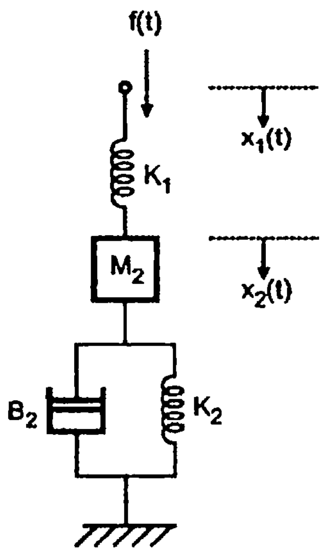
$$0 = \frac{1}{R_2}[V_2(s) - V_1(s)] + C_2\, s\, V_2(s) + \frac{1}{sL}\,V_2(s) \qquad \text{...Node 2}$$

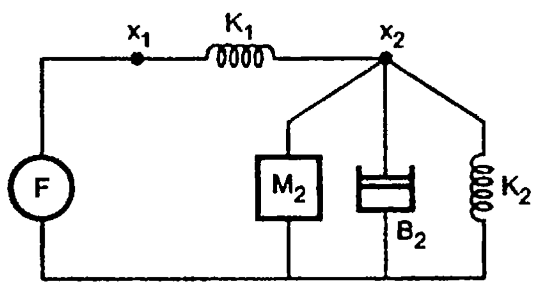
Hence,



. Number of node voltages equal to number of displacements.

Example 4.2 : *Draw the equivalent mechanical system and analogous systems based on F-V and F-I methods for the given system.*





Equivalent system

Solution : **Two displacements** : No element under $x_1(t)$ alone as force is directly applied to a spring K_1. So it will store energy and hence is the cause to change the force applied to M_2. Hence displacement of M_2 is x_2 and as B_2 and K_2 are connected to fixed supports both are under $x_2(t)$ only as shown in the equivalent system.

At node 1, $F = K_1 (X_1 - X_2)$ ... (1)

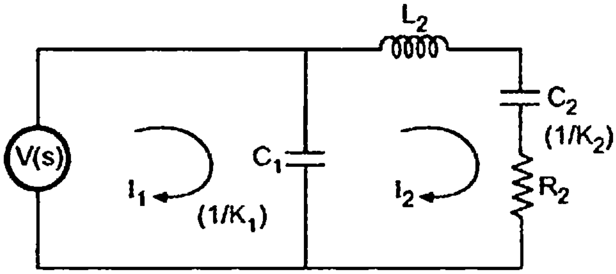
At node 2, $0 = K_1 (X_2 - X_1) + M_2 s^2 X_2 + K_2 X_2 + B_2 s X_2$ ... (2)

M_2 , B_2 , K_2 are under same displacement.

(i) F-V analogy : M → L B → R K → 1/C

$$V = \frac{1}{C_1}(q_1 - q_2) \quad \text{... (3) Loop (1)}$$

$$0 = \frac{1}{C_1}(q_2 - q_1) + L_2 s^2 q_2 + \frac{1}{C_2} q_2 + R_2 s q_2 \quad \text{... (4) Loop (2)}$$



Same displacement same current.

Equations in terms of I_1 and I_2 can be written by using $i(t) = \frac{dq}{dt}$ i.e. $I(s) = s\,Q(s)$ as explained earlier.

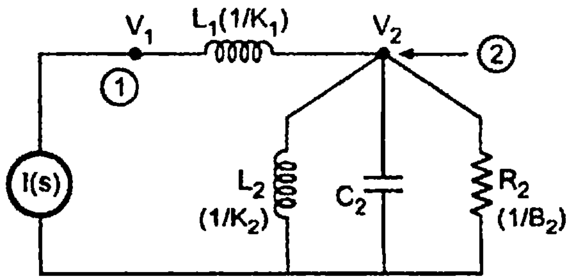
(ii) F - I analogy : M → C , B → 1/R, K → 1/L

$$\therefore \quad I(s) = \frac{1}{L_1}(\phi_1 - \phi_2) \quad \text{... (1) Node (1)}$$

$$0 = \frac{1}{L_1}(\phi_2 - \phi_1) + C_2 s^2 \phi_2 + \frac{1}{R_2} s \phi_2 + \frac{1}{L_2}\phi_2 \quad \text{... (2) Node (1)}$$

Equations in terms of $V_1(s)$ and $V_2(s)$ can be obtained by using the relation,

$$v(t) = \frac{d\phi}{dt} \quad \text{i.e.} \quad V(s) = s\,\phi(s) \quad \text{as explained earlier.}$$



Same displacement-same voltage.

4.2 Analysis of Mechanical Systems

In mechanical systems, motion can be of different types i.e. Translational, Rotational or combination of both. The equations governing such motion in mechanical systems are often directly or indirectly governed by **Newton's laws of motion**

4.2.1 Translational Motion

Consider a mechanical system in which motion is taking place along a straight line. Such systems are of translational type. These systems are characterised by displacement, linear velocity and linear acceleration.

Key Point : *According to Newton's law of motion, sum of forces applied on rigid body or system must be equal to sum of forces consumed to produce displacement, velocity and acceleration in various elements of the system.*

The following elements are dominantly involved in the analysis of translational motion systems.

i) Mass ii) Spring iii) Friction

4.2.2 Mass (M)

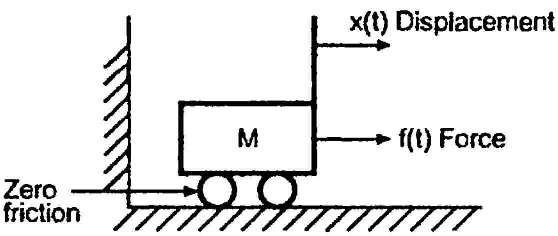


Fig. 4.1

This is the property of the system itself which stores the kinetic energy of the translational motion. Mass has no power to store the potential energy. It is measured in kilograms (kg). The displacement of mass always takes place in the direction of the applied force results in inertial force. This force is always proportional to the acceleration produced in mass (M) by the applied force.

Consider a mass 'M' as shown in the Fig. 4.1 having zero friction with surface, shown by rollers.

The applied force f(t) produces displacement x(t) in the direction of the applied force f(t). Force required for the same is proportional to acceleration.

$$\therefore \quad f(t) = M \times \text{acceleration} = M \frac{d^2x(t)}{dt^2}$$

Taking Laplace and neglecting initial conditions we can write,

$$f(s) = Ms^2 X(s)$$

Also mass cannot store potential energy so there cannot be consumption of force in the mass e.g. if two masses are directly connected to each other as shown in the Fig. 4.2 and if force f(t) is applied to mass M_1 then mass M_2 will also displace by same amount as M_1.

4.10 Servomotors

The servosystem is one in which the output is some mechanical variable like position, velocity or acceleration. Such systems are generally automatic control systems which work on the error signals. The error signals are amplified to drive the motors used in such systems. These motors used in servosystems are called servomotors. These motors are usually coupled to the output shaft i.e. load through gear train for power matching.

These motors are used to convert electrical signal applied, into the angular velocity or movement of shaft.

4.10.1 Requirements of Good Servomotor

The servomotors which are designed for use in feedback control systems must have following requirements :

i) Linear relationship between electrical control signal and the rotor speed over a wide range.

ii) Inertia of rotor should be as low as possible. A servomotor must stop running without any time delay, if control signal to it is removed. For low inertia, it is designed with large length to diameter ratio, for rotors. Compared to its frame size, the rotor of a servomotor has very small diameter.

iii) Its response should be as fast as possible. For quickly changing error signals, it must react with good response. This is achieved by keeping torque to weight ratio high.

iv) It should be easily reversible.

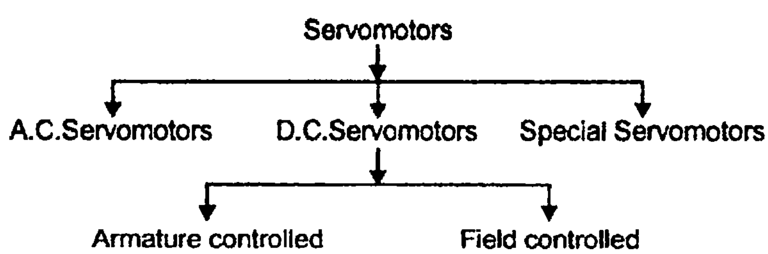
v) It should have linear torque - speed characteristics.

vi) Its operation should be stable without any oscillations or overshoots.

4.11 Types of Servomotors

The servomotors are basically classified depending upon the nature of the electric supply to be used for its operation.

The types of servomotors are as shown in the following chart :



4.12 D.C. Servomotor

Basically d.c. servomotor is more or less same as normal d.c.motor. There are some minor differences between the two. All d.c. servomotors are essentially separately excited type. This ensures linear torque-speed characteristics.

The control of d.c. servomotor can be from field side or from armature side. Depending upon this, these are classified as field controlled d.c. servomotor and armature controlled d.c. servomotor.

4.12.1 Field Controlled D.C. Servomotor

In this motor, the controlled signal obtained from the servoamplifier is applied to the field winding. With the help of constant current source, the armature current is maintained constant. The arrangement is shown in the Fig. 4.26.

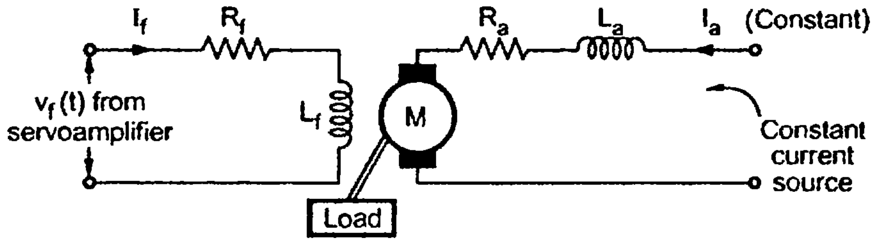


Fig. 4.26 Field controlled d.c. servomotor

This type of motor has large L_f/R_f ratio where L_f is reactance and R_f is resistance of field winding. Due to this the time constant of the motor is high. This means it can not give rapid response to the quick changing control signals hence this is uncommon in practice.

4.12.1.1 Features of Field Controlled D.C. Servomotor

It has following features :

i) Preferred for small rated motors.

ii) It has large time constant.

iii) It is open loop system. This means any change in output has no effect on the input.

iv) Control circuit is simple to design.

4.12.2 Armature Controlled D.C. Servomotor

In this type of motor, the input voltage 'V_a' is applied to the armature with a resistance of R_a and inductance L_a. The field winding is supplied with constant current I_f. Thus armature input voltage controls the motor shaft output. The arrangement is shown in the Fig. 4.27.

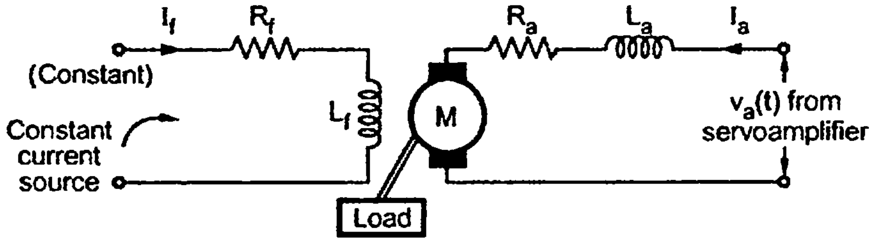


Fig. 4.27 Armature controlled d.c. servomotor

The constant field can be supplied with the help of parmanent magnets. In such case no field coils are necessary.

4.12.2.1 Features of Armature Controlled D.C. Servomotor

It has following features :

i) Suitable for large rated motors.

ii) It has small time constant hence its response is fast to the control signal.

iii) It is closed loop system.

iv) The back e.m.f. provides internal damping which makes motor operation more stable.

v) The efficiency and overall performance is better than field controlled motor.

As the armature controlled d.c. servomotor is closed loop system, in comparison with open loop field controlled system, generally armature controlled motors are used.

4.12.3 Characteristics of D.C. Servomotors

The characteristics of d.c. servomotors are mainly similar to the torque-speed characteristics of a.c. servomotor. The characteristics are shown in the Fig. 4.28.

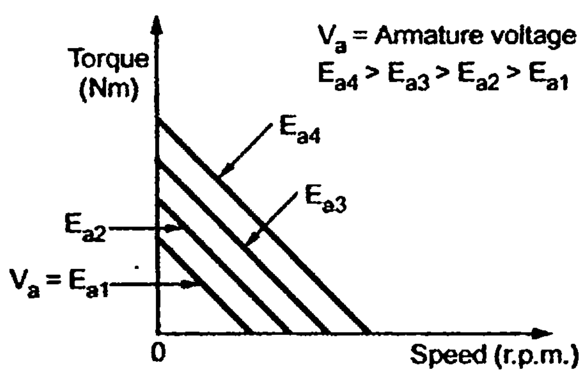


Fig. 4.28 Torque-speed characteristics for an armature controlled d.c. servomotor

4.12.4 Applications of D.C. Servomotor

These are widely used in air craft control systems, electromechanical actuators, process controllers, robotics, machine tools etc.

4.13 Transfer Function of Field Controlled D.C. Motor

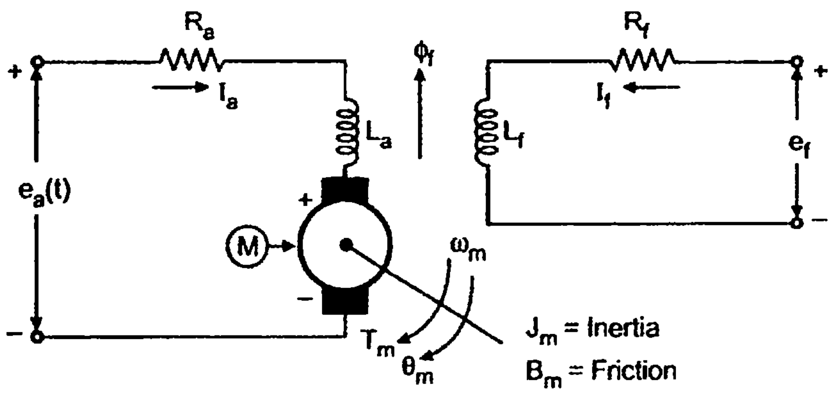


Fig. 4.29

Assumptions :

(1) Constant armature current is fed into the motor.

(2) $\phi_f \propto I_f$. Flux produced is proportional to field current.

$$\therefore \quad \boxed{\phi_f = K_f I_f}$$

(3) Torque is proportional to product of flux and armature current.

$$\boxed{T_m \propto \phi I_a}$$

$$\therefore \quad T_m = K' \phi I_a = K' K_f I_f I_a$$

$$\boxed{T_m = K_m K_f I_f} \quad \text{... (1)}$$

Where $K_m = K' I_a$ = Constant

Apply Kirchhoff's law to field circuit.

$$L_f \frac{di_f}{dt} + R_f I_f = e_f \quad \text{... (2)}$$

Now shaft torque T_m is used for driving load against the inertia and frictional torque.

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} \quad \text{... (3)}$$

$$\text{Inertia force} = J_m \frac{d^2\theta_m}{dt^2} \text{ similar to } m \frac{d^2x}{dt^2}$$

$$\text{Frictional force} = B_m \frac{d\theta_m}{dt} \text{ similar to } B \frac{dx}{dt}$$

Finding Laplace Transforms of equations (1), (2) and (3) we get,

$$T_m(s) = K_m K_f I_f(s) \quad \text{... (4)}$$

$$E_f(s) = (sL_f + R_f) I_f(s) \quad \text{... (5)}$$

$$T_m(s) = J_m s^2 \theta_m(s) + B_m s \theta_m(s) \quad \text{... (6)}$$

Eliminate $I_f(s)$ from equations (4) and (5)

$$T_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)} \quad \text{... (7)}$$

Eliminate $T_m(s)$ from equations (6) and (7),

$$(s^2 J_m + sB_m)\theta_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)}$$

$$\text{Input} = E_f(s)$$

$$\text{Output} = \text{Rotational displacement } \theta_m(s)$$

$$\therefore \text{ Transfer function} = \frac{\theta_m(s)}{E_f(s)}$$

$$\frac{\theta_m(s)}{E_f(s)} = \frac{K_m K_f}{(J_m s^2 + sB_m)(R_f + sL_f)}$$

$$= \frac{K_m K_f}{sR_f B_m [1 + s\tau_m][1 + s\tau_f]}$$

Where $\tau_m = \frac{J_m}{B_m}$ = Motor time constant

$\tau_f = \frac{L_f}{R_f}$ = Field time constant

$$\text{T.F.} = \frac{\theta_m(s)}{E_f(s)} = \frac{K_f}{R_f[1 + s\tau_f]} \cdot \frac{K_m}{B_m(1 + s\tau_m)} \cdot \frac{1}{s}$$

Block diagram for field controlled d.c. motor is as shown in Fig. 4.30.

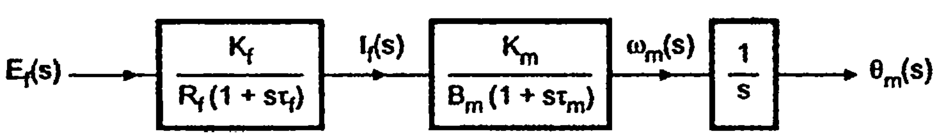


Fig. 4.30 Block diagram

4.14 Transfer Function of Armature Controlled D.C. Motor

Assumptions :

(i) Flux is directly proportional to current through field winding,

$$\varphi_m = K_f I_f = \text{Constant}$$

(ii) Torque produced is proportional to product of flux and armature current.

$$T = K'_m \phi I_a$$

$$T = K'_m K_f I_f I_a$$

(iii) Back e.m.f. is directly proportional to shaft velocity ω_m, as flux ϕ is constant.

as $\omega_m = \frac{d\theta(t)}{dt}$

$$E_b = K_b \omega_m(s) = K_b s\theta_m(s)$$

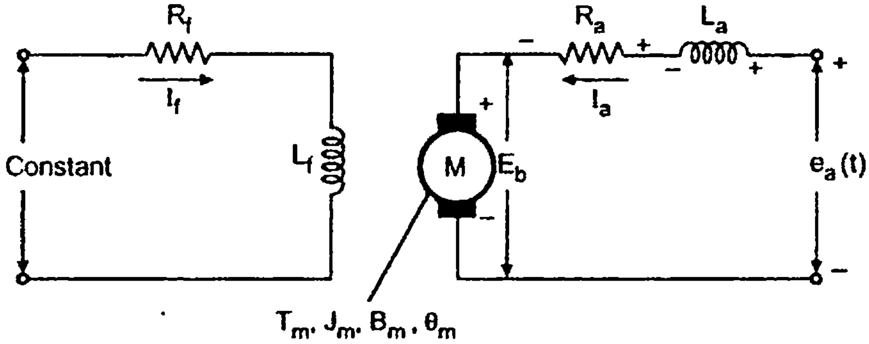


Fig. 4.31

Apply Kirchhoff's law to armature circuit :

$$e_a = E_b + I_a(R_a) + L_a \frac{di_a}{dt}$$

Take Laplace transform,

$$\therefore \quad E_a(s) = E_b(s) + I_a(s)[R_a + sL_a]$$

$$\therefore \quad I_a(s) = \frac{E_a(s) - E_b(s)}{R_a + sL_a}$$

$$I_a(s) = \frac{E_a(s) - K_b s\theta_m(s)}{R_a + sL_a}$$

Now $\quad T_m = K'_m K_f I_f I_a$

$$T_m = K'_m K_f I_f \left\{ \frac{E_a - K_b s\theta_m(s)}{R_a + sL_a} \right\}$$

Also $T_m = \{J_m s^2 + s B_m\} \theta_m(s)$ ... from equation (3)

Equating equations of T_m,

$$\frac{K'_m K_f I_f E_a(s)}{(R_a + s L_a)} = \frac{K'_m K_f I_f K_b s \theta_m(s)}{(R_a + s L_a)} + (J_m s^2 + s B_m) \theta_m(s)$$

$$\therefore \frac{K'_m K_f I_f}{(R_a + s L_a)} E_a(s) = \left[\frac{K'_m K_f I_f K_b s}{(R_a + s L_a)} + J_m s^2 + s B_m\right] \theta_m(s)$$

$$\therefore \frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_m}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}}{1 + \dfrac{K_m \cdot s K_b}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}} = \frac{G(s)}{1 + G(s)H(s)}$$

where $\tau_m = J_m / B_m$ and $\tau_a = \dfrac{L_a}{R_a}$

$$K_m = K'_m K_f$$

$$G(s) = \frac{K_m}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}$$

$$H(s) = s K_b$$

Therefore can be represented in its block diagram form as in Fig. 4.32.

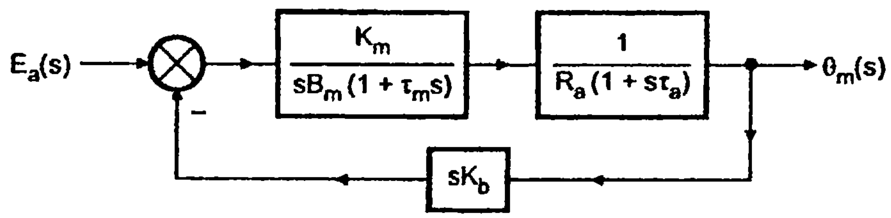


Fig. 4.32 Block diagram

Key Point : *Field controlled d.c. motor is open loop while armature controlled is closed loop system. Hence armature controlled d.c. motors are preferred over field controlled type.*

4.15 A.C. Servomotor

Most of the servomotors used in low power servomechanisms are a.c. servomotors. The a.c. servomotor is basically two phase induction motor. The output power of a.c. servomotor varies from fraction of watt to few hundred watts. The operating frequency is 50 Hz to 400 Hz.

4.15.1 Construction

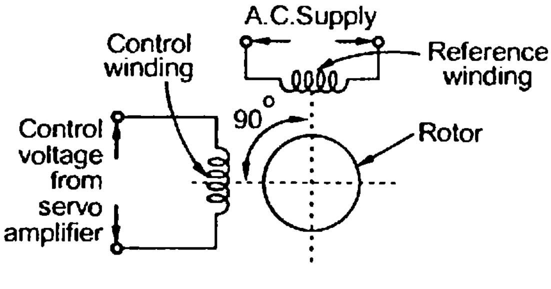


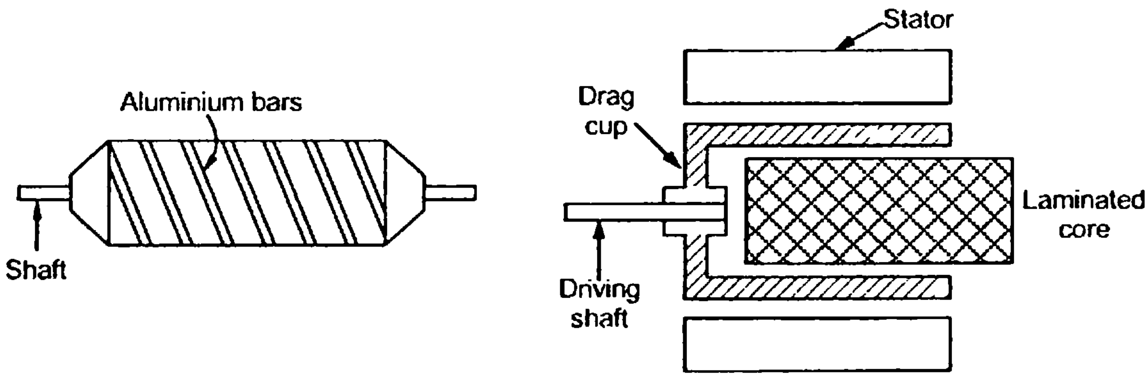
Fig. 4.33 Stator of A.C. servomotor

It is mainly divided into two parts namely stator and rotor.

The stator carries two windings, uniformly distributed and displaced by 90°, in space. One winding is called **main winding** or **fixed winding** or **reference winding**. This is excited by a constant voltage a.c. supply. The other winding is called **control winding**. It is excited by variable control voltage, which is obtained from a servoamplifier. This voltage is 90° out of phase with respect to the voltage applied to the reference winding. This is necessary to obtain rotating magnetic field. The schematic stator is shown in the Fig 4.33.

4.15.2 Rotor

The rotor is generally of two types. The one is usual squirrel cage rotor. This has small diameter and large length. Aluminium conductors are used to keep weight small. Its resistance is very high to keep torque-speed characteristics as linear as possible. Air gap is kept very small which reduces magnetising current. This cage type of rotor is shown with skewed bars in the Fig. 4.34 (a). The other type of rotor is drag cup type. There are two air gaps in such construction. Such a construction reduces inertia considerably and hence



(a) Squirrel cage rotor **(b) Drag cup type rotor**

Fig. 4.34

such type of rotor is used in very low power applications. The aluminium is used for the cup construction. The construction is shown in the Fig. 4.34 (b).

4.15.3 Torque-speed Characteristics

The torque-speed characteristics of a two phase induction motor, mainly depends on the ratio of reactance to resistance. For small X to R ratio i.e. high resistance low reactance

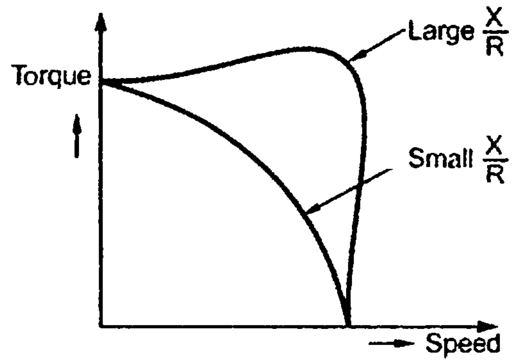


Fig. 4.35

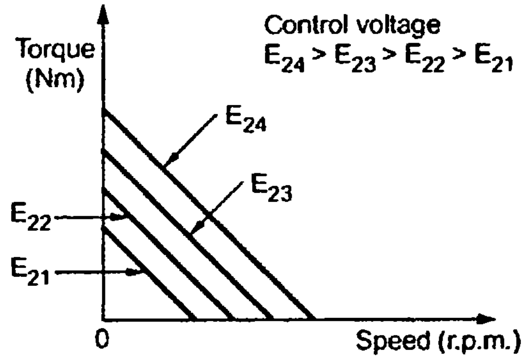


Fig. 4.36

motor, the characteristics is much more linear while it is nonlinear for large X to R ratio as shown in the Fig. 4.35.

In practice, design of the motor is so as to get almost linear torque-speed characteristics. The Fig. 4.36 shows the torque-speed characteristics for various control voltages. The torque varies almost linearly with speed. All the characteristics are equally spaced for equal increments of control voltage. It is generally operated with low speeds.

4.15.4 Features of A. C. Servomotor

The a.c. servomotor has following features :

i) Light in weight. ii) Robust construction. iii) Reliable and stable operation. iv) Smooth and noise free operation. v) Large torque to weight ratio. vi) Large R to X ratio i.e. small X to R ratio. vii) No brushes or slip rings hence maintenance free. viii) Simple driving circuits.

4.15.5 Applications

Due to the above features it is widely used in instrument servomechanisms, remote positioning devices, process control systems, self balancing recorders, computers, tracking and guidance systems, robotics, machine tools etc.

4.15.6 Transfer Function of A.C. Servomotor

The various **approximations** to derive transfer function are,

(i) A servomotor rarely operates at high speeds. Hence for a given value of control voltage, $T \propto N$ characteristics are perfectly linear.

(ii) In order that $T \propto N$ characteristics are directly proportional to voltage applied to its control phase, we assume $T \propto N$ characteristics are straight lines and equally spaced.

Torque at any speed 'N' is,

$$T_m = K_{tm} E_{2t} + m \frac{d\theta_m}{dt} \qquad \text{... (1)}$$

where, $\frac{d\theta_m}{dt}$ is speed of motor.

If load consists inertia J_m and friction B_m we can write,

$$T_m(s) = J_m s^2 \theta_m + B_m s \theta_m \qquad \text{.... (2)}$$

Now Laplace transform of equation (1) is

$$T_m(s) = K_{tm} E_2(s) + m s \theta_m(s) \qquad \text{... (3)}$$

Equating equations (2) and (3)

$$\therefore K_{tm} E_2(s) + m s \theta_m(s) = J_m s^2 \theta_m(s) + B_m s \theta_m(s)$$

$$\therefore \quad \frac{\theta_m(s)}{E_2(s)} = \frac{K_{tm}}{s(s J_m - m + B_m)} = \frac{K_{tm}}{s(B_m - m)\left[1 + \frac{s J_m}{(B_m - m)}\right]}$$

$$\therefore \quad \boxed{\frac{\theta_m(s)}{E_2(s)} = \frac{K_m}{s(1 + \tau_m s)}}$$

where $\quad K_m = \frac{K_{tm}}{B_m - m}$

and $\quad \tau_m = \frac{J_m}{B_m - m}$

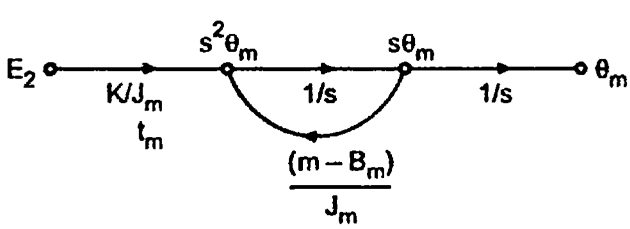


Fig. 4.37 Signal flow graph of a.c. servomotor

Key Point : *As slope is negative, in the above equation* $[B_m - m]$ *shows that total friction increases due to m. As it adds more friction, the damping improves, improving stability of the motor. This is called* ***Internal Electric Damping*** *of 2 ph A.C. servomotor.*

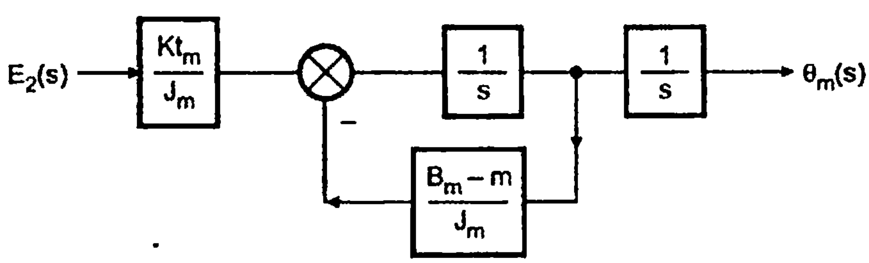
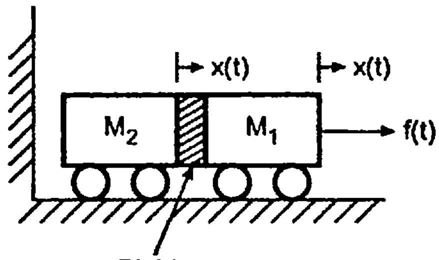


Fig. 4.38 Block diagram of a.c. servomotor

Due to mass, there cannot be any change in force from one mass to other hence no change in displacement.



(No friction and no elastic action)

Fig. 4.2

Key Point : *The displacement of rigidly connected masses is always same.*

4.2.3 Linear Spring

In actual mechanical system there may be an actual spring or indication of spring action because of elastic cable or a belt. Now spring has a property to store the potential energy. The force required to cause the displacement is proportional to the net displacement in the spring. All springs are basically nonlinear in nature but for small deformations their behaviours can be approximated as linear one. Hence assuming linear spring constant `K' for the spring, we can write equation for the spring in the system.

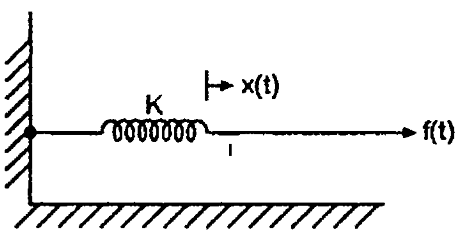


Fig. 4.3

Consider a spring having negligible mass and connected to a rigid support. Its spring constant be `K' as shown in the Fig. 4.3.

$\therefore$ Force required to cause displacement x(t) in the spring is proportional to displacement.

$$f(t) = K\, x(t)$$

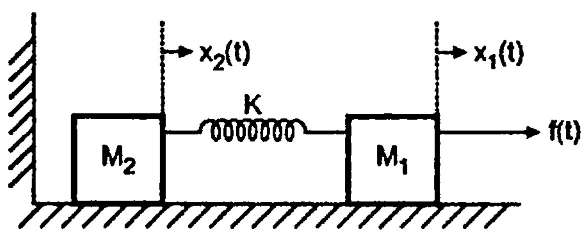


Fig. 4.4

Now consider the spring connected between the two moving elements having masses M_1 and M_2 as shown in the Fig. 4.4 where force is applied to mass M_1.

Signal flow graph for A.C. servomotor is as shown in the Fig. 4.37. Hence block diagram of A.C. servomotor is as shown in Fig. 4.38

4.16 Comparison of Servomotors

4.16.1 Comparison between A.C. and D.C. Servomotor

Sr. No.	A.C. Servomotor	D.C. Servomotor
1)	Low power output of about $\frac{1}{2}$ W to 100 W.	Deliver high power output
2)	Efficiency is less about 5 to 20 %.	High efficiency.
3)	Due to absence of commutator maintenance is less.	Frequent maintenance required due to commutator.
4)	Stability problems are less.	More problems of stability.
5)	No radio frequency noise	Brushes produce radio frequency noise.
6)	Relatively stable and smooth operation.	Noisy operation.
7)	A.C. amplifiers used have no drift.	Amplifiers used have a drift.

4.16.2 Comparison between Armature Controlled and Field Controlled D.C. Servomotors

Sr. No.	Field Controlled	Armature Controlled
1)	Due to low power requirement amplifiers are simple to design.	High power amplifiers are required to design.
2)	Control voltage is applied to the field.	Control voltage is applied to the armature
3)	Time constant is large.	Time constant is small.
4)	This is open loop system.	This is closed loop system.
5)	Armature current is kept constant.	Field current is kept constant.
6)	Poor efficiency.	Better efficiency.
7)	Suitable for small rated motors	Suitable for large rated motors.
8)	Costly as field coils are must	Permanent magnet can be used instead of field coils which makes the motor less expensive.

4.17 Models of Commonly used Electromechanical Systems

In this article the transfer functions of very commonly used systems are derived. This will help the reader to find out the transfer functions of the different practical systems.

4.17.1 Generators

Consider a separately excited generator which is many times used in various practical mechanical systems. Generators are required to drive the motors because vacuum diodes, transistor amplifiers are not suitable because of their low ratings. Consider a generator as shown in the figure.

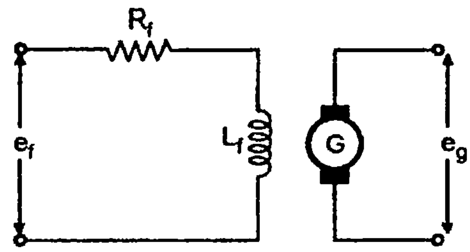


Fig. 4.39 Generator

R_f = Field resistance

L_f = Field inductance

e_f = Applied voltage (input)

e_g = Generated voltage (output)

Now for a generator,

$$e_g \propto \phi \qquad \text{where } \phi = \text{flux}$$

Flux is directly proportional to current passing through the field winding say i_f.

$$\therefore \quad e_g \propto i_f$$

Let K_a be the generator constant in V/A

$$\therefore \quad e_g = K_a \; i_f \qquad \text{... (1)}$$

Applying Kirchhoff's Law to field circuit.

$$e_f = i_f \; R_f + L_f \frac{di_f}{dt} \qquad \text{... (2)}$$

Taking Laplace of both the equations (1) and (2)

$$E_g(s) = K_a \; I_f(s)$$

$$E_f(s) = R_f \; I_f(s) + L_f(s) \; I_f(s) \qquad \text{neglecting } I_f(0)$$

$$\therefore \quad I_f(s) = \frac{E_f(s)}{R_f + s\,L_f}$$

$$\therefore \quad E_g(s) = \frac{K_a \; E_f(s)}{R_f + s\,L_f}$$

$$\therefore \quad \boxed{\frac{E_g(s)}{E_f(s)} = \frac{K_a}{R_f + s\,L_f}}$$

This is the T.F. of separately excited generator.

4.17.2 Generator Driving Motor

It is very common to find generator driving motor in practical mechanical systems. So let us discuss the T.F. of a system with generator driving motor.

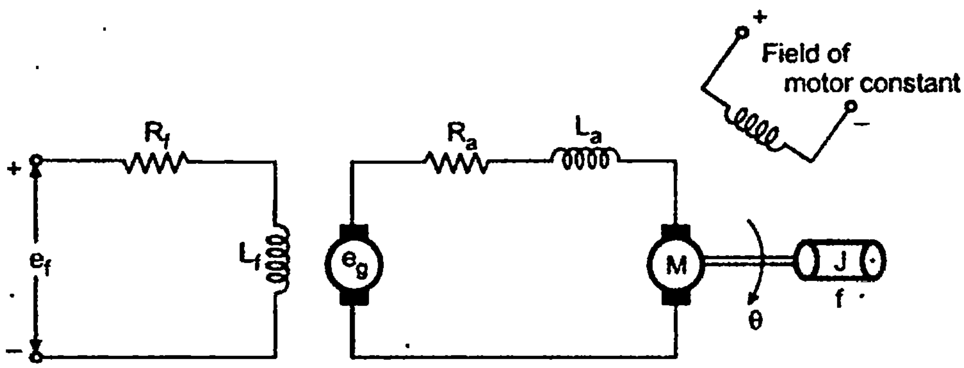


Fig. 4.40 Generator driving motor

R_f and L_f - Resistance and inductance of generator field

e_g - Generated voltage

K_g - Generator constant in V/A

R_a and L_a - Resistance and inductance of motor armature

J - M.I. of Load and f is frictional force

Now T. F. of generator is,

$$\frac{E_g(s)}{E_f(s)} = \frac{K_g}{R_f + s L_f}$$

Now consider armature controlled motor,

Torque produced by motor is dependent on i_a and let K_T be torque constant.

$$T = K_T \, i_a$$

This torque is utilised to drive a load.

$$\therefore \quad T = J\frac{d^2\theta}{dt^2} + f\,\frac{d\theta}{dt}$$

Now E_g is voltage applied to armature.

$$\therefore \quad E_g = i_a R_a + L_a \frac{di_a}{dt} + e_b$$

where e_b is back e.m.f.

$$e_b \propto \omega \propto \frac{d\theta}{dt}$$

Let K_b be the back e.m.f. constant

$$\therefore \quad e_b = K_b \frac{d\theta}{dt}$$

$$\therefore \quad K_T i_a = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

Taking Laplace transform

$$K_T I_a(s) = J s^2 \theta(s) + f s \theta(s) \qquad \text{... (3)}$$

$$E_g = i_a R_a + L_a \frac{di_a}{dt} + K_b \frac{d\theta}{dt}$$

Taking Laplace transform

$$E_g(s) = I_a(s) R_a + L_a s I_a(s) + K_b s \theta(s) \qquad \text{... (4)}$$

finding $I_a(s)$ from equation (3)

$$I_a(s) = \frac{(J s^2 + f s) \theta(s)}{K_T}$$

Substituting in equation (4)

$$E_g(s) = \frac{s \theta(s)(J s + f)}{K_T}(R_a + s L_a) + K_b s \theta(s)$$

$$E_g(s) = \theta(s) s \left[\frac{(f + J s)(R_a + s L_a)}{K_T} + K_b\right]$$

$$E_g(s) - s K_b \theta(s) = \frac{s(R_a + s L_a)(s J + f)\theta(s)}{K_T}$$

$$\therefore \quad \frac{\theta(s)}{E_g(s) - s K_b \theta(s)} = \frac{K_T}{s(R_a + s L_a)(s J + f)}$$

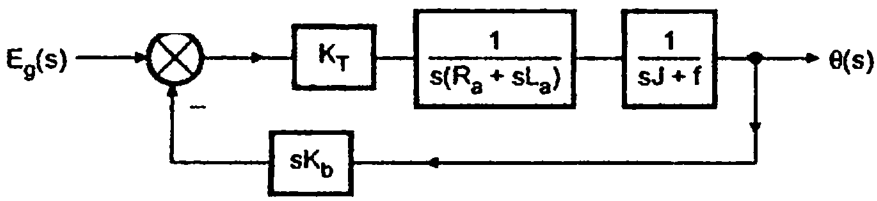


Fig. 4.41

Therefore using both the transfer functions of generator and armature controlled motor we can develop the combined block diagram as below :

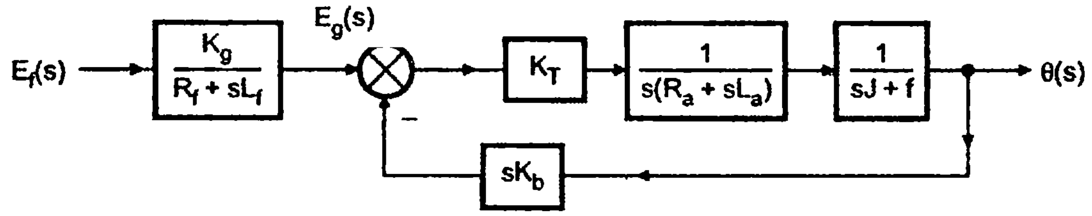


Fig. 4.42

Reducing the block diagram, solving feedback loop of motor which is

$$= \frac{\dfrac{K_T}{s(R_a + s\,L_a)(J\,s + f)}}{1 + \dfrac{s\,K_b\;K_T}{s(R_a + s\,L_a)(J\,s + f)}} = \frac{K_T}{s(R_a + s\,L_a)(J\,s + f) + s\,K_b\,K_T}$$

$$= \frac{K_T}{s[(R_a + s\,L_a)(J\,s + f) + K_b\,K_T]}$$

$$\therefore \quad \boxed{\frac{\theta(s)}{E_f(s)} = \frac{K_T\;K_g}{(R_f + s\,L_f)\{s[(R_a + s\,L_a)(J\,s + f) + K_b\,K_T]\}}}$$

4.17.3 Position Control System

Another very common system used in practice is position control system. This is used to control position of shaft, by use of potentiometer as error detector. The error is to be amplified by amplifier and then must be given to armature controlled motor whose shaft position will get controlled as per the controlled signal. The motor shaft is coupled to the load through gearing arrangement with ratio N_1/N_2.

Load has M.I. J and friction as f while θ_r is the reference position while θ_a is the actual position of shaft.

The circuit diagram can be drawn as below :

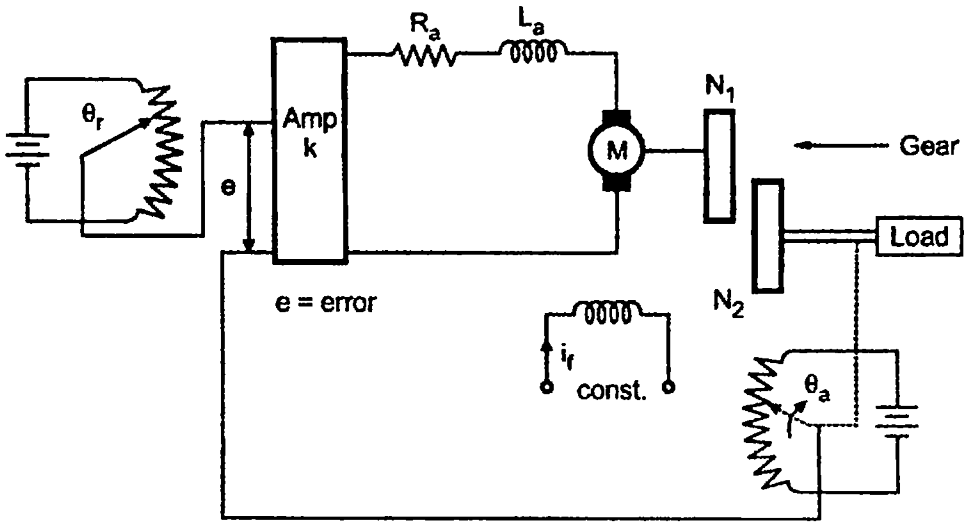


Fig. 4.43 Position control system

Let K_p be the potentiometer sensitivity, in V/rad

The corresponding block diagram can be drawn as shown in the Fig. 4.44.

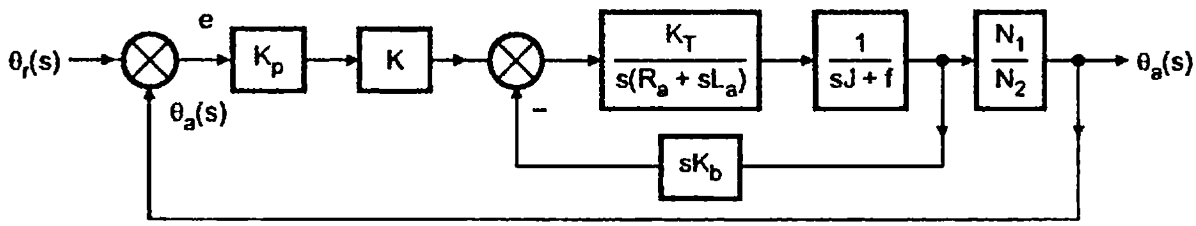


Fig. 4.44

Reducing the block diagram as shown in the Fig. 4.45.

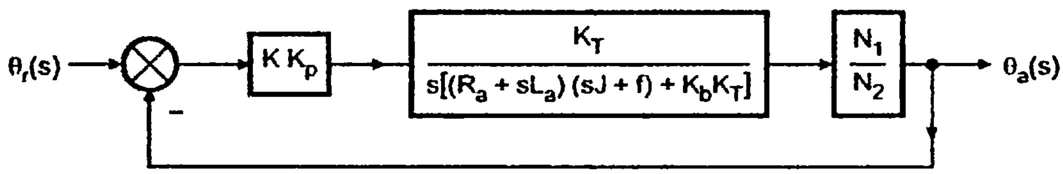


Fig. 4.45

$$G(s) = \frac{K K_P K_T (N_1/N_2)}{s[(R_a + s L_a)(J s + f) + K_b K_T]}$$

$$K K_P K_T (N_1/N_2) = K_s = \text{System gain constant}$$

$$H(s) = 1$$

∴ Over all T. F. can be calculated as below,

$$\frac{\theta_a(s)}{\theta_r(s)} = \frac{\dfrac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T]}}{1 + \dfrac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T]}}$$

$$\therefore \quad \boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T] + K_s}}$$

4.17.4 Position Control with Field Controlled Motor

In the above case if instead of armature controlled motor, field controlled motor is used then derive the T.F. of overall closed loop system.

Consider field controlled motor,

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

Now $\quad T \propto \phi\, i_a$ and i_a is constant

$$\therefore \quad T \propto \phi \propto i_f$$

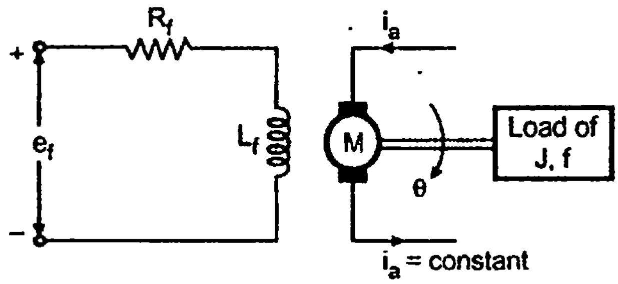


Fig. 4.46

Let K_f be constant in N-m/A

$$\therefore \quad T = K_f \, i_f$$

This torque is utilised to drive a load of moment of inertia J and friction f.

$$\therefore \quad T = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

$$\therefore \quad K_f \, i_f = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

Finding Laplace of all the equations,

$$E_f(s) = I_f(s) \, R_f + s \, L_f \, I_f(s)$$

and $$K_f \, I_f(s) = J s^2 \theta(s) + s f \theta(s)$$

$$\therefore \quad I_f(s) = \frac{s\theta(s) \, [sJ + f]}{K_f}$$

Substituting in $E_f(s)$

$$E_f(s) = \frac{s\theta(s) \, [sJ + f] \, [R_f + sL_f]}{K_f}$$

$$\therefore \quad \frac{\theta(s)}{E_f(s)} = \frac{K_f}{s[sJ + f][R_f + sL_f]}$$

Now using the same block diagram as derived in case (iii) replacing armature controlled motor T. F. by the field controlled we can get the new block diagram.

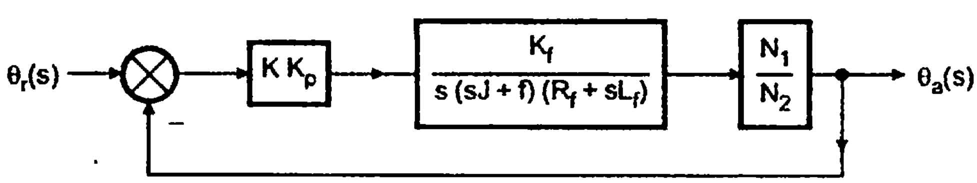


Fig. 4.47

Let $$K_s = K\, K_p\, K_f \left(\frac{N_1}{N_2}\right)$$

$$G(s) = \frac{K_s}{s(sJ+f)(R_f+sL_f)} \qquad H(s) = 1$$

$$\therefore \quad \frac{\theta_a(s)}{\theta_r(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K_s}{s(sJ+f)(R_f+sL_f)}}{1+\dfrac{K_s}{s(sJ+f)(R_f+sL_f)}}$$

$$\boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_s}{s(sJ+f)(R_f+sL_f)+K_s}}$$

4.17.5 Speed Control System

In some of the practical applications it is necessary to drive a load at a desired speed ω rad/sec. For this application an electromechanical system can be used which is shown as below.

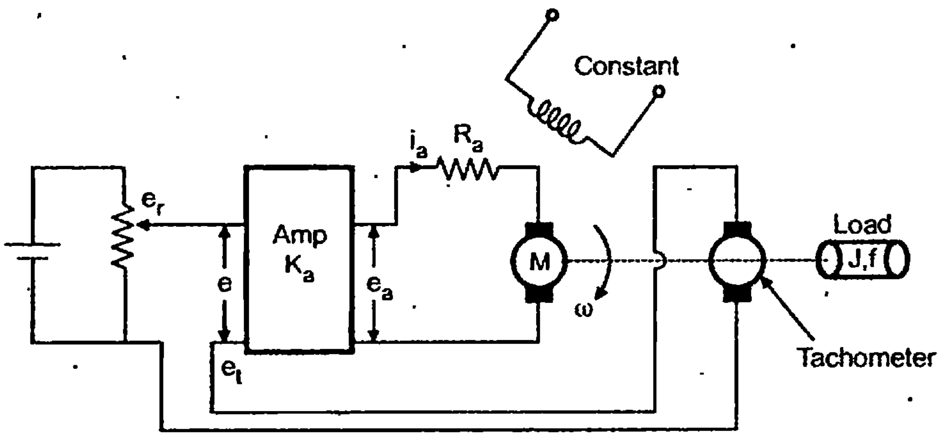


Fig. 4.48 Speed control system

System uses armature controlled motor and tachometer feedback. Let K_A be the amplifier gain. Let us derive the transfer function of this system. The objective of system is to move the load at a desired speed.

The d.c. tachometer output voltage is proportional to output speed ω.

e_r is reference voltage and e_t is tachometer voltage.

error $$e = e_r - e_t$$

$$e_a = K_a \times e = K_a\,(e_r - e_t) \qquad \text{... (5)}$$

$$e_t = K_t\,\omega \quad \text{where } K_t \text{ is constant.} \qquad \text{... (6)}$$

Neglecting inductance of armature

$$e_a = i_a\, R_a + e_b$$

$$e_b = K_b \omega$$

$$e_a = i_a R_a + K_b \omega \quad \text{substituting from equation (5),}$$

$$\therefore \quad K_a (e_r - e_t) = i_a R_a + K_b \omega \quad \text{substituting from equation (6),}$$

$$K_a e_r - K_a K_t \omega = i_a R_a + K_b \omega$$

Taking Laplace

$$K_a E_r(s) = (K_a K_t + K_b)\,\omega(s) + i_a(s) R_a \qquad \text{... (7)}$$

Now torque produced by motor

$$T \propto i_a$$

$$\therefore \quad T = K_T i_a \qquad \text{... (8)}$$

Now this drives a load

$$\therefore \quad K_T i_a = J\frac{d\omega}{dt} + f\omega \qquad \text{where } \omega = \frac{d\theta}{dt} \qquad \text{... (9)}$$

$\therefore$ Taking Laplace

$$K_T i_a(s) = Js\,\omega(s) + f\,\omega(s)$$

$$\therefore \quad i_a(s) = \frac{[J s + f]}{K_T} + f\,\omega(s) \qquad \text{... (10)}$$

Substituting in equation (7)

$$K_a E_r(s) = (K_a K_t + K_b)\,\omega(s) + \frac{[J s + f]\,\omega(s)}{K_T} R_a$$

$$\therefore \quad E_r(s) = \frac{(K_a K_t + K_b)\,\omega(s)}{K_a} + \frac{[J s + f]\,\omega(s) R_a}{K_a K_T}$$

$$\therefore E_r(s) - \left(K_t + \frac{K_b}{K_a}\right)\omega(s) = \frac{[J s + f] R_a}{K_a K_T}\,\omega(s)$$

$$\therefore \quad E_r(s) = \left\{K_t + \frac{K_b}{K_a} + \frac{[J s + f] R_a}{K_a K_T}\right\}\omega(s)$$

$$\therefore \quad \boxed{\frac{\omega(s)}{E_r(s)} = \frac{K_a K_T}{K_a K_T K_t + K_b K_T + [J s + f] R_a}}$$

This is the required transfer function of the speed control system.

4.17.6 Speed Control using Generator Driving Motor

A position control system described in Fig. 4.49 in which the armature of motor is applied with a control voltage through a generator. The field current of generator controls the voltage generated by the generator.

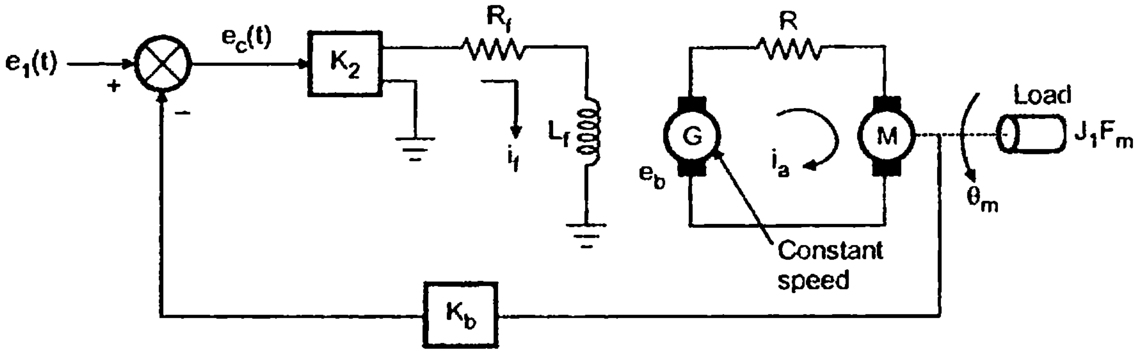


Fig. 4.49 Speed control system

Let us determine the transfer function of the system

Input is e_i (t).

Now $$e_c(t) = e_1(t) - K_b \frac{d(\theta)}{dt} \quad \text{... (11)}$$

Taking Laplace transform

$$E_c(s) = E_1(s) - K_b\, s\,\theta(s)$$

Now $$E_c(t) \cdot K_2 = L_f \frac{di_f}{dt} + R_f\, i_f \quad \text{... (12)}$$

applying Kirchhoff's law to the field circuit of generator.

Now $e_a = K_a\, i_f$ where K_a is constant.

Taking Laplace transform

$$E_a(s) = K_a\, I_f(s) \quad \text{... (13)}$$

Taking Laplace transform of (12)

$$E_c(s)\, K_2 = I_f(s)\left[R_f + sL_f\right]$$

Eliminating I_f (s)

$$\frac{E_c(s)\, K_2}{(R_f + s\,L_f)} = I_f(s)$$

$$E_a(s) = \frac{K_a\, E_c(s)\, K_2}{(R_f + s\,L_f)}$$

$$\frac{E_a(s)}{E_c(s)} = \frac{K_a\, K_2}{(R_f + sL_f)}$$

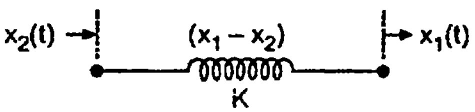


Fig. 4.5 Spring between two moving points causes change in displacement

Now mass M_1 will get displaced by $x_1(t)$ but mass M_2 will get displaced by $x_2(t)$ as spring of constant K will store some potential energy and will be the cause for change in displacement. Consider free body diagram of spring as shown in the Fig. 4.5.

Net displacement in the spring is $x_1(t) - x_2(t)$ and opposing force by the spring is proportional to the net displacement i.e. $x_1(t) - x_2(t)$.

$$\therefore \quad F_{spring} = K[x_1(t) - x_2(t)]$$

Taking Laplace,

$$F_{spring} = K[X_1(s) - X_2(s)]$$

Key Point : *The spring between the moving points causes a change in displacement from one point to another.*

Spring behaves exactly same in rotational systems, only the linear spring constant becomes torsional spring constant but denoted as 'K' only.

4.2.4 Friction

Whenever there is a motion, there exists a friction. Friction may be between moving element and fixed support or between two moving surfaces. Friction is also nonlinear in nature. It can be divided into three types,

i) Viscous friction ii) Static friction iii) Coulomb friction

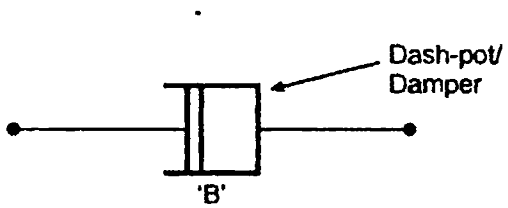


Fig. 4.6

Viscous friction as dominant out of the three is generally considered, neglecting other two types. Viscous friction is assumed to be linear, with frictional constant 'B'. This has linear relationship with relative velocity between two moving surfaces.

The friction is generally shown by a dash-pot or a damper as shown in the Fig. 4.6.

This is the symbolic representation of a friction.

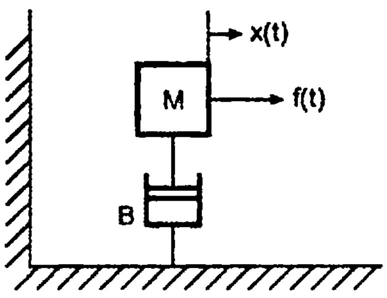


Fig. 4.7

Consider a mass M as shown in the Fig. 4.7 having friction with a support with a constant 'B' represented by a dash-pot.

Friction will oppose the motion of mass M and opposing force is proportional to velocity of mass M.

Now consider field constant armature controlled motor. Armature is energized by output of the generator.

Applying Kirchhoff's law to armature circuit,

$$E_a(s) = E_b(s) + I_a(s)\left[R_a + s L_a\right]$$

Back e.m.f. is proportional to θ_m

$$\therefore \quad E_b(s) = K_b \frac{d\theta_m}{dt}$$

Taking Laplace Transform

$$E_b(s) = K_b\, s\,\theta_m(s)$$

$$\therefore \quad E_a(s) = K_b\, s\,\theta_m(s) + I_a(s)[R_a + s L_a]$$

In armature controlled d.c. motor as field current is constant so torque produced is directly proportional to the armature current.

$$T = K I_a(s) \text{ where K is motor constant.}$$

This torque is utilized to drive the load with moment of inertia J and friction F_m.

$$T = J\frac{d^2\theta_m}{dt^2} + F_m \frac{d\theta_m}{dt} \quad \text{Taking Laplace transform from}$$

$$\therefore \quad K I_a(s) = J s^2 \theta_m(s) + s F_m \theta_m(s)$$

$$I_a(s) = \frac{s}{K}(Js + F_m)\theta_m(s)$$

Substituting in equation for $E_a(s)$

$$E_a(s) = K_b\, s\,\theta_m(s) + \frac{s}{K}(Js + F_m)\theta_m(s)(R_a + s L_a)$$

$$\therefore \quad \frac{\theta_m(s)}{E_a(s)} = \frac{K}{s K K_b + s(R_a + s L_a)(Js + F_m)}$$

$$= \frac{K_m}{s[1 + T_m s]} \quad \text{... Neglecting } L_a \text{ of armature}$$

$$K_m = \frac{K}{[KK_b + R_a F_m]} = \text{Motor constant}$$

$$T_m = \frac{R_a J}{[KK_b + R_a F_m]} = \text{Time constant of motor}$$

Neglecting 'L_a' of armature motor.

$$\therefore \quad \frac{\theta_m(s)}{E_a(s)} = \frac{K_m}{s(1 + T_m s)}$$

$$\frac{E_a(s)}{E_c(s)} = \frac{K_a\ K_2}{[R_f + s\,L_f]}$$

$$\therefore \qquad E_1(s) = \frac{K_a\ K_2\ E_c(s)}{(R_f + s\,L_f)} = \frac{\theta_m(s)\,s\,(1+T_m\,s)}{K_m}$$

and

$$E_c(s) = E_1(s) - K_b\,s\,\theta_m$$

$$\therefore \frac{K_a\ K_2\,[E_1(s) - K_b\,s\,\theta_m(s)]}{[R_f + s\,L_f]} = \frac{\theta_m(s)\,s\,(1+T_m\,s)}{K_m}$$

$$\frac{K_a\ K_2\ K_m}{[R_f + s\,L_f]}\,E_1(s) = \theta_m(s)\left[s\,(1+T_m\,s) + \frac{K_o\ K_a\ K_2\ K_m\ s}{(R_f + s\,L_f)}\right]$$

$$\therefore \quad \frac{K_a\ K_2\ K_m}{[1+s\,T_f]}\,E_1(s) = \theta_m(s)\left[\frac{R_f\ s\,(1+T_m\,s)\,(1+T_f\ s) + K_b\ K_a\ K_2\ K_m\ s}{[1+T_f(s)]}\right]$$

$$\frac{\theta_m(s)}{E_i(s)} = \frac{K_a\ K_2\ K_m}{s\,[K_o\ K_a\ K_2\ K_m + R_f\,(1+T_m\,s)\,(1+T_f\ s)]}$$

$$\boxed{\frac{\theta_m(s)}{E_i(s)} = \frac{K_2\ K_m\ K_g}{s\,[K_g\ K_b\ K_2\ K_m + (1+T_m\,s)\,(1+T_f\ s)]}}$$

where $K_g = \frac{K_a}{R_f}$ generator constant.

4.17.7 A Typical Position Control System used in Industry

A position control feedback system has a potentiometer bridge with a sensitivity of K_p V/radian as error detector. It feeds a d.c. amplifier with an open circuit gain K. This Supplies to a field of generator which has resistance R_f and inductance L_f. Generator constant is K_gV/A. This is connected to an armature controlled d.c. motor with motor torque constant K_T and back e.m.f constant K_b V/rad/sec. It drives a load of inertia J and friction f.

Draw the simplified block diagram and hence calculate its transfer function.

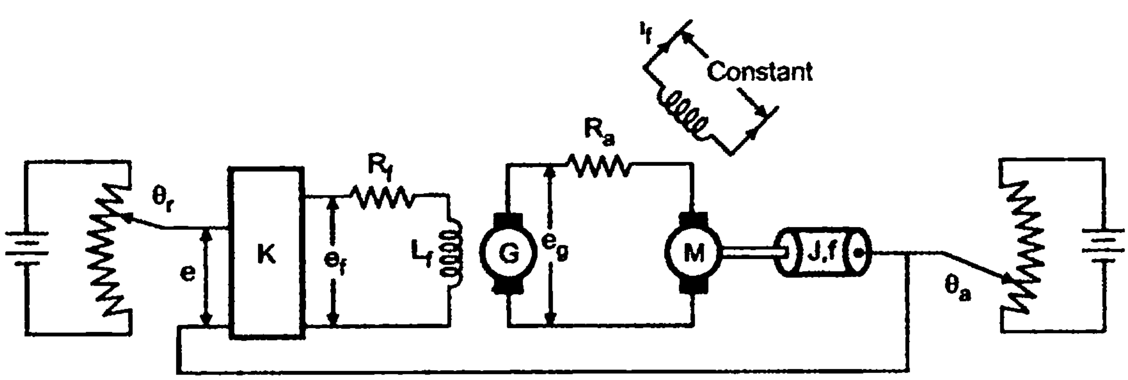


Fig. 4.50 Typical position control system

θ_r = Reference position , θ_a = Actual position

T. F of generator $\quad \dfrac{E_g(s)}{E_f(s)} = \dfrac{K_g}{R_f + s\,L_f}$

Block diagram can be drawn from the analysis of the different cases discussed.

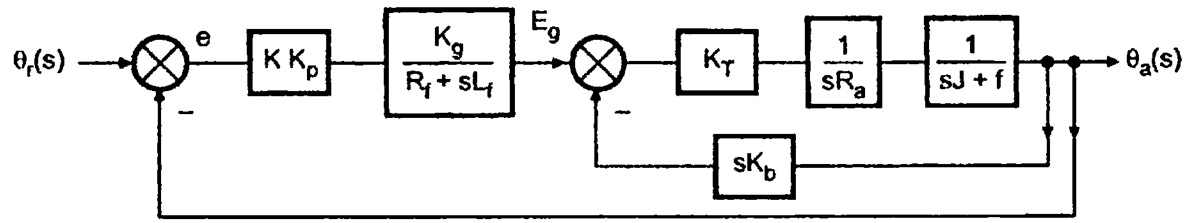


Fig. 4.51

Neglecting the inductance of armature winding of motor.

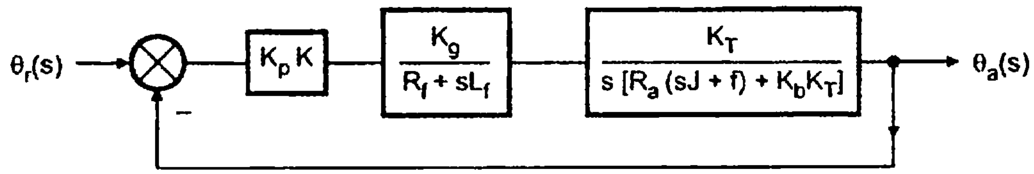


Fig. 4.52

$$G(s) = \frac{K_p\ K\ K_g\ K_T}{s\,(R_f + sL_f)\ [R_a\ (Js + f) + K_b\ K_T\ ]} \qquad H(s) = 1$$

Let $\quad K_R = K_p\ K\ K_g\ K_T$

$$\therefore \qquad T.\,F = \frac{\dfrac{K_R}{s\,(R_f + sL_f)\,[\,R_a\ (Js + f) + K_b\ K_T\,]}}{1 + \dfrac{K_R}{s\,[(R_a\ (Js + f) + K_b\ K_T\ ]\ (R_f + sL_f)}}$$

$$\boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_R}{s\,(R_f + sL_f)\,[R_a\ (Js + f) + K_b\ K_T\ ] + K_R}}$$

4.18 D.C. Motor Position Control System

In industry to conrol the position of the shaft, a d.c. motor position control system is commonly used.

4.18.1 Transfer Function of D.C. Motor Position Control System

Consider the D.C. position control system which is controlling position of the shaft. Assume that the input and output of the system are the input shaft position and output shaft position respectively.

Assume following system constants,

r = Angular displacement of the reference input shaft

c = Angular displacement of the output shaft

θ = Angular displacement of the d.c. motor shaft used

K_1 = Gain of potentiometric error detector

K_p = Amplifier gain

e_a = Applied armature voltage

e_b = Back e.m.f.

R_a = Armature winding resistance

L_a = Armature winding inductance

i_a = Armature winding current

K_b = Back e.m.f. constant

K = Motor torque constant

J_m = Moment of inertia of motor.

b_m = Viscous friction coefficient of motor

J_L = Moment of inertia of load

b_L = Viscous friction coefficient of load

n = Gear ratio N_1/N_2

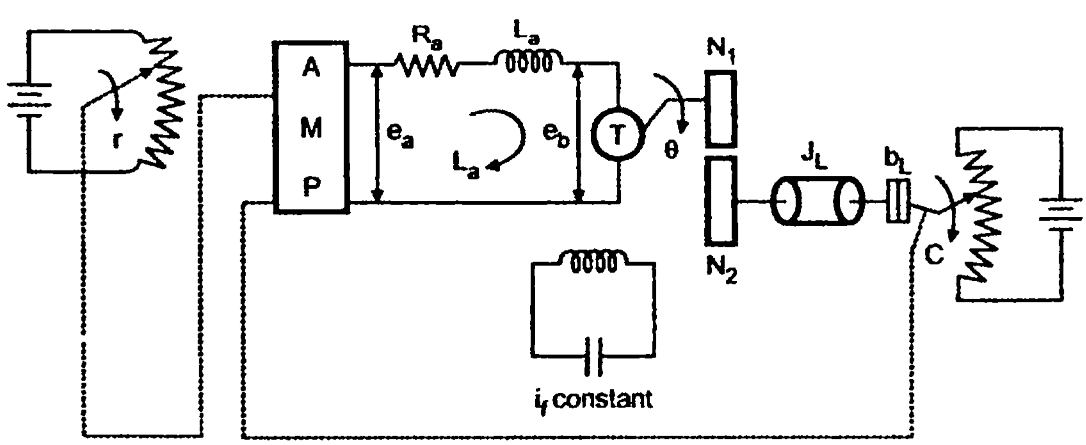


Fig. 4.53 D.C. Motor position control system

Equations describing above system can be written as follows :

Output shaft position is to be controlled so as to keep that at required position. Output is sensed by angular displacement 'c' and compared with input which is r. Error is amplified by amplifier with gain 'K_p' and given as input to the d.c. motor which in turn controls the angular position of the shaft of the motor 'θ' which in turn controls output position of shaft 'c' so as to modify the error.

For potentiometric error detector we can write.

$$E(s) = K_1 [R(s) - C(s)]$$

For amplifier

$$E_a(s) = K_p E(s)$$

For armature controlled d.c. motor

$$i_f = \text{constant so flux } \phi \text{ is constant}$$

$$T = K i_a \qquad \text{where } K = \text{motor torque constant}$$

$$e_b \propto \theta$$

$$\therefore \quad e_b = K_b \frac{d\theta}{dt}$$

For armature circuit

$$e_a = e_b + L_a \frac{di_a}{dt} + i_a R_a$$

Taking Laplace transforms

$$E_b(s) = K_b s\theta(s)$$

$$E_a(s) = E_b(s) + I_a(s)[R_a + sL_a]$$

$$T = K I_a(s)$$

Now torque is utilised to drive load + shaft of motor.

$$\therefore \quad J_{eq} = J_m + n^2 J_L = \text{Equivalent moment of inertia}$$

$$b_{eq} = b_m + n^2 b_L = \text{Equivalent frictional coefficient}$$

$$T = KI_a(s) = J_{eq} \frac{d^2\theta}{dt^2} + b_{eq} \frac{d\theta}{dt}$$

$$= [J_{eq} s^2 + b_{eq} s]\theta(s) \qquad \text{Laplace transform}$$

$$\therefore \quad I_a(s) = \frac{1}{K}[J_{eq} s^2 + b_{eq} s]\theta(s)$$

$$\therefore \quad E_a(s) = K_b s\theta(s) + \frac{\theta(s)}{K}[J_{eq} s^2 + b_{eq} s][R_a + sL_a]$$

$$\therefore \quad \frac{\theta(s)}{E_a(s)} = \frac{K}{s[J_{eq}s + b_{eq}][R_a + s L_a] + KK_b\, s}$$

$$= \frac{K}{s[KK_b + La\, J_{eq}s^2 + s(L_a\, b_{eq} + R_a\, J_{eq}) + b_{eq}R_a]}$$

'L_a' is generally small hence neglected.

$$= \frac{K}{s[KK_b + s\, R_a\, J_{eq} + b_{eq}R_a]}$$

$$= \frac{K_m}{s(1 + T_m\, s)}$$

$$K_m = \frac{K}{(b_{eq}R_a + KK_b)} = \text{Motor constant}$$

$$T_m = \frac{R_a\, J_{eq}}{(J_{eq}R_a + KK_b)} = \text{Motor time constant}$$

Now $\quad C(s) = n\,\theta(s)$

$$\therefore \quad \frac{C(s)}{E_a(s)} = n\,\frac{\theta(s)}{E_a(s)}$$

$$\therefore \quad \frac{C(s)}{E_a(s)} = \frac{n\,K_m}{s(1 + T_m\, s)}$$

$$\frac{C(s)}{K_pE(s)} = \frac{n\,K_m}{s(1 + T_m\, s)} \qquad \text{where } E_a(s) = K_p\, E(s)$$

$$C(s) = \frac{n\,K_m\,K_p\,E(s)}{s(1 + T_m\, s)}$$

$$C(s) = \frac{n\,K_m\,K_p\,K_1\,[R(s) - C(s)]}{s(1 + T_m\, s)}$$

$$\therefore \quad C(s) + \frac{n\,K_m\,K_p\,K_1}{s(1 + T_m\, s)}\,C(s) = \frac{n\,K_m\,K_p\,K_1\,R(s)}{s(1 + T_m\, s)}$$

$$\therefore C(s)\left[\frac{s(1 + T_m\, s) + n\,K_m\,K_p\,K_1}{s(1 + T_m\, s)}\right] = \frac{n\,K_m\,K_p\,K_1\,R(s)}{s(1 + T_m\, s)}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{\dfrac{nK_mK_pK_1}{s(1 + T_m s)}}{1 + \dfrac{nK_mK_pK_1}{s(1 + T_m s)}} = \frac{G(s)}{1 + G(s)H(s)}}$$

Block diagram :

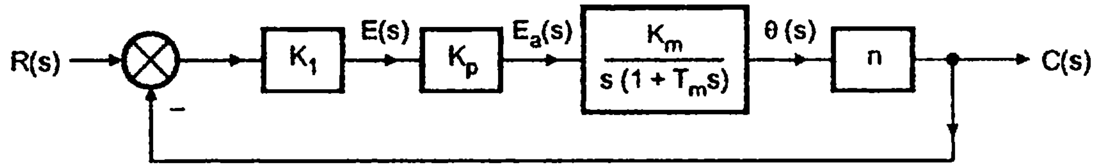
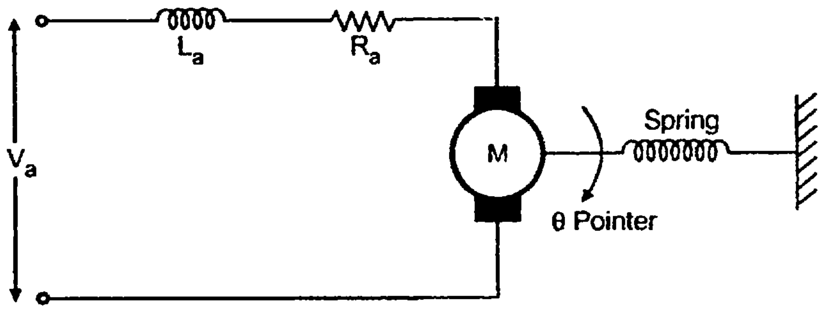


Fig. 4.54

Example 4.3 : *A d.c. motor drives a pointer which is spring loaded, to return to the reference position. If K_b = Back e.m.f. constant, K_T = Torque constant K_s = Spring constant and J = Moment of inertia. Find the transfer function.*



(M.U. : Nov.-93, Nov.-94)

Solution : Writing the system equations.

$$V_a(t) = I_a R_a + L_a \frac{dI_a}{dt} + E_b$$

Taking Laplace,

$$V_a(s) = E_b(s) + I_a(s)\left[R_a + s L_a\right] \quad \text{... (1)}$$

$$\therefore \quad I_a(s) = \frac{V_a(s) - E_b(s)}{(R_a + sL_a)} \quad \text{.... (2)}$$

Now $T_m = K_T I_a$, T_m = Motor torque ... (3)

and $E_b(s) = K_b\, s\theta(s)$ as $E_b \propto \frac{d\theta}{dt}$ ... (4)

$$\therefore \quad I_a(s) = \frac{V_a(s) - \theta(s)}{(R_a + s L_a)}$$

and $$T_m(s) = K_T \left[\frac{V_a(s) - K_b s\theta(s)}{(R_a + s L_a)}\right] \quad \text{.. (5)}$$

This torque is used to drive a pointer with inertia J and spring load of constant K_s.

$$\therefore \quad T_m = J\frac{d^2\theta(t)}{dt^2} + K_s\,\theta(t)$$

Taking Laplace

$$T_m(s) = J\,s^2\,\theta(s) + K_s\,\theta(s) \quad \text{...(6)}$$

Equating (5) and (6)

$$\therefore \; J\,s^2\,\theta(s) + K_s\,\theta(s) = K_T\left[\frac{V_a(s) - K_b\,s\,\theta(s)}{R_a + sL_a}\right]$$

$$J\,s^2\,\theta(s) + K_s\,\theta(s) + \frac{K_T\,K_b\,s\,\theta(s)}{(R_a + sL_a)} = \frac{K_T}{(R_a + s\,L_a)}\,V_a(s)$$

$$\therefore \; \theta(s)\left[\frac{Js^2\,(R_a + s\,L_a) + K_s\,(R_a + sL_a) + K_T\,K_b s}{(R_a + s\,L_a)}\right] = \frac{K_T}{(R_a + s\,L_a)}\,V_a(s)$$

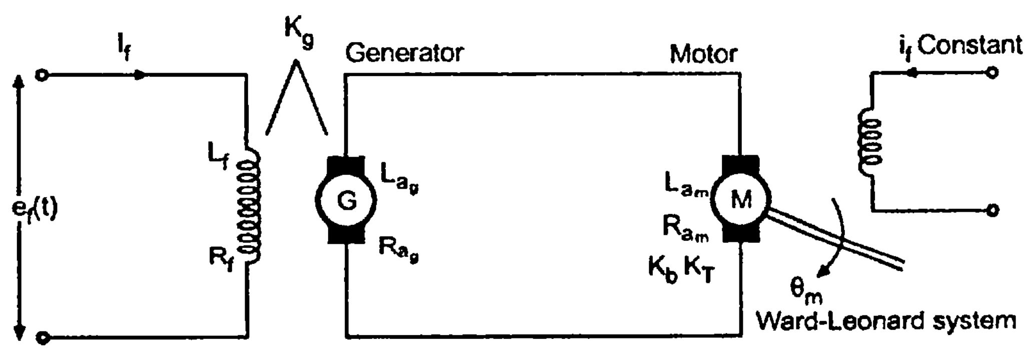
So transfer function is,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{s^2\,J\,(R_a + sL_a) + K_s\,(R_a + sL_a) + K_T\,K_b\,s}$$

$$\therefore \quad \frac{\theta(s)}{V_a(s)} = \frac{K_T}{(R_a + sL_a)\,[s^2\,J + K_s] + K_T\,K_b\,s}$$

Example 4.4 : *A d.c. generator supplies its output to a separately excited d.c. motor. The field current of the motor is constant. The voltage applied to the field circuit of d.c. generator is $e_f(t)$. Write the differential equations of this Ward-Leonard system relating the input $e_f(t)$ to the output $\theta_m(t)$. (Refer to figure shown below) Hence obtain expression for $\theta_m(s)$.*

(M.U.-May-96[PTDC] and April-96[Electrical])



Solution : Torque produced by motor,

$$T = K_T\,I_a(t) \quad \text{... (1)}$$

Voltage applied to the motor

$$E_g(t) = I_a\,R_a + L\,\frac{dI_a}{dt} + E_b(t) \quad \text{... (2)}$$

Back emf is proportional to the angular velocity

$$\therefore \quad E_b(t) = K_b \frac{d\theta_m(t)}{dt} \quad \text{... (3)}$$

Torque produced T is used to drive a load of inertia J and friction B

$$\therefore \quad T = J\frac{d^2\theta_m}{dt^2} + B\frac{d\theta_m}{dt} \quad \text{... (4)}$$

Equating (1) and (4)

$$K_T I_a(t) = J\frac{d^2\theta_m}{dt^2} + B\frac{d\theta_m}{dt}$$

$$\therefore \quad I_a(t) = \frac{J}{K_T}\frac{d^2\theta_m}{dt^2} + \frac{B}{K_T}\frac{d\theta_m}{dt} \quad \text{... (5)}$$

Substituting in equation (2) we can get resultant differential equation relating $E_g(t)$ and $\theta_m(t)$.

Now
$$E_f(t) = I_f R_f + L_f \frac{dI_f}{dt} \quad \text{... (6)}$$

and
$$E_g(t) = K_g I_f(t) \quad \text{... (7)}$$

$\therefore$ Substituting $I_f(t) = \frac{E_g(t)}{K_g}$ in equation (6) we can get, equation relating $E_f(t)$ and $E_g(t)$.

Substituting this in equation (2) we can obtain the final differential equation relating $E_f(t)$ and $\theta_m(t)$. It is very difficult to obtain in time domain so let us obtain it in Laplace domain.

Taking Laplace transform of all the equations

$$T(s) = K_T I_a(s) \quad \text{... (1)}$$

$$E_g(s) = I_a(s)[R_a + s L_a] + E_b(s) \quad \text{... (2)}$$

$$E_b(s) = K_b S\theta_m(s) \quad \text{... (3)}$$

$$T(s) = \theta_m(s)[Js^2 + Bs] \quad \text{... (4)}$$

$$\therefore \quad K_T I_a(s) = \theta_m(s)[Js + B]s$$

$$\therefore \quad I_a(s) = \frac{\theta_m(s)\, s[Js + B]}{K_T} \quad \text{... (5)}$$

$$\therefore \quad E_g(s) = \frac{\theta_m(s)\, s[Js + B][R_a + s L_a]}{K_T} + K_b s\theta_m(s) \quad \text{... (6)}$$

and
$$E_f(s) = I_f(s)[R_f + s L_f] \quad \text{... (7)}$$

$$E_g(s) = K_g\, I_f(s)$$

So
$$I_f(s) = \frac{E_g(s)}{K_g}$$

$\therefore$
$$E_f(s) = \frac{E_g(s)\ [R_f + sL_f]}{K_g} \quad \text{... (8)}$$

Substituting $E_g(s)$ in equation (8) we get

$$\frac{E_f(s)\, K_g}{[R_f + s\, L_f]} = \theta_m(s)\left[\frac{s(Js+B)(R_a + s\, L_a)}{K_T} + K_b s\right]$$

$\therefore$
$$\frac{\theta_m(s)}{E_f(s)} = \frac{K_g\, K_T}{(R_f + s\, L_f)\ \{s(R_a + s\, L_a)(Js+B) + sK_bK_T\}}$$

$\therefore$
$$\theta_m(s) = \frac{K_g\, K_T\, E_f(s)}{(R_f + s\, L_f)\ \{\, s[(R_a + s\, L_a)(Js+B) + K_b\, K_T]\}}$$

4.19 Models of Thermal Systems

4.19.1 Heat Transfer System

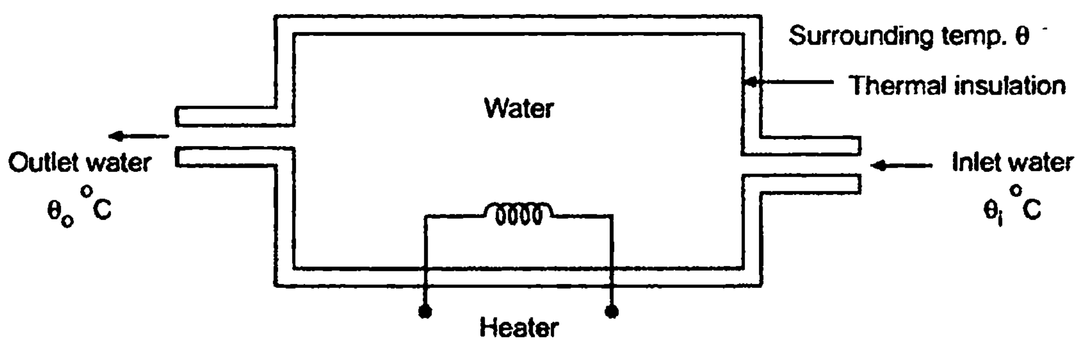


Fig. 4.55

A thermal system used for heating flow of water is shown below.

Electric heating element is provided in the tank to heat the water. The tank is insulated to reduce heat to the surroundings.

The necessary simplifying assumptions are :

1) There is no heat storage in the insulation.

2) All the water in the tank is perfectly mixed and hence at a uniform temperature.

$$\therefore \quad F_{frictional} = B\frac{d\,x(t)}{dt}$$

Taking Laplace and neglecting initial conditions,

$$F_{frictional}(s) = Bs\,X(s)$$

Similarly if friction is between two moving surfaces, it is shown in the Fig. 4.8.

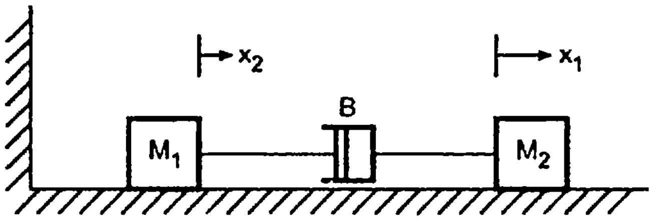


Fig. 4.8 Friction between two moving points causes change in displacement

In such a case, opposing force is given by,

$$F_{frictional} = B\left[\frac{d\,x_1(t)}{dt} - \frac{d\,x_2(t)}{dt}\right]$$

Taking Laplace,

$$F_{frictional}(s) = Bs\left[X_1(s) - X_2(s)\right]$$

Thus if the force applied to mass M_2 is f(t) then due to friction between the masses M_1 and M_2, the force getting transmitted to M_1 is always less than f(t). Hence the displacement of mass M_1 is different than the displacement of mass M_2.

Key Point : *The friction between two moving points, causes a change in displacement from one point to other.*

Frictional force also behaves exactly in same manner, in rotational systems, only linear frictional constant becomes torsional frictional constant but denoted by same symbol 'B' only.

4.3 Rotational Motion

This is the motion about a fixed axis. In such systems, the force gets replaced by a moment about the fixed axis i.e. (force × distance from fixed axis) which is called **Torque.**

So extension of Newton's law states that the sum of the torques applied to a rigid body or a system must be equal to sum of the torques consumed by the different elements of the system in order to produce angular displacement (θ), angular velocity (ω) and angular acceleration (α) in them. As previously stated, spring and friction behaves in same manner in rotational systems. The property of system which stores kinetic energy in

θ_i = Inlet water temperature in °C.

θ_o = Outlet water temperature in °C.

θ = Surrounding temperature.

q = Rate of heat flow from heating element in J/sec.

q_i = Rate of heat flow to the water.

q_t = Rate of heat flow through tank insulation.

C = Thermal capacity in J/°C.

R = Resistance of thermal insulation.

So rate of heat flow for the water in tank is,

$$q_i = C\frac{d\theta_o}{dt} \quad \text{... (1)}$$

The rate of heat flow from the water to the surrounding atmosphere through insulation is,

$$q_t = \frac{\theta_o - \theta}{R} \quad \text{... (2)}$$

As per the heat transfer principles,

$$q = q_i + q_t \quad \text{... (3)}$$

Substituting equation (1) and (2)

$$q = C\frac{d\theta_o}{dt} + \frac{\theta_o - \theta}{R} \quad \text{... (4)}$$

Neglecting the term θ/R from the equation (4) this is because the variation of water temperature θ_o is over and above ambient temperature θ_ω.

$$\therefore \quad q = C\frac{d\theta_o}{dt} + \frac{\theta_o}{R}$$

Taking Laplace transform,

$$Q(s) = Cs\theta_o(s) + \frac{\theta_o(s)}{R}$$

∴ Transfer function is,

$$\boxed{\frac{\theta_o(s)}{Q(s)} = \frac{R}{1+sCR}}$$

The time constant of the system is RC.

4.19.2 Thermometer

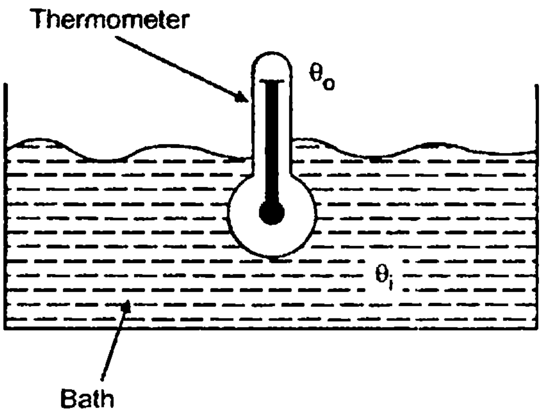


Fig. 4.56

Consider a thermometer placed in a water bath having temperature θ_i, as shown.

θ_o is the temperature indicated by the thermometer. The rate of heat flow into the thermometer through its wall is,

$$\frac{dq}{dt} = \frac{\theta_i - \theta_o}{R}$$

Where R = Thermal resistance of the thermometer wall.

The indicated temperature, rises at a rate of

$$\frac{d\theta_o}{dt} = \frac{1}{C}\frac{dq}{dt}$$

where C is thermal capacity of the thermometer.

$$\therefore \quad \frac{d\theta_o}{dt} = \frac{1}{C}\cdot\left[\frac{\theta_i - \theta_o}{R}\right]$$

Taking Laplace of the equation,

$$\therefore \quad s\theta_o(s) = \frac{1}{RC}\left[\theta_i(s) - \theta_o(s)\right]$$

$$\therefore \quad \boxed{\frac{\theta_o(s)}{\theta_i(s)} = \frac{1}{1 + sRC}}$$

The time constant is RC.

4.20 Actuators

The actuator is a device which receives input signal from the controller and it produces the input signal to the plant according to control signal so that the output will approach the reference input signal to reduce the error to zero. Thus an actuator is generally after the controller and before the plant in the control system. An actuator can be of two types :

i) Hydraulic actuator 2) Pneumatic actuator

4.20.1 Hydraulic Actuator

The structure of an hydraulic actuator is shown in the Fig. 4.57.

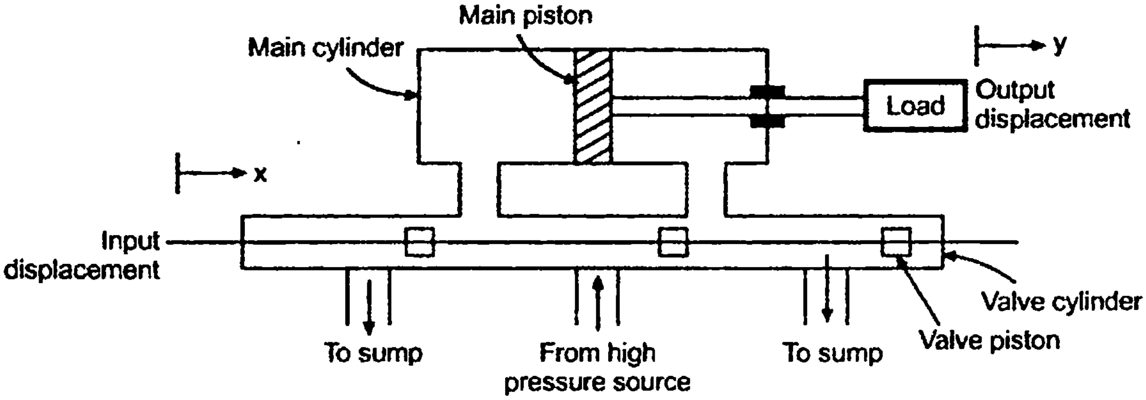


Fig. 4.57

They are piston devices in which motion of the spool regulates the oil flow to either side of the power cylinder. When the spool moves to the right, the high pressure oil enters the power cylinder to the left of the piston.

The differential pressure produced causes the power piston to move to the right, pushing the oil infront of it into the sump.

The load coupled rigidly to the piston moves a distance y from its reference position corresponding to the displacement x of the valve piston from its neutral position. The oil is pressurised by a pump and is recalculated in the system.

Equation of motion and transfer function

The rate of flow of oil into the piston is proportional to the rate of the movement of the piston.

$$Q = A\frac{dy}{dt}, \quad A = \text{area of piston} \qquad \dots (1)$$

If P is the differential pressure across the piston then the force on the piston is AP. This moves the load of mass M against friction B.

$$A \times P = M\frac{d^2 y}{dt^2} + B\frac{dy}{dt} \qquad \dots (2)$$

For small values of the displacement x, if P is the differential pressure on the piston and Q is the oil flow then,

$$K_2 P = K_1 x - Q \qquad \dots (3)$$

$$K_2\left[\frac{M}{A}\frac{d^2 y}{dt^2} + \frac{B}{A}\frac{dy}{dt}\right] = K_1 x - A\frac{dy}{dt}$$

Taking Laplace transform,

$$\frac{K_2 M}{A} s^2 Y(s) + \frac{K_2 B}{A} s Y(s) = K_1 X(s) - A Y(s) s$$

$$Y(s) \left[s^2 K_2 M + K_2 Bs + A^2 s\right] = K_1 A X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{K_1 A}{s^2 K_2 M + s K_2 B + A^2 s}$$

$$\therefore \quad \boxed{\frac{Y(s)}{X(s)} = \frac{\left(\frac{K_1 A}{K_2}\right)}{s\left[Ms + \left(B + \frac{A^2}{K_2}\right)\right]}}$$

This transfer function is similar to the electric motors.

4.20.2 Pneumatic Actuator

Pneumatic acting valve is used to obtain linear displacement of a plunger with pressurised air as input.

The air at pressure P is injected through inlet. Pressurised air pushes the diaphragm and plunger. The plunger has a mass M and friction on B with spring constant K.

Let A be the area of diaphragm then transfer function can be obtained as below.

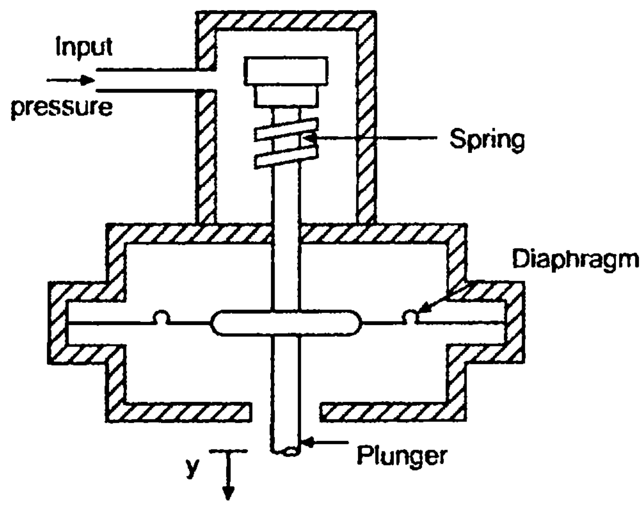


Fig. 4.58

Force exerted on the plunger is A × P. This force is opposed by mass, friction and spring.

$$A \times P = M \frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky$$

Taking Laplace transform,

$$A\,P(s) = Ms^2 Y(s) + Bs\,Y(s) + K\,Y(s)$$

$$\therefore \quad \boxed{\frac{Y(s)}{P(s)} = \frac{A}{Ms^2 + Bs + K}}$$

The advantages of pneumatic systems are fire proof, explosion proof, simplicity and easy to maintain.

4.20.3 Comparison between Pneumatic and Hydraulic Systems

Sr.No.	Pneumatic systems	Hydraulic systems
1.	The fluid used is air.	The fluid used is oil.
2.	Air is compressible.	Oil is incompressible.
3.	Air does not have lubricating property.	Oil acts as a lubricator.
4.	The output power is much less compared to hydraulic.	The output power is much higher than pneumatic.
5.	At low velocities, the accuracy is poor.	At all velocities, the accuracy is satisfactory.
6.	No return pipes are required when air is used.	The return pipes are must.
7.	Can be operated for the temperature range of 0°C to 200°C. It is insensitive to temperature changes.	It is sensitive to the temperature changes and the range is 20°C to 70°C.
8.	These are fire proof and explosion proof.	These are not fire and explosion proof.
9.	Easy from maintenance point of view.	Difficult from maintenance point of view.

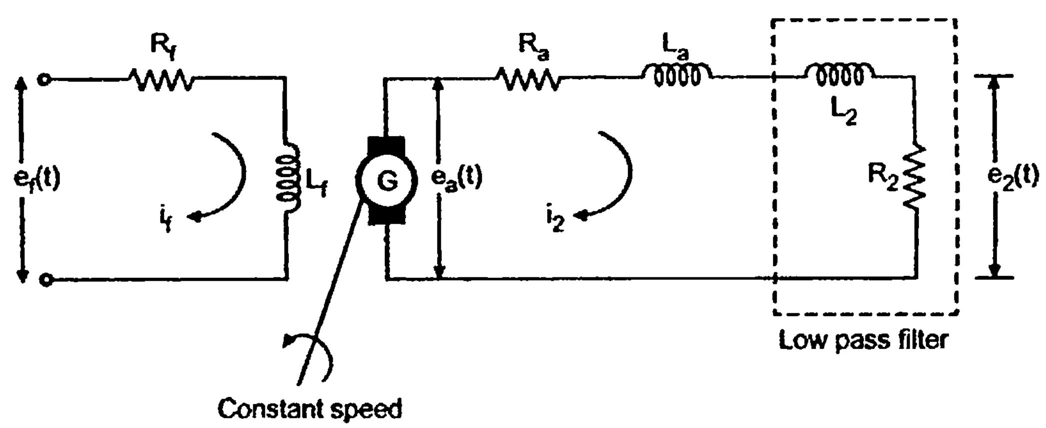
Note : The state space method of modeling the systems is separately covered in the chapter 15.

Examples with Solutions

Example 4.5 : *The voltage generated by a d.c. generator is filtered by a low pass filter consisting R_2 and L_2 as shown below. This filtered voltage is controlled by the voltage applied to the field of generator.*

If $R_f = 40\ \Omega$, $L_f = 60\ H$, $R_a = 0.5\ \Omega$, $L_a = 1H$, $L_2 = 1H$, $R_2 = 2\Omega$ and generator constant $Kg = 120\ V / \text{field Amp}$. Determine the transfer function $\frac{E_2(s)}{E_f(s)}$ of the system.

(M.U.: Dec.-97)



Solution : For field circuit, $e_f(t) = i_f(t) R_f + L_f \frac{d i_f}{dt}$ ... (1)

For armature circuit, $e_a(t) = i_2(t) R_a + i_2(t) R_2 + L_a \frac{di_2}{dt} + L_2 \frac{di_2}{dt}$ ... (2)

For generator, $e_a(t) = K_g \, i_f(t)$ ... (3)

For output , $e_2(t) = i_2(t) \, R_2$ ... (4)

Taking Laplace transform of all the equations,

$$E_f(s) = I_f(s) \left[R_f + s L_f\right] \quad \text{... (5)}$$

$$E_a(s) = I_2(s)\left[(R_a + R_2) + s(L_a + L_2)\right] \quad \text{... (6)}$$

$$E_a(s) = K_g \, I_f(s) \quad \text{... (7)}$$

$$E_2(s) = I_2(s) R_2 \quad \text{... (8)}$$

Equation (6) and (7),

$$K_g \, I_f(s) = I_2(s) \left[(R_a + R_2) + s(L_a + L_2)\right]$$

Substituting $I_a(s)$ from equation (8) and $I_f(s)$ from equation (5),

$$K_g \left[\frac{E_f(s)}{R_f + s L_f}\right] = \frac{E_2(s)}{R_2} \left[(R_a + R_2) + s(L_a + L_2)\right]$$

Substituting $I_2(s)$ from equation (8) and $I_f(s)$ from (5),

$$\frac{E_2(s)}{E_f(s)} = \frac{K_g \, R_2}{(R_f + s L_f) \left[(R_a + R_2) + s(L_a + L_2)\right]}$$

Substituting the values,

$$\therefore \quad \boxed{\frac{E_2(s)}{E_f(s)} = \frac{120 \times 2}{(40 + 60s)\left[(2.5 + 2s)\right]} = \frac{2.4}{(1 + 1.5s)(1 + 0.8s)}}$$

Example 4.6 : *A high gain speed control system uses a tachogenerator for speed sensing. The tachogenerator produces 5 V per 100 r.p.m. This voltage is compared with reference voltage to produce error signal. If the reference voltage is set to 10.8 volt. What is the value of expected speed?* **(M.U. : Nov.-94)**

Solution : The tachogenerator produces 5 V per 100 r.p.m.,

$\therefore$ Its constant is $\frac{5}{100} = 0.05$ V/r.p.m.

Now for the expected speed, error should be zero,

i.e V_r – tachogenerator output = error

∴ Tachogenerator output = V_r

Let N be expected speed in r.p.m.

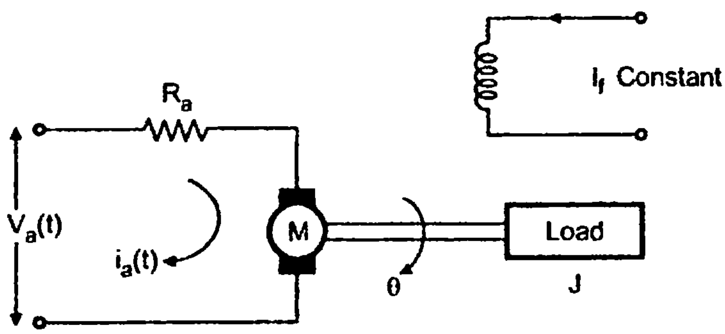
∴ $N \times 0.05 = 10.8$

∴ $N = $ **216 r.p.m.** ... Expected speed

Example 4.7 : *An armature controlled d.c. motor has an armature resistance of 0.37 Ω. The moment of inertia is 2.5×10^{-6} kg - m². A back emf of 2.09 V is generated per 100 r.p.m. of the motor speed. The torque constant of the motor is 0.2 N–m /A. Determine the transfer function of the motor relating the motor shaft shift and the input voltage.*

(M.U. : Dec.-96)

Solution : The motor can be shown as,



Now back emf is 2.09 V per 100 r.p.m. , N = 100 r.p.m. is,

$$\omega = \frac{2\pi N}{60} = 10.4719 \text{ rad/sec}$$

$$\therefore \text{Back emf constant } K_b = \frac{-2.09}{10.4719} = 0.1995 \text{ V/rad/sec}$$

$$K_T = 0.2 \text{ N-m/A}$$

For armature circuit $V_a(t) = I_a R_a + E_b$ ... (1)

Taking Laplace $V_a(s) = I_a(s) R_a + E_b(s)$

$$E_b(t) = K_b \frac{d\theta}{dt} \quad \text{... (2)}$$

Taking Laplace $E_b(s) = K_b s \theta(s)$

$$T_m = K_T I_a(t) \quad \text{... (3)}$$

Taking Laplace $T_m(s) = K_T\, I_a(s)$

This torque drives a load of inertia J.

$$\therefore \quad T_m(t) = J\frac{d^2\theta}{dt^2} \qquad \ldots (4)$$

Taking Laplace $T_m(s) = J\, s^2\, \theta(s)$

$$\therefore \quad K_T\, I_a(s) = J\, s^2\, \theta(s)$$

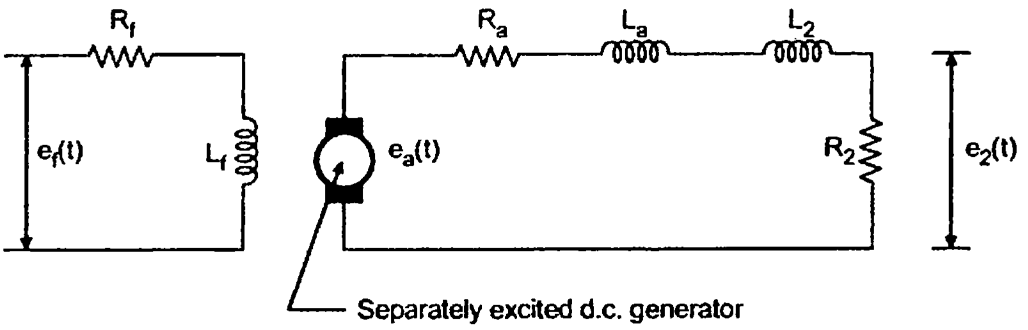
$$\therefore \quad I_a(s) = \frac{2.5\times 10^{-6}}{0.2}\, s^2\, \theta(s) = 1.25\times 10^{-5}\, s^2\, \theta(s)$$

$$\therefore \quad V_a(s) = 1.25\times\, s^2\, 10^{-5}\, \theta(s)\times 0.37 + K_b\, s\theta(s)$$

$$\therefore \quad V_a(s) = 4.625\times 10^{-6}\, s^2\, \theta(s) + 0.1995\, s\theta(s)$$

$$\therefore \quad \boxed{\frac{\theta(s)}{V_a(s)} = \frac{1}{4.625\times 10^{-6}\, s^2 + 0.1995\, s} = \frac{216216.22}{s(s+43135.135)}}$$

Example 4.8 : *For the system shown below determine* $\frac{E_2(s)}{E_f(s)}$. **(M.U. : Dec.-97)**



Solution :

For field circuit, $\quad e_f(t) = i_f(t)\, R_f + L_f\frac{di_f}{dt}$

For armature circuit, $\quad e_a(t) = i_2(t)\, R_a + i_2(t)\, R_2 + L_a\frac{di_2}{dt} + L_2\frac{di_2}{dt}$

For generator, $\quad e_a(t) = K_g\, i_f(t)$

For output, $\quad e_2(t) = i_2(t)\, R_2$

Taking Laplace of all the equations we get,

$$E_f(s) = I_f(s)[R_f + s L_f]$$

$$E_a(s) = I_2(s)\left[(R_a + R_2) + s(L_a + L_2)\right] = K_g \, I_f(s)$$

$$E_2(s) = I_2(s)\, R_2$$

Hence equating $E_a(s)$ equations,

$$K_g \, I_f(s) = I_2(s) \left[I_2(s)(R_a + R_2) + s(L_a + L_2)\right]$$

and hence using values of $I_f(s)$ and $I_2(s)$ from remaining equations we get,

$$K_g \times \frac{E_f(s)}{[R_f + s L_f]} = \frac{E_2(s)}{R_2} \times [(R_a + R_2) + d(L_a + L_2)]$$

$$\therefore \quad \boxed{\frac{E_2(s)}{E_f(s)} = \frac{K_g R_2}{(R_f + sL_f)[(R_a + R_2) + s(L_a + L_2)]}}$$

Example 4.9 : *A 50 Hz, 2 phase a.c. servomotor has the following parameters :*

Starting torque = 0.186 Nm

Rotor inertia = 1×10^{-5} *kg − m²*

Supply voltage = 120 V

No load angular velocity = 304 rad/s

Assuming straight line torque-speed characteristics of the motor and zero friction, obtain its transfer function. **(Gate)**

Solution : The starting torque is nothing but locked rotor torque.

$$\therefore \quad K_{tm} = \frac{\text{Locked rotor torque}}{\text{Rated voltage}} = \frac{0.186}{120}$$

$$= 1.55 \times 10^{-3}$$

Let m be the slope of linearised torque-speed characteristics.

$$\therefore \quad m = \frac{\text{Locked rotor torque}}{\text{No load angular speed}} = \frac{0.186}{304}$$

$$= -6.118 \times 10^{-4}$$

The torque at any angular speed ω is given by,

$$T_m = K_{tm} E_{2t} + m\frac{d\theta}{dt} \quad \text{... (1)}$$

where E_{2t} = rotor voltage

Taking Laplace, $T_m(s) = K_{tm} E_{2t}(s) + m s\theta(s)$ ... (2)

This torque is used to drive load of inertia J_m. The friction is given zero.

$$\therefore \quad T_m = J_m \frac{d^2\theta}{dt^2} \qquad \text{... (3)}$$

$$\therefore \quad T_m(s) = s^2 J_m \theta(s) \qquad \text{... (4)}$$

Equating (2) and (4),

$$K_{tm}E_{2t}(s) + m s\theta(s) = s^2 J_m \theta(s)$$

$$\therefore \quad K_{tm} E_{2t}(s) = s\theta(s)[sJ_m - m]$$

Hence the transfer function of the motor is,

$$\frac{\theta(s)}{E_{2t}(s)} = \frac{K_{tm}}{s[sJ_m - m]} = \frac{1.55\times10^{-3}}{s[1\times10^{-5}\, s-(-6.118\times10^{-4})]}$$

$$\therefore \quad \boxed{\frac{\theta(s)}{E_{2t}(s)} = \frac{155}{s(s+61.18)}}$$

This is the required transfer function.

Example 4.10 : *A two phase a.c. servomotor having a torque constant of 0.045 Nm/V controls a position load through a gear ratio of 10:1. The effective moment of inertia and coefficient of viscous friction referred to load side are 0.25 kg-m² and 1.0 N-m/(rad/sec). The synchro error detector produces an error signal of 0.1V per degree error in misalignment. Develop the block diagram representation of the control system and there from obtain the transfer function.* **(M.U. : May-99)**

Solution : For the error detector, K_1 = 0.1 V/degree error

$$\therefore \quad K_1 = 0.1\times\frac{180}{\pi}$$

$$= 5.7295 \text{ V/radian}$$

$$K_{tm} = \text{Torque constant}$$

$$= 0.045 \text{ Nm/V}$$

$$\frac{N_1}{N_2} = \frac{1}{10}$$

$$J_L = 0.25 \text{ kg - m}^2 \text{ on load side}$$

$$\therefore \quad J_{eq} = \text{Motor side} = \left(\frac{N_1}{N_2}\right)^2 \times J_L = \frac{1}{100}\times 0.25$$

rotational system is called **Inertia** and denoted by 'J' i.e. moment of inertia. Opposing torque due to inertia 'J' is proportional to the angular acceleration (α) of that inertia.

$$\therefore \quad T_{\text{due to inertia}} = J\frac{d^2\theta(t)}{dt^2} \qquad \text{where } \alpha = \frac{d^2\theta}{dt^2}$$

Taking Laplace,

$$T_{\text{due to inertia}}(s) = J\,s^2\,\theta(s)$$

Sr. No.	Translational Motion	Rotational Motion
1	Mass (M)	Inertia (J)
2	Friction (B)	Friction (B)
3	Spring (K)	Spring (K)
4	Force (F)	Torque (T)
5	Displacement (x)	Angular displacement (θ)
6	Velocity $v = \left(\frac{dx}{dt}\right)$	Angular velocity $\left(\omega = \frac{d\theta}{dt}\right)$
7	Acceleration $\left(\frac{d^2x}{dt^2}\right)$	Angular acceleration $\left(\alpha = \frac{d^2\theta}{dt^2}\right)$

Table 4.1 Analogous Elements

4.4 Equivalent Mechanical System (Node Basis)

While drawing analogous networks, it is always better to draw the equivalent mechanical system from the given mechanical system. To draw such system use following steps :

Step 1 : Due to applied force, identify the displacements in the mechanical system.

Step 2 : Identify the elements which are under the influence of different displacements.

Step 3 : Represent each displacement by a separate node, using Nodal Analysis.

Step 4 : Show all the elements in parallel under the respective nodes which are under the influence of respective displacements.

Step 5 : Elements causing same change in displacement will get connected in parallel in between the respective nodes.

$$= 2.5 \times 10^{-3} \text{ kg-}m^2$$

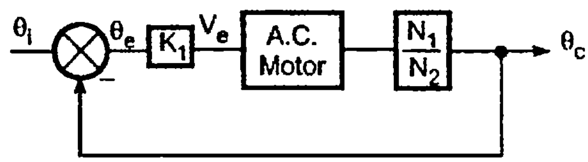
$$B_L = 1 \text{ N-m/(rad/sec)}$$

$$\therefore \quad B_{eq} = \text{Motor side} = \left(\frac{N_1}{N_2}\right)^2 \times B_l = \frac{1}{100} \times 1$$

$$= 0.01 \text{ N-m/ (rad/sec)}$$

The block diagram of the system is,

$$G(s) = K_l \times \text{motor T.F.} \times \frac{N_1}{N_2}$$



For motor, $T_m = K_{tm} E_2$

$$\therefore \quad T_m(s) = K_{tm} E_2(s) \quad \text{...(1)}$$

and $T_m(t) = J_{eq} \frac{d^2\theta}{dt^2} + B_{eq} \frac{d\theta}{dt}$

$$\therefore \quad T_m(s) = \left(s^2 J_{eq} + s B_{eq}\right)\theta(s) \quad \text{...(2)}$$

Equating equations (1) and (2),

$$K_{tm} E_2(s) = \left(s^2 J_{eq} + s B_{eq}\right)\theta(s)$$

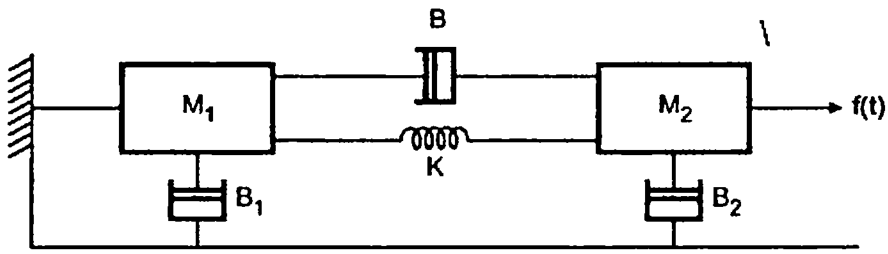
$$\therefore \quad \frac{\theta(s)}{E_2(s)} = \frac{K_{tm}}{s\left(s J_{eq} + B_{eq}\right)} = \frac{0.045}{s\left(s \times 2.5 \times 10^{-3} + 0.01\right)} = \frac{18}{s(s+4)}$$

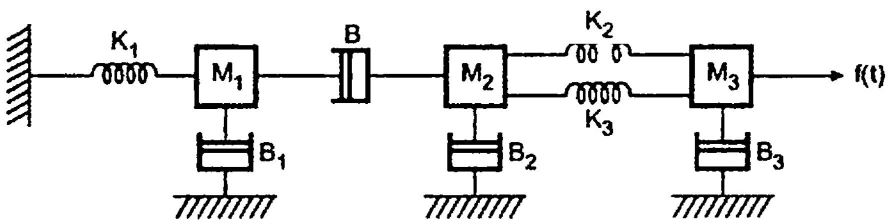
$$\therefore \quad G(s) = 5.7295 \times \frac{18}{s(s+4)} \times \frac{1}{10} = \frac{10.3131}{s(s+4)}$$

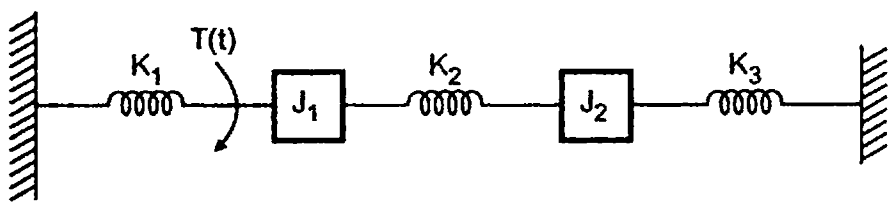
$$\therefore \quad \boxed{\frac{\theta_c(s)}{\theta_i(s)} = \frac{\frac{10.3131}{s(s+4)}}{1 + \frac{10.3131}{s(s+4)}} = \frac{10.3131}{s^2 + 4s + 10.3131}}$$

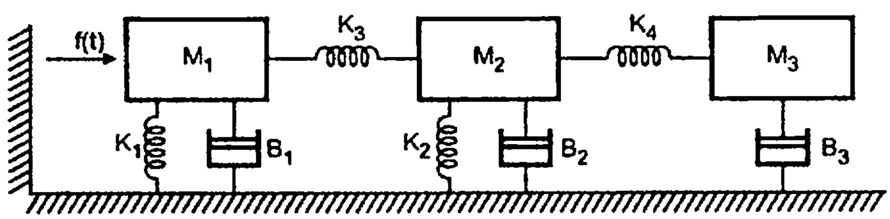
Review Questions

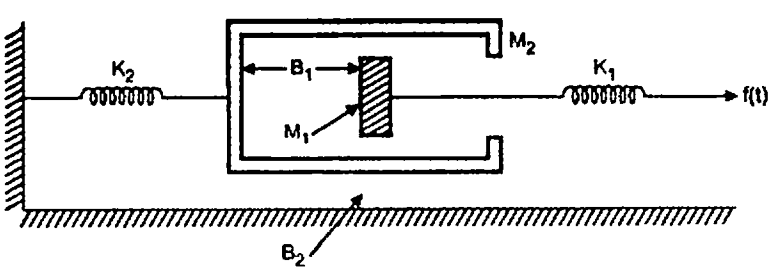
1. *Explain the derivation of analogous networks using*
   i) Force-voltage *ii) Force-current analogy*
2. *Write a short note on direct and inverse analogous networks.*
3. *Draw the analogous electrical networks based on*
   a) F - V analogy *b) F - I analogy of the following mechanical systems*











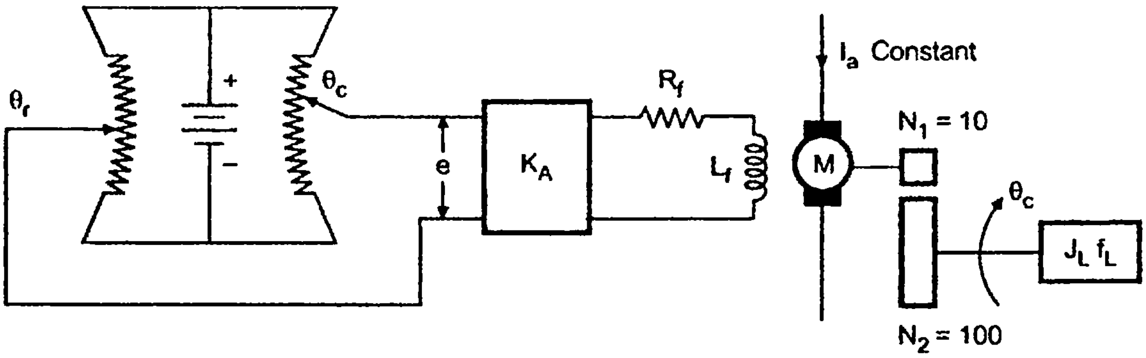
4. *Distinguish between A.C. servomotor and D.C. servomotor.*
5. *Derive transfer function of a.c. servomotor stating the assumption made.*
6. *State the applications of a.c. servomotor.*
7. *Derive the transfer function of field controlled d.c. servomotor.*
8. *Answer the following giving reasons :*

   a) A.C. servomotor has a smaller diameter and more length.

   b) Field controlled D.C. servomotor is preferred than armature controlled.
9. *Develop block diagram for armature controlled D.C. servomotor and find its transfer function.*
10. *An armature controlled d.c. motor is supplied in series with a resistance from a 24 V d.c. supply. The motor takes current of 5A on stalling and the stalling torque being 0.915 N-m. The motor runs at 1000 r.p.m. taking a current of 1 A. The value of $J = 4 \times 10^{-3}$ Kg - m^2 and friction constant as 1.5×10^{-3} Nm/ (rad/sec)*

    Determine the transfer function of the motor. **(Ans. : $\dfrac{\theta(s)}{V(s)} = \dfrac{2.9}{s(1+0.3\,s)}$)**
11. *For the closed loop system shown below, draw the block diagram and determine transfer function $\theta_c(s) / \theta_r(s)$. The given values are Error detector gain K_e = 8 V/rad, Amplifier gain K_A = 10 V/A R_f = 5 Ω, L_f = 0.25 H, K_f = 0.05 Nm/A, J_{motor} = 0.02 Kg – m^2, J_L = 3 Kg - m^2. f_{motor} = 0.03 Nm/(rad /sec).*

    f_L = 5.5 Nm/ (rad/sec)

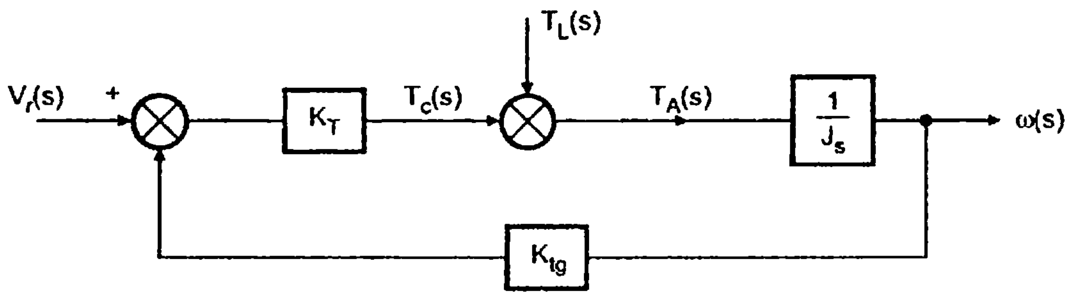


(Ans. : $\dfrac{\theta_c(s)}{\theta_r(s)} = \dfrac{32.12}{(s^3+21.5s^2+30.35+32.12)}$)

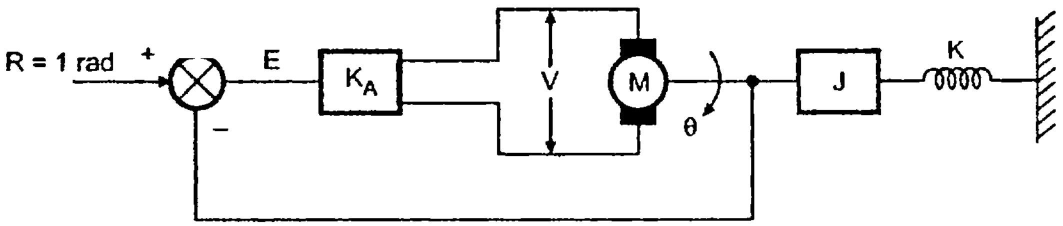
12. *The moment of inertia J_m and the coefficient of viscous friction f_m for a field controlled d.c. motor are motor respectively 5×10^{-4} kg - m^2 and 12.5×10^{-4}Nm (rad/sec). The motor torque constant K_f being 2.5 Nm/A. Determine the transfer function relating the angular speed of the shaft and the field current.* **(Ans. : $\dfrac{\omega(s)}{I_f(s)} = \dfrac{2000}{1+0.4s}$)**

13. *Figure given below shows a block diagram of speed regulator system. The accelerating torque is the difference between the torque developed by the controller and load* T_L. *The controller gain is 0.00102 Nm/rad and the tachogenerator constant is 0.191 V/rad/sec. The load speed is adjusted to 1000 r.p.m. The moment of inertia is J and friction is negligible.*

Calculate a) The reference voltage. b) The speed if a constant load torque of 0.001 Nm is suddenly applied. **(Ans. : a) 20 V b) 951 r.p.m.)**



14. *An instrument servo consisting of motor, spring loaded shaft, etc is shown below*



Where V = *Voltage in volts*

R = *Motor resistance 1 Ω*

L = *Motor inductance 0.1 H*

K_A = *Amplifier gain = 10 V/V*

K = *Spring constant = 0.001 Nm/rad*

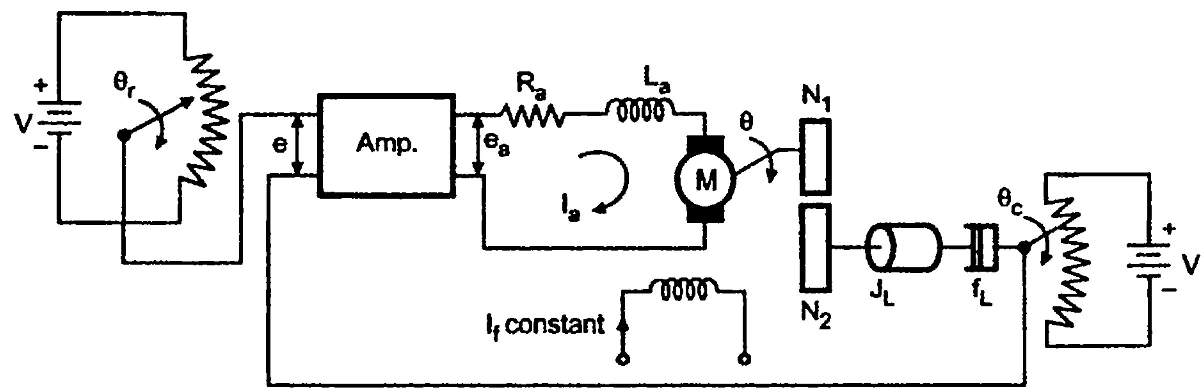
K_b = *Back e.m.f. constant = 0.01 V (rad/sec)*

K_T = *Torque content = 0.01 N-m/A*

J = *Moment of inertia = 0.005 N-m-s²*

If the input is 1 radians, what is the steady state error? **(Ans. :** $e_{ss} = \dfrac{1}{\left[1 + \dfrac{K_T\, K_A}{(K \times R)}\right]} = \dfrac{1}{101}$**)**

15. *The positions servomechanism is shown in figure below.* θ_r *is reference position and* θ_c *actual angular displacement of shaft in radians. Obtain its closed loop transfer function* $\frac{\theta_c(s)}{\theta_r(s)}$



Given

θ = *angular displacement of the motor shaft*

K_f = *gain f error detector = 7.64* V/rad

R_a = 2 Ω, L_a = *negligible*

K_b = 5.5×10^{-2} V/rad/sec

K_T = 6×10^{-5} *Nm/A*

J_m = 1×10^{-5} Kg - m^2

f_m = *negligible*

J_L = 4.4×10^{-3} Kg - m^2

f_L = 4×10^{-2} Nm/ (rad/sec)

N_1 = *1*

N_2 = *10*

(**Ans.:** $\frac{\theta_c(s)}{\theta_r(s)} = \frac{42.3}{s^2 + 7.7\, s + 42.3}$)

16. *Derive the transfer function of a typical d.c. position control system.*
17. *Derive the transfer function of a typical d.c. speed control system.*
18. *Obtain the mathematical model of heat transfer system.*
19. *Obtain the mathematical model of thermometer.*
20. *What is actuator ? Explain hydraulic and pneumatic actuator systems.*
21. *Compare hydraulic and pneumatic actuators.*

4.5 Remarks on Nodal Method

a) The terms for an element connected to a node 'X' and stationary surface (reference) is,

For mass $\rightarrow M \dfrac{d^2x}{dt^2}$

For friction $\rightarrow B \dfrac{dx}{dt}$

For spring $\rightarrow Kx$

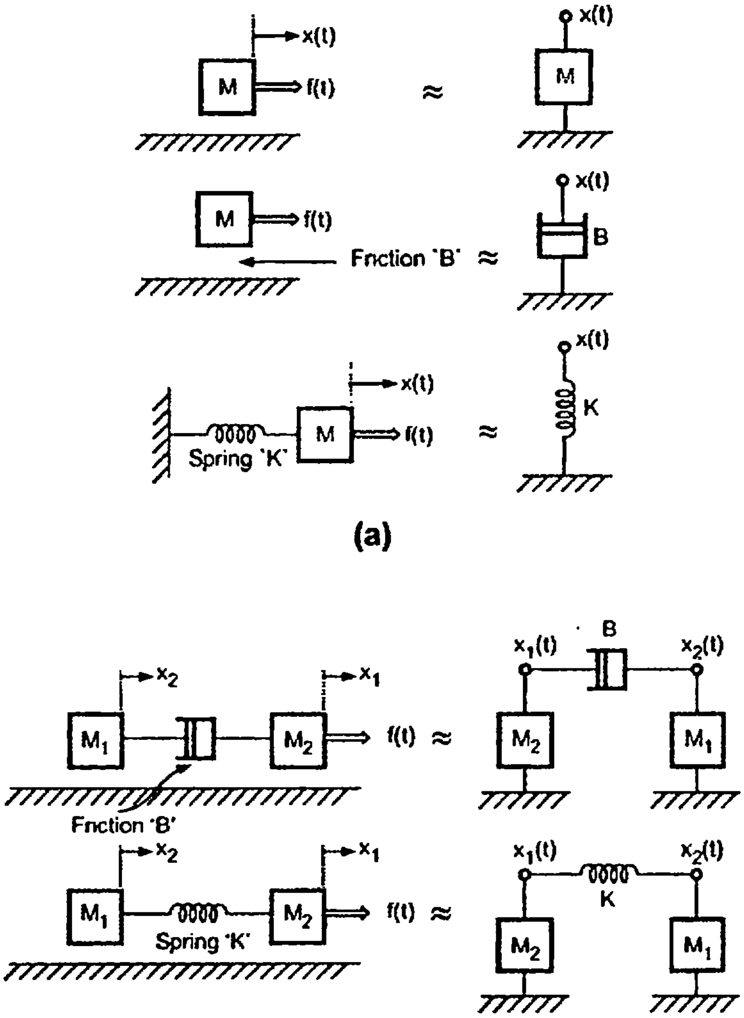
b) The term for an element connected between the two nodes 'X_1' and 'X_2' i.e. between two moving surfaces is,

For friction $\rightarrow B\left[\dfrac{dx_1}{dt} - \dfrac{dx_2}{dt}\right]$

For spring $\rightarrow K\,[\,x_1 - x_2]$

No mass can be between the two nodes as due to mass there cannot be change in force as mass cannot store potential energy.

c) All elements which are under the influence of same displacement get connected in parallel under that node indicating the corresponding displacement.

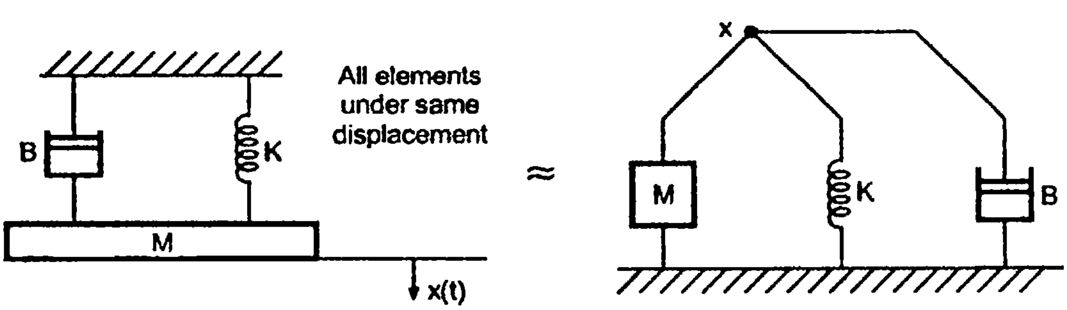


(a)

(b)

Fig. 4.9

e.g. consider the part of the system, shown in the Fig. 4.10(a).



(a) (b)

Fig. 4.10

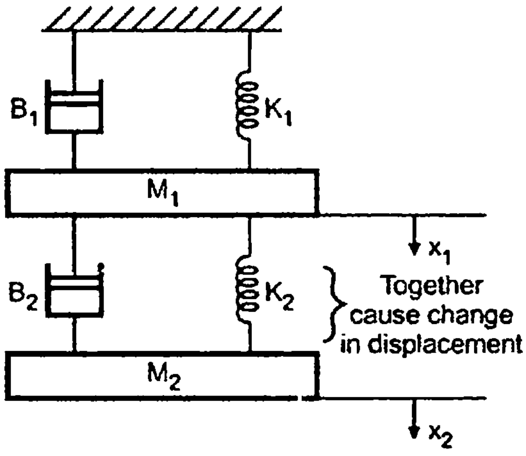


Fig. 4.11

Here M, B and K all are under the influence of x(t). Hence in equivalent system all of them will get connected in parallel under the node `X'. Consider another example of the system shown in the Fig. 4.11. In this system M_1 , B_1 and K_1 all are under the influence of displacement X_1. This is because all are connected to rigid support.

While there is change from X_1 to X_2 due to simultaneous effect of B_2 and K_2. So B_2 and K_2 are under the influence of $(X_1 - X_2)$. But mass M_2 is under the influence of X_2 alone. **Mass cannot be under the influence of difference between displacements.** So in equivalent system the elements B_1 , K_1 and M_1, all in parallel under X_1 while B_2 and K_2 in parallel between X_1 and X_2 and element M_2 is under node X_2 as shown in the Fig. 4.12 .

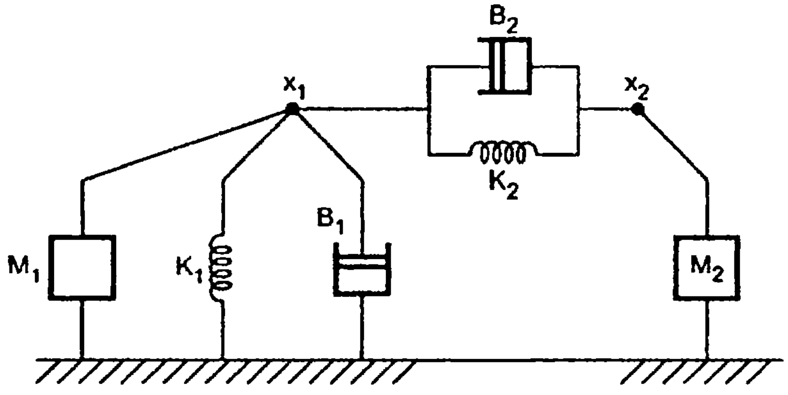


Fig. 4.12

4.6 Gear Trains

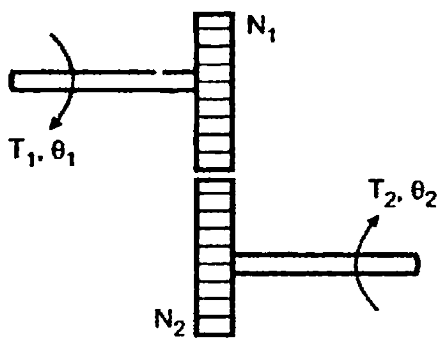


Fig. 4.13 Gear system

A gear train is a mechanical device that transmits energy from one part of a system to another in such a way that force, torque, speed and displacement may be altered. The inertia and friction of the gears are neglected in the ideal case. Consider a gear system as shown in the Fig. 4.13.

The number of teeth on the surface of the gears is proportional to the radii r_1 and r_2 of the gears.

i.e. $$r_1 N_2 = r_2 N_1$$

The distance travelled along the surface of each gear is same.

i.e. $$\theta_1 r_1 = \theta_2 r_2$$

The work done by one gear is same as the other.

i.e. $T_1 \theta_1 = T_2 \theta_2$

$\therefore$ we can say $$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Remarks :

1) The numbers of teeth N are proportional to the radius r of a gear.
2) The distance travelled on each gear is same.
3) Work done = Tθ by each gear is same.

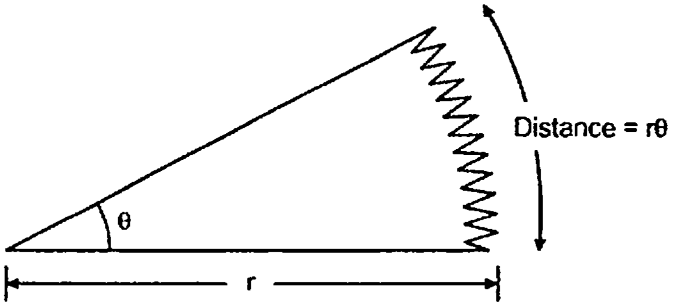


Fig. 4.14

4.6.1. Gear Train with Inertia and Friction

In practice, gears do have inertia and friction which can not be neglected. Consider such practical gear arrangement connected to the load, shown below.

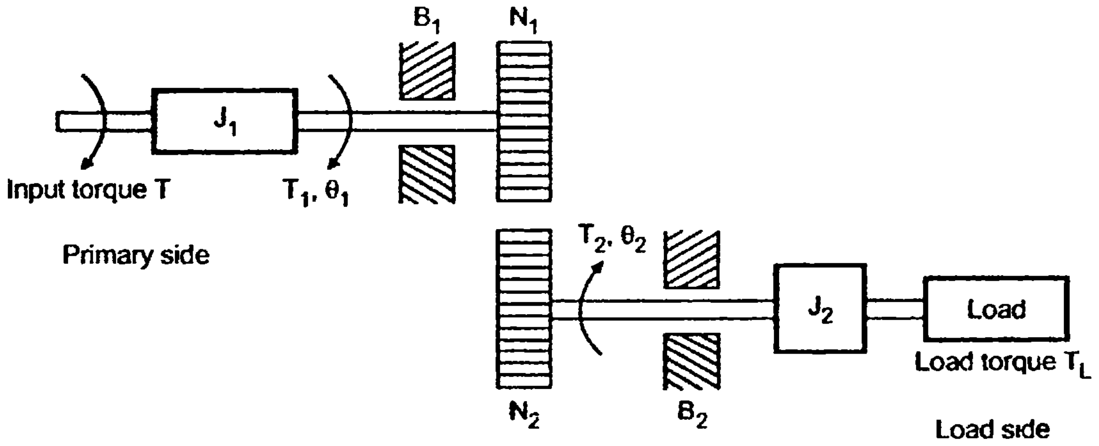


Fig. 4.15

T = Applied torque

θ_1, θ_2 = Angular displacements

T_1, T_2 = Torque transmitted to gears

J_1, J_2 = Inertia of gears.

N_1, N_2 = Number of teeth

B_1, B_2 = Friction coefficients

The distance travelled along the surface of each gear is same.

i.e. $$\theta_1 r_1 = \theta_2 r_2$$

The work done by one gear is same as the other.

i.e. $$T_1 \theta_1 = T_2 \theta_2$$

$\therefore$ we can say $$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Remarks :

1) The numbers of teeth N are proportional to the radius r of a gear.
2) The distance travelled on each gear is same.
3) Work done = $T\theta$ by each gear is same.

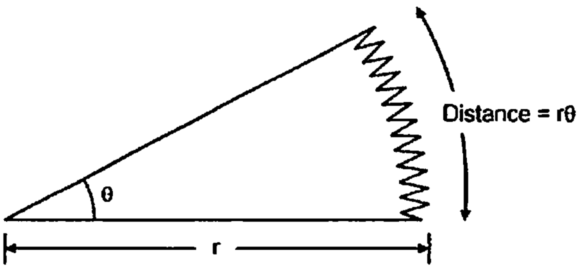


Fig. 4.14

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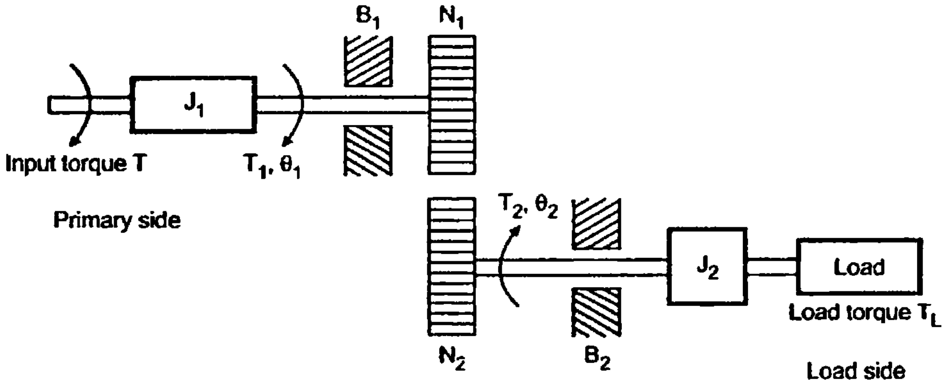


Fig. 4.15

T = Applied torque

T_1 , T_2 = Torque transmitted to gears

N_1 , N_2 = Number of teeth

θ_1 , θ_2 = Angular displacements

J_1 , J_2 = Inertia of gears.

B_1 , B_2 = Friction coefficients

Torque equation of side 1 is,

$$T = J_1 \frac{d^2 \theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + T_1(t) \quad \text{... (1)}$$

Torque equation of side 2 is,

$$T_2 = J_2 \frac{d^2 \theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + T_L(t) \quad \text{... (2)}$$

Now $\dfrac{T_1}{T_2} = \dfrac{N_1}{N_2} = \dfrac{\theta_2}{\theta_1}$

$\therefore \quad T_2 = \dfrac{N_2}{N_1} T_1$

Substituting in equation (2)

$$\therefore \quad \frac{N_2}{N_1} T_1 = J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_L$$

$$\therefore \quad T_1 = \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L \quad \text{... (3)}$$

Substituting value of T_1 in equation (1)

$$T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt_2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L$$

Substituting $\theta_2 = \dfrac{N_1}{N_2} \theta_1$

$$\therefore T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{N_1}{N_2} \frac{d^2\theta_1}{dt_2} + \frac{N_1}{N_2} \cdot B_2 \frac{N_1}{N_2} \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$$\therefore T = \left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right] \frac{d^2\theta_1}{dt^2} + \left[B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\right] \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$\therefore \quad J_{1e}$ = Equivalent inertia referred to primary side

$$\therefore \quad \boxed{J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2}$$

and $\quad B_{1e}$ = Equivalent friction referred to primary side

$$\therefore \quad \boxed{B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2}$$

$$\therefore \quad \boxed{T = J_{1e} \frac{d^2\theta_1}{dt_2} + B_{1e} \frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right) T_L}$$

Similarly the equation can be written referred to load side also, where applied torque gets transferred to load as $\left(\frac{N_2}{N_1} T\right)$.

$$\left(\frac{N_2}{N_1}\right) T = J_{2e} \frac{d^2 \theta_2}{dt^2} + B_{2e} \frac{d\theta_2}{dt} + T_L$$

$$\text{where } J_{2e} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1 \text{ and } B_{2e} = B_2 + \left(\frac{N_2}{N_1}\right)^2 B_1$$

4.6.2. Belt or Chain Drives

Belt and chain drives perform same function as that of gear train. Assuming that there is no slippage between belt and pulleys we can write,

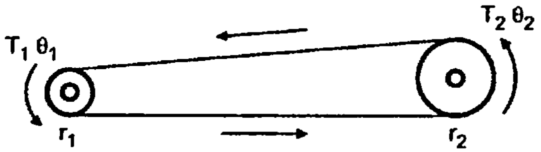


Fig. 4.16

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$$ for such drive

4.6.3 Levers

The lever system is shown in the Fig. 4.17. This transmits translational motion and forces, similar to gear trains.

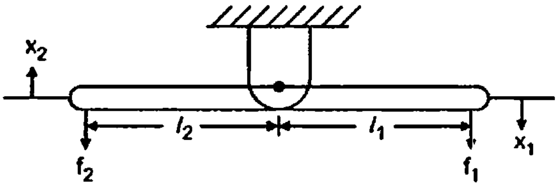


Fig. 4.17

By law of moment,

$$f_1 l_1 = f_2 l_2$$

By work done $f_1 x_1 = f_2 x_2$

Hence $$\frac{f_1}{f_2} = \frac{l_2}{l_1} = \frac{x_2}{x_1}$$

4.7 Electrical Systems

Similar to the mechanical systems, very commonly used systems are of electrical type. The behaviour of such systems is governed by Ohm's law. The dominant elements of an electrical system are,

i) Resistor ii) Inductor iii) Capacitor

i) Resistor : Consider a resistance carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = IR$$

Suppose it carries a current $(I_1 - I_2)$ then for the polarity of the voltage drop shown its equation is,

$$V = (I_1 - I_2)\, R$$

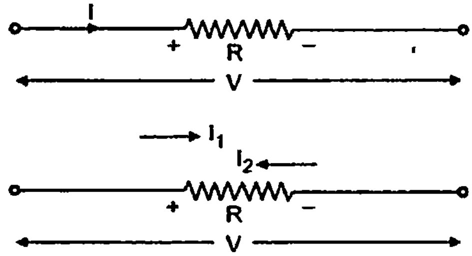


Fig. 4.18

ii) Inductor : Consider an inductor carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = L\frac{dI}{dt} \quad \text{or} \quad I = \frac{1}{L}\int V\, dt$$

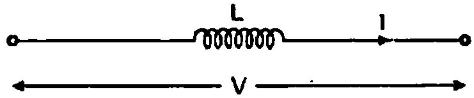


Fig. 4.19

If it carries a current $(I_1 - I_2)$ then for the polarity shown its voltage equation is,

$$V = L\frac{d(I_1 - I_2)}{dt}$$

or

$$(I_1 - I_2) = \frac{1}{L}\int V\, dt$$

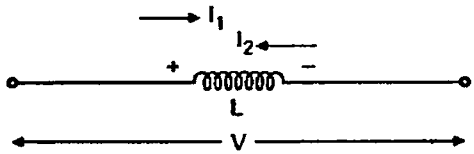


Fig. 4.20

iii) Capacitor : Consider a capacitor carrying current 'I' as shown, then the voltage drop across it can be written as,

$$V = \frac{1}{C}\int I\, dt$$

or

$$I = C\frac{dV}{dt}$$

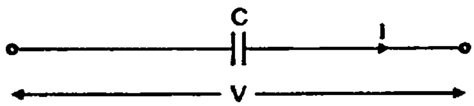


Fig. 4.21

If it carries a current $(I_1 - I_2)$ then for the polarity shown its voltage equation is,

$$V = \frac{1}{C}\int (I_1 - I_2)\, dt$$

or

$$(I_1 - I_2) = C\frac{dV}{dt}$$

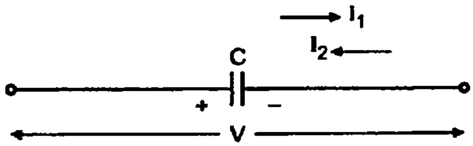


Fig. 4.22

4.8 Analogous Systems

In between electrical and mechanical systems there exists a fixed analogy and their exists a similarity between their equilibrium equations. Due to this, it is possible to draw an electrical system which will behave exactly similar to the given mechanical system, this is called electrical analogous of given mechanical system and vice versa. It is always advantageous to obtain **electrical analogous** of the given mechanical system as we are well familiar with the methods of analysing electrical network than mechanical systems.

There are two methods of obtaining electrical analogous networks, namely

1) Force - Voltage Analogy i.e. Direct Analogy.

2) Force - Current Analogy i.e. Inverse Analogy.

4.8.1 Mechanical System

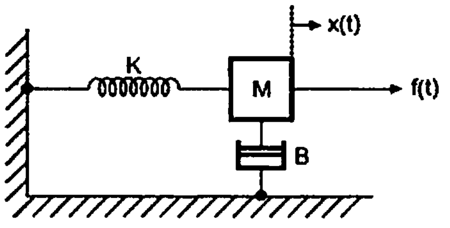


Fig. 4.23

Consider simple mechanical system as shown in the Fig. 4.23.

Due to the applied force, mass M will displace by an amount x(t) in the direction of the force f(t) as shown in the Fig. 4.23.

According to Newton's law of motion, applied force will cause displacement x(t) in spring, acceleration to mass M against frictional force having constant B

$$\therefore \quad f(t) = Ma + Bv + K\,x(t)$$

Where, a = Acceleration, v = Velocity

$$\therefore \quad f(t) = M\frac{d^2\,x(t)}{dt^2} + B\frac{d\,x(t)}{dt} + K\,x(t)$$

Taking Laplace,

$$F(s) = Ms^2\,X(s) + Bs\,X(s) + K\,X(s)$$

This is equilibrium equation for the given system.

Now we will try to derive analogous electrical network.

4.8.2 Force Voltage Analogy (Loop Analysis)

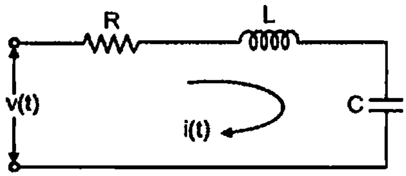


Fig. 4.24

In this method, to the force in mechanical system, voltage is assumed to be analogous one. Accordingly we will try to derive other analogous terms. Consider electric network as shown in the Fig. 4.24.

The equation according to Kirchhoff's law can be written as,

$$v(t) = i(t)\,R + L\,\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

Taking Laplace,

$$V(s) = I(s)\,R + Ls\,I(s) + \frac{I(s)}{sC}$$

But we cannot compare F(s) and V(s) unless we bring them into same form.

For this we will use current as rate of flow of charge.

$$\therefore \quad i(t) = \frac{dq}{dt}$$

i.e. $$I(s) = s\,Q(s) \quad \text{or} \quad Q(s) = \frac{I(s)}{s}$$

Replacing in above equation,

$$\boxed{V(s) = L\,s^2Q(s) + Rs\,Q(s) + \frac{1}{C}Q(s)}$$

Comparing equations for F(s) and V(s) it is clear that,

i) Inductance `L' is analogous to mass M

ii) Resistance `R' is analogous to friction B.

iii) Reciprocal of capacitor i.e. 1/C is analogous to spring of constant K.

Translational	Rotational	Electrical
Force	Torque T	Voltage V
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant K Nm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x} = \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Current $i = \frac{dq}{dt}$

Table 4.2 Tabular form of force-voltage analogy

4.8.3 Force Current Analogy (Node Analysis)

In this method, current is treated as analogous quantity to force in the mechanical system. So force shown is replaced by a current source in the system shown in the Fig. 4.25.

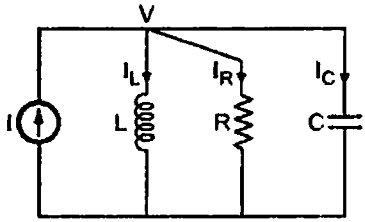


Fig. 4.25

The equation according to Kirchhoff's current law for above system is,

$$I = I_L + I_R + I_C$$

Let node voltage be V,

$$\therefore \quad I = \frac{1}{L}\int V\,dt + \frac{V}{R} + C\frac{dV}{dt}$$

Taking Laplace,

$$I(s) = \frac{V(s)}{sL} + \frac{V(s)}{R} + sC\,V(s)$$

But to get this equation in the similar form as that of F(s) we will use,

$$v(t) = \frac{d\phi}{dt} \qquad \text{where } \phi = \text{flux}$$

$$\therefore \quad V(s) = s\,\phi(s) \qquad \text{i.e. } \phi(s) = \frac{V(s)}{s}$$

Substituting in equation for I(s)

$$\therefore \quad \boxed{I(s) = Cs^2\,\phi(s) + \frac{1}{R}\,s\,\phi(s) + \frac{1}{L}\,\phi(s)}$$

Comparing equations for F(s) and I(s) it is clear that,

i) Capacitor `C' is analogous to mass M.

ii) Reciprocal of resistance $\frac{1}{R}$ is analogous to frictional constant B.

iii) Reciprocal of inductance $\frac{1}{L}$ is analogous to spring constant K.

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'e' $= \frac{d\phi}{dt}$

Table 4.3 Tabular form of force-current analogy

Key Point : *The elements which are in series in F - V analogy, get connected in parallel in F – I analogous network and which are in parallel in F - V analogy, get connected in series in F – I analogous network.*

4.9 Steps to Solve Problems on Analogous Systems

Step 1 : Identify all the displacements due to the applied force. The elements spring and friction between two moving surfaces cause change in displacement.

Step 2 : Draw the equivalent mechanical system based on node basis. The elements under same displacement will get connected in parallel under that node. Each displacement is represented by separate node. Element causing change in displacement (either friction or spring) is always between the two nodes.

Step 3 : Write the equilibrium equations. At each node algebraic sum of all the forces acting at the node is zero.

Step 4 : In F-V analogy, use following replacements and rewrite equations,

$$F \to V, \quad M \to L, \quad B \to R, \quad K \to 1/C, \quad x \to q, \quad \dot{x} \to i \text{ (current)}$$

Step 5 : Simulate the equations using loop method. Number of displacements equal to number of loop currents.

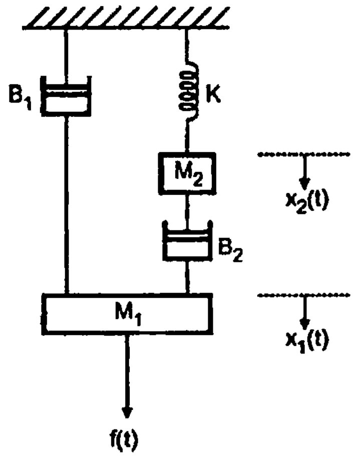
Step 6 : In F-I analogy, use following replacements and rewrite equations,

$$F \to I, \quad M \to C, \quad B \to 1/R, \quad K \to 1/L, \quad x \to \phi, \quad \dot{x} = e(\text{e.m.f.})$$

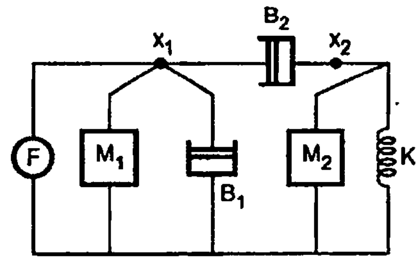
Step 7 : Simulate the equations using node basis. Number of displacements equal to number of node voltages. Infact the system will be exactly same as equivalent mechanical system obtained in step 2 with appropriate replacements.

➠ **Example 4.1 :** *Draw the equivalent mechanical system of the given system. Hence write the set of equilibrium equations for it and obtain electrical analogous circuits using,*

i) F-V Analogy and ii) F-I Analogy



Solution : The displacement of M_1 is $x_1(t)$ and as B_1 is between M_1 and fixed support hence it is also under the influence of $x_1(t)$. While B_2 changes the displacement from $x_1(t)$ to $x_2(t)$ as it is between two moving points. And M_2 and K are under the displacement $x_2(t)$ as K is between mass and fixed support.



Equivalent system

$$\Sigma F = 0$$

At node 1, $$F = M_1 s^2 X_1 + B_1 s X_1 + B_2 s(X_1 - X_2) \quad \text{... (1)}$$

At node 2, $$0 = M_2 s^2 X_2 + K X_2 + B_2 s(X_2 - X_1) \quad \text{... (2)}$$

Now (i) F - V Analogy $M \rightarrow L \quad B \rightarrow R \quad K \rightarrow 1/C \quad x \rightarrow q$

$$\therefore \quad V(s) = L_1 s^2 q_1 + R_1 s q_1 + R_2 s(q_1 - q_2) \quad \text{... (3)}$$

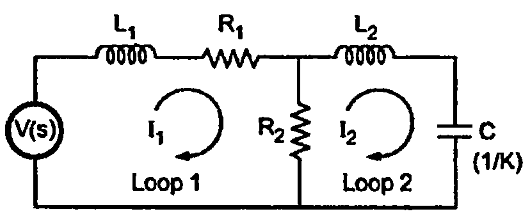
$$0 = L_2 s^2 q_2 + (1/C) q_2 + R_2 s(q_2 - q_1) \quad \text{... (4)}$$

Replacing $$\frac{I(s)}{s} = Q(s) \text{ i.e. } \quad I(s) = s\,Q(s)$$

$$V(s) = L_1 s I_1(s) + R_1 I_1(s) + R_2 [I_1(s) - I_2(s)] \rightarrow \text{Loop 1}$$

$$0 = L_2 s I_2(s) + \frac{1}{sC} I_2(s) + R_2 [I_2(s) - I_1(s)] \rightarrow \text{Loop 2}$$

Hence,



Number of loop currents equal to number of displacements.

(II) F - I Analogy

$F \rightarrow I \quad M \rightarrow C \quad B \rightarrow 1/R \quad K \rightarrow 1/L \quad x \rightarrow \phi$

$$I(s) = C_1 s^2 \phi_1 + \frac{1}{R_1} s\,\phi_1 + \frac{1}{R_2} s\,(\phi_1 - \phi_2) \quad \text{... (1)}$$

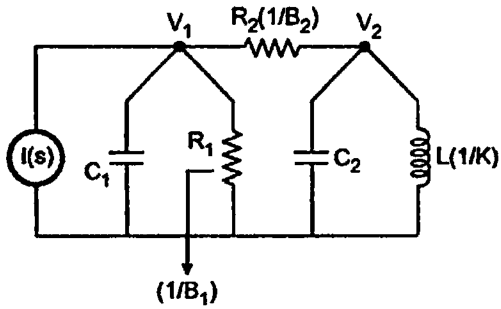
$$0 = \frac{1}{R_2} s(\phi_2 - \phi_1) + C_2 s^2 \phi_2 + \frac{1}{L} \cdot \phi_2 \quad \text{... (2)}$$

Replacing $$s\,\phi(s) = V(s)$$

$$I(s) = C_1 s V_1(s) + \frac{V_1(s)}{R_1} + \frac{1}{R_2}[V_1(s) - V_2(s)] \quad \text{...Node 1}$$

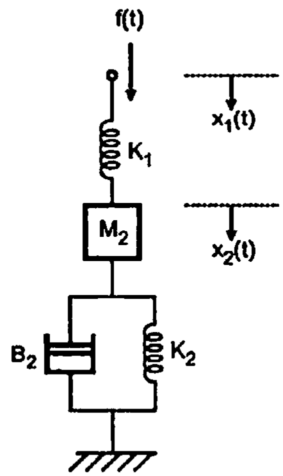
$$0 = \frac{1}{R_2}[V_2(s) - V_1(s)] + C_2 s V_2(s) + \frac{1}{sL} V_2(s) \quad \text{...Node 2}$$

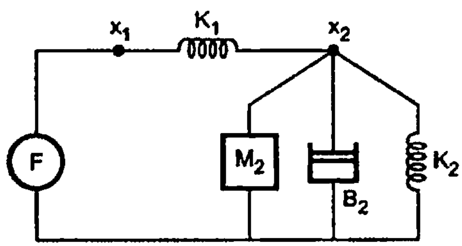
Hence,



Number of node voltages equal to number of displacements.

Example 4.2 : *Draw the equivalent mechanical system and analogous systems based on F-V and F-I methods for the given system.*





Equivalent system

Solution : **Two displacements :** No element under $x_1(t)$ alone as force is directly applied to a spring K_1. So it will store energy and hence is the cause to change the force applied to M_2. Hence displacement of M_2 is x_2 and as B_2 and K_2 are connected to fixed supports both are under $x_2(t)$ only as shown in the equivalent system.

At node 1, $F = K_1(X_1 - X_2)$ ... (1)

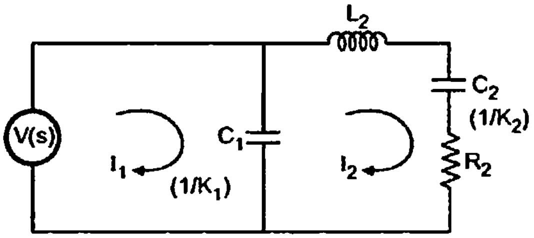
At node 2, $0 = K_1(X_2 - X_1) + M_2 s^2 X_2 + K_2 X_2 + B_2 s X_2$ ... (2)

M_2, B_2, K_2 are under same displacement.

(i) F-V analogy : M → L B → R K → 1/C

$$V = \frac{1}{C_1}(q_1 - q_2) \quad \text{... (3) Loop (1)}$$

$$0 = \frac{1}{C_1}(q_2 - q_1) + L_2 s^2 q_2 + \frac{1}{C_2} q_2 + R_2 s q_2 \quad \text{... (4) Loop (2)}$$



Same displacement same current.

Equations in terms of I_1 and I_2 can be written by using $i(t) = \frac{dq}{dt}$ i.e. $I(s) = s\,Q(s)$ as explained earlier.

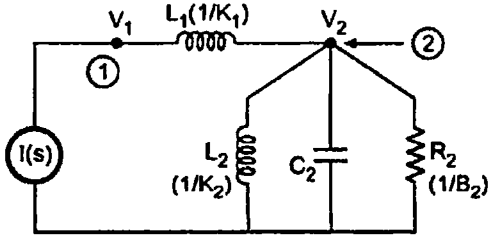
(ii) F - I analogy : M → C , B → 1/R, K → 1/L

$$\therefore \quad I(s) = \frac{1}{L_1}(\phi_1 - \phi_2) \quad \text{... (1) Node (1)}$$

$$0 = \frac{1}{L_1}(\phi_2 - \phi_1) + C_2 s^2 \phi_2 + \frac{1}{R_2} s \phi_2 + \frac{1}{L_2} \phi_2 \quad \text{... (2) Node (1)}$$

Equations in terms of $V_1(s)$ and $V_2(s)$ can be obtained by using the relation,

$$v(t) = \frac{d\phi}{dt} \quad \text{i.e.} \quad V(s) = s\,\phi(s) \quad \text{as explained earlier.}$$



Same displacement-same voltage.

4.10 Servomotors

The servosystem is one in which the output is some mechanical variable like position, velocity or acceleration. Such systems are generally automatic control systems which work on the error signals. The error signals are amplified to drive the motors used in such systems. These motors used in servosystems are called servomotors. These motors are usually coupled to the output shaft i.e. load through gear train for power matching.

These motors are used to convert electrical signal applied, into the angular velocity or movement of shaft.

4.10.1 Requirements of Good Servomotor

The servomotors which are designed for use in feedback control systems must have following requirements :

i) Linear relationship between electrical control signal and the rotor speed over a wide range.

ii) Inertia of rotor should be as low as possible. A servomotor must stop running without any time delay, if control signal to it is removed. For low inertia, it is designed with large length to diameter ratio, for rotors. Compared to its frame size, the rotor of a servomotor has very small diameter.

iii) Its response should be as fast as possible. For quickly changing error signals, it must react with good response. This is achieved by keeping torque to weight ratio high.

iv) It should be easily reversible.

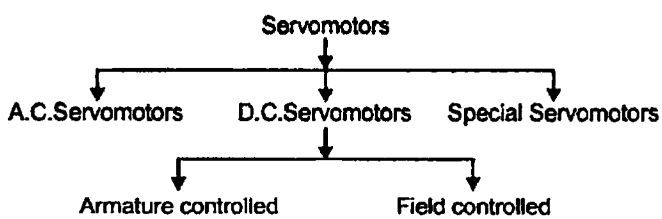
v) It should have linear torque - speed characteristics.

vi) Its operation should be stable without any oscillations or overshoots.

4.11 Types of Servomotors

The servomotors are basically classified depending upon the nature of the electric supply to be used for its operation.

The types of servomotors are as shown in the following chart :



4.12 D.C. Servomotor

Basically d.c. servomotor is more or less same as normal d.c.motor. There are some minor differences between the two. All d.c. servomotors are essentially separately excited type. This ensures linear torque-speed characteristics.

The control of d.c. servomotor can be from field side or from armature side. Depending upon this, these are classified as field controlled d.c. servomotor and armature controlled d.c. servomotor.

4.12.1 Field Controlled D.C. Servomotor

In this motor, the controlled signal obtained from the servoamplifier is applied to the field winding. With the help of constant current source, the armature current is maintained constant. The arrangement is shown in the Fig. 4.26.

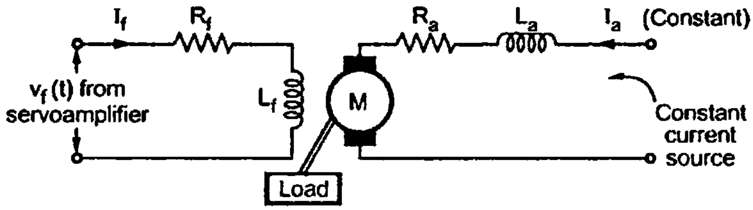


Fig. 4.26 Field controlled d.c. servomotor

This type of motor has large L_f / R_f ratio where L_f is reactance and R_f is resistance of field winding. Due to this the time constant of the motor is high. This means it can not give rapid response to the quick changing control signals hence this is uncommon in practice.

4.12.1.1 Features of Field Controlled D.C. Servomotor

It has following features :

i) Preferred for small rated motors.

ii) It has large time constant.

iii) It is open loop system. This means any change in output has no effect on the input.

iv) Control circuit is simple to design.

4.12.2 Armature Controlled D.C. Servomotor

In this type of motor, the input voltage 'V_a' is applied to the armature with a resistance of R_a and inductance L_a. The field winding is supplied with constant current I_f. Thus armature input voltage controls the motor shaft output. The arrangement is shown in the Fig. 4.27.

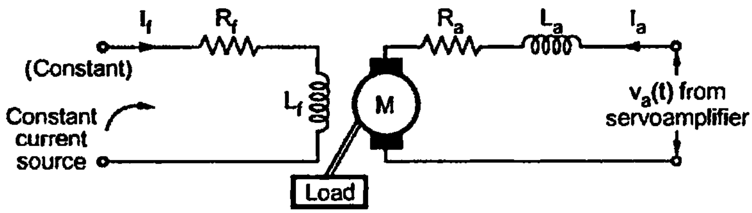


Fig. 4.27 Armature controlled d.c. servomotor

The constant field can be supplied with the help of parmanent magnets. In such case no field coils are necessary.

4.12.2.1 Features of Armature Controlled D.C. Servomotor

It has following features :

i) Suitable for large rated motors.

ii) It has small time constant hence its response is fast to the control signal.

iii) It is closed loop system.

iv) The back e.m.f. provides internal damping which makes motor operation more stable.

v) The efficiency and overall performance is better than field controlled motor.

As the armature controlled d.c. servomotor is closed loop system, in comparison with open loop field controlled system, generally armature controlled motors are used.

4.12.3 Characteristics of D.C. Servomotors

The characteristics of d.c. servomotors are mainly similar to the torque-speed characteristics of a.c. servomotor. The characteristics are shown in the Fig. 4.28.

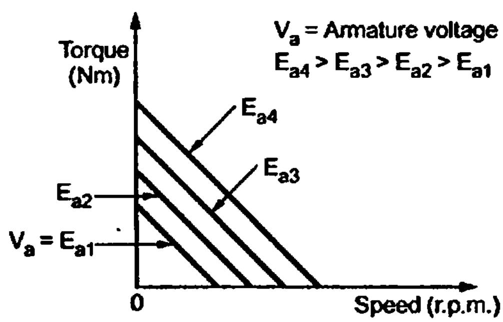


Fig. 4.28 Torque-speed characteristics for an armature controlled d.c. servomotor

4.12.4 Applications of D.C. Servomotor

These are widely used in air craft control systems, electromechanical actuators, process controllers, robotics, machine tools etc.

4.13 Transfer Function of Field Controlled D.C. Motor

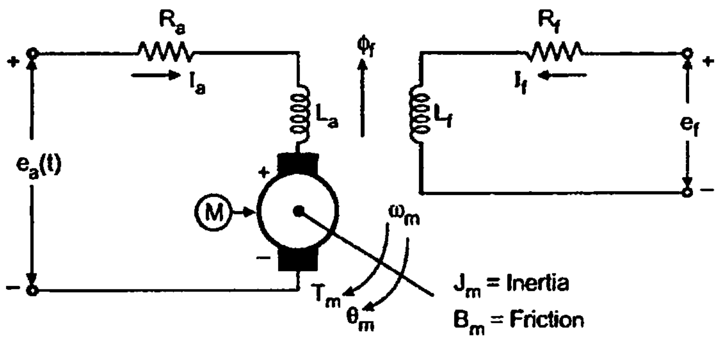


Fig. 4.29

Assumptions :

(1) Constant armature current is fed into the motor.

(2) $\phi_f \propto I_f$. Flux produced is proportional to field current.

$$\therefore \qquad \boxed{\phi_f = K_f I_f}$$

(3) Torque is proportional to product of flux and armature current.

$$\boxed{T_m \propto \phi I_a}$$

$$\therefore \qquad T_m = K' \phi I_a = K' K_f I_f I_a$$

$$\boxed{T_m = K_m K_f I_f} \qquad \text{... (1)}$$

Where $K_m = K' I_a$ = Constant

Apply Kirchhoff's law to field circuit.

$$L_f \frac{di_f}{dt} + R_f I_f = e_f \qquad \text{... (2)}$$

Now shaft torque T_m is used for driving load against the inertia and frictional torque.

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} \qquad \text{... (3)}$$

$$\text{Inertia force} = J_m \frac{d^2\theta_m}{dt^2} \text{ similar to } m \frac{d^2x}{dt^2}$$

$$\text{Frictional force} = B_m \frac{d\theta_m}{dt} \text{ similar to } B\frac{dx}{dt}$$

Finding Laplace Transforms of equations (1), (2) and (3) we get,

$$T_m(s) = K_m K_f I_f(s) \quad \text{... (4)}$$

$$E_f(s) = (sL_f + R_f)\, I_f(s) \quad \text{... (5)}$$

$$T_m(s) = J_m s^2 \theta_m(s) + B_m s\, \theta_m(s) \quad \text{... (6)}$$

Eliminate $I_f(s)$ from equations (4) and (5)

$$T_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)} \quad \text{... (7)}$$

Eliminate $T_m(s)$ from equations (6) and (7),

$$(s^2 J_m + sB_m)\theta_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)}$$

$$\text{Input} = E_f(s)$$

$$\text{Output} = \text{Rotational displacement } \theta_m(s)$$

$$\therefore \text{Transfer function} = \frac{\theta_m(s)}{E_f(s)}$$

$$\frac{\theta_m(s)}{E_f(s)} = \frac{K_m K_f}{(J_m s^2 + sB_m)\,(R_f + sL_f)}$$

$$= \frac{K_m K_f}{sR_f B_m [1 + s\tau_m][1 + s\tau_f]}$$

Where $\tau_m = \frac{J_m}{B_m}$ = Motor time constant

$\tau_f = \frac{L_f}{R_f}$ = Field time constant

$$\text{T.F.} = \frac{\theta_m(s)}{E_f(s)} = \frac{K_f}{R_f[1 + s\tau_f]} \cdot \frac{K_m}{B_m(1 + s\tau_m)} \cdot \frac{1}{s}$$

Block diagram for field controlled d.c. motor is as shown in Fig. 4.30.

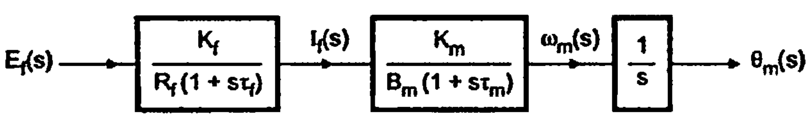


Fig. 4.30 Block diagram

4.14 Transfer Function of Armature Controlled D.C. Motor

Assumptions :

(i) Flux is directly proportional to current through field winding,

$$\boxed{\phi_m = K_f I_f = \text{Constant}}$$

(ii) Torque produced is proportional to product of flux and armature current.

$$T = K'_m \phi I_a$$
$$T = K'_m K_f I_f I_a$$

(iii) Back e.m.f. is directly proportional to shaft velocity ω_m, as flux ϕ is constant.

as $$\omega_m = \frac{d\theta(t)}{dt}$$

$$\boxed{E_b = K_b \omega_m(s) = K_b s\theta_m(s)}$$

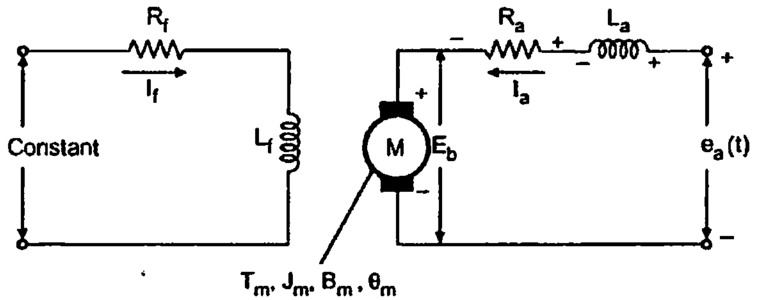


Fig. 4.31

Apply Kirchhoff's law to armature circuit :

$$e_a = E_b + I_a(R_a) + L_a \frac{di_a}{dt}$$

Take Laplace transform,

$$\therefore \quad E_a(s) = E_b(s) + I_a(s)[R_a + s L_a]$$

$$\therefore \quad I_a(s) = \frac{E_a(s) - E_b(s)}{R_a + s L_a}$$

$$I_a(s) = \frac{E_a(s) - K_b s\theta_m(s)}{R_a + s L_a}$$

Now $$T_m = K'_m K_f I_f I_a$$

$$T_m = K'_m K_f I_f \left\{\frac{E_a - K_b s\theta_m(s)}{R_a + s L_a}\right\}$$

Also $\quad T_m = \{J_m s^2 + s B_m\} \theta_m(s)$ ... from equation (3)

Equating equations of T_m,

$$\frac{K'_m K_f I_f E_a(s)}{(R_a + s L_a)} = \frac{K'_m K_f I_f K_b s \theta_m(s)}{(R_a + s L_a)} + (J_m s^2 + s B_m) \theta_m(s)$$

$$\therefore \frac{K'_m K_f I_f}{(R_1 + s L_a)} E_a(s) = \left[\frac{K'_m K_f I_f K_b s}{(R_a + s L_a)} + J_m s^2 + s B_m\right] \theta_m(s)$$

$$\therefore \frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_m}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}}{1 + \dfrac{K_m \cdot s K_b}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}} = \frac{G(s)}{1 + G(s)H(s)}$$

where $\tau_m = J_m / B_m$ and $\tau_a = \frac{L_a}{R_a}$

$$K_m = K'_m K_f$$

$$G(s) = \frac{K_m}{s R_a B_m (1 + s\tau_m)(1 + s\tau_a)}$$

$$H(s) = s K_b$$

Therefore can be represented in its block diagram form as in Fig. 4.32.

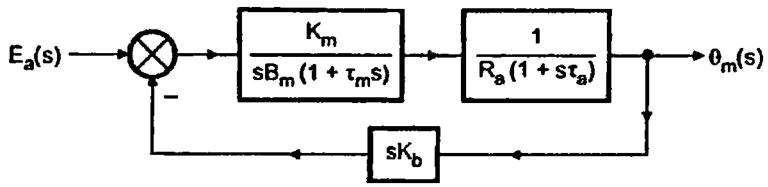


Fig. 4.32 Block diagram

Key Point : *Field controlled d.c. motor is open loop while armature controlled is closed loop system. Hence armature controlled d.c. motors are preferred over field controlled type.*

4.15 A.C. Servomotor

Most of the servomotors used in low power servomechanisms are a.c. servomotors. The a.c. servomotor is basically two phase induction motor. The output power of a.c. servomotor varies from fraction of watt to few hundred watts. The operating frequency is 50 Hz to 400 Hz.

4.15.1 Construction

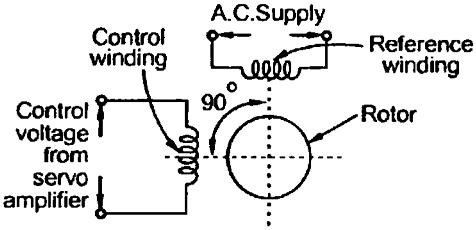


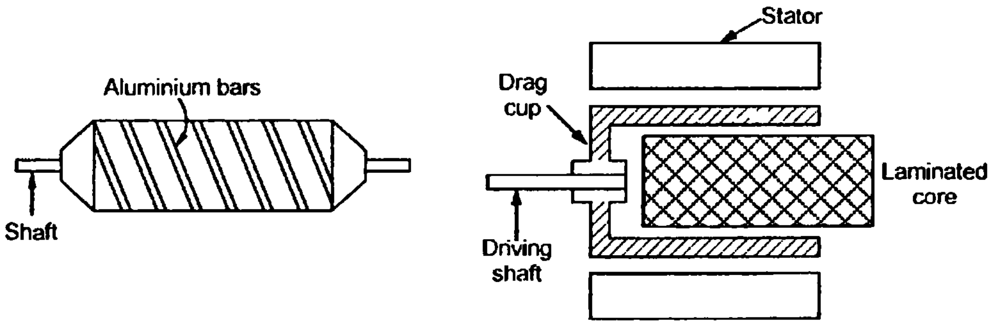
Fig. 4.33 Stator of A.C. servomotor

It is mainly divided into two parts namely stator and rotor.

The stator carries two windings, uniformly distributed and displaced by 90°, in space. One winding is called **main winding** or **fixed winding** or **reference winding**. This is excited by a constant voltage a.c. supply. The other winding is called **control winding.** It is excited by variable control voltage, which is obtained from a servoamplifier. This voltage is 90° out of phase with respect to the voltage applied to the reference winding. This is necessary to obtain rotating magnetic field. The schematic stator is shown in the Fig 4.33.

4.15.2 Rotor

The rotor is generally of two types. The one is usual squirrel cage rotor. This has small diameter and large length. Aluminium conductors are used to keep weight small. Its resistance is very high to keep torque-speed characteristics as linear as possible. Air gap is kept very small which reduces magnetising current. This cage type of rotor is shown with skewed bars in the Fig. 4.34 (a). The other type of rotor is drag cup type. There are two air gaps in such construction. Such a construction reduces inertia considerably and hence such type of rotor is used in very low power applications. The aluminium is used for the cup construction. The construction is shown in the Fig. 4.34 (b).



(a) Squirrel cage rotor **(b) Drag cup type rotor**

Fig. 4.34

4.15.3 Torque-speed Characteristics

The torque-speed characteristics of a two phase induction motor, mainly depends on the ratio of reactance to resistance. For small X to R ratio i.e. high resistance low reactance

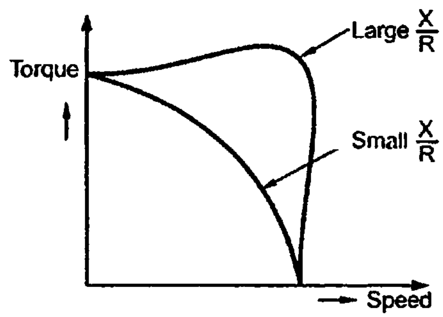


Fig. 4.35

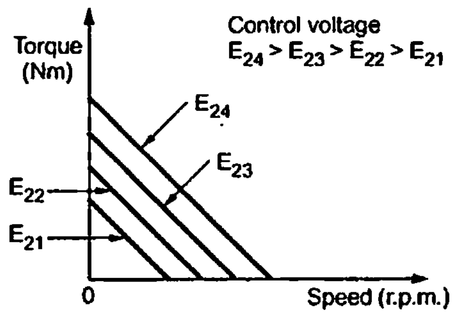


Fig. 4.36

motor, the characteristics is much more linear while it is nonlinear for large X to R ratio as shown in the Fig. 4.35.

In practice, design of the motor is so as to get almost linear torque-speed characteristics. The Fig. 4.36 shows the torque-speed characteristics for various control voltages. The torque varies almost linearly with speed. All the characteristics are equally spaced for equal increments of control voltage. It is generally operated with low speeds.

4.15.4 Features of A. C. Servomotor

The a.c. servomotor has following features :

i) Light in weight. ii) Robust construction. iii) Reliable and stable operation. iv) Smooth and noise free operation. v) Large torque to weight ratio. vi) Large R to X ratio i.e. small X to R ratio. vii) No brushes or slip rings hence maintenance free. viii) Simple driving circuits.

4.15.5 Applications

Due to the above features it is widely used in instrument servomechanisms, remote positioning devices, process control systems, self balancing recorders, computers, tracking and guidance systems, robotics, machine tools etc.

4.15.6 Transfer Function of A.C. Servomotor

The various approximations to derive transfer function are,

(i) A servomotor rarely operates at high speeds. Hence for a given value of control voltage, $T \propto N$ characteristics are perfectly linear.

(ii) In order that $T \propto N$ characteristics are directly proportional to voltage applied to its control phase, we assume $T \propto N$ characteristics are straight lines and equally spaced.

Torque at any speed 'N' is,

$$T_m = K_{tm} E_{2t} + m \frac{d\theta_m}{dt} \qquad \text{... (1)}$$

where, $\frac{d\theta_m}{dt}$ is speed of motor.

If load consists inertia J_m and friction B_m we can write,

$$T_m(s) = J_m s^2 \theta_m + B_m s \theta_m \qquad \text{.... (2)}$$

Now Laplace transform of equation (1) is

$$T_m(s) = K_{tm} E_2(s) + m s \theta_m(s) \qquad \text{... (3)}$$

Equating equations (2) and (3)

$$\therefore K_{tm} E_2(s) + m s \theta_m(s) = J_m s^2 \theta_m(s) + B_m s \theta_m(s)$$

$$\therefore \quad \frac{\theta_m(s)}{E_2(s)} = \frac{K_{tm}}{s(s J_m - m + B_m)} = \frac{K_{tm}}{s(B_m - m)\left[1 + \frac{s J_m}{(B_m - m)}\right]}$$

$$\therefore \quad \boxed{\frac{\theta_m(s)}{E_2(s)} = \frac{K_m}{s(1 + \tau_m s)}}$$

where $\quad K_m = \frac{K_{tm}}{B_m - m}$

and $\quad \tau_m = \frac{J_m}{B_m - m}$

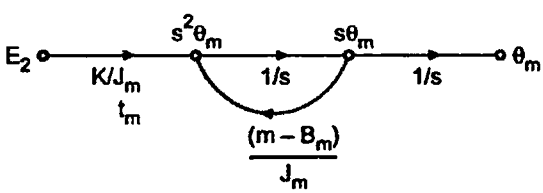


Fig. 4.37 Signal flow graph of a.c. servomotor

Key Point : *As slope is negative, in the above equation* $[B_m - m]$ *shows that total friction increases due to m. As it adds more friction, the damping improves, improving stability of the motor. This is called* ***Internal Electric Damping*** *of 2 ph A.C. servomotor.*

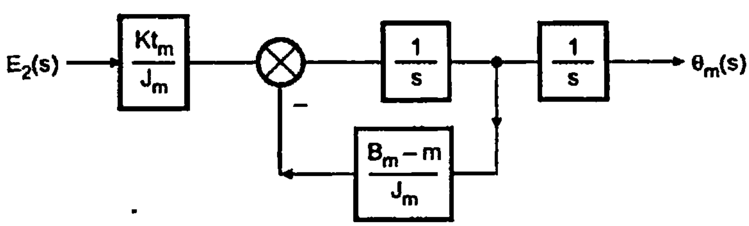


Fig. 4.38 Block diagram of a.c. servomotor

Signal flow graph for A.C. servomotor is as shown in the Fig. 4.37. Hence block diagram of A.C. servomotor is as shown in Fig. 4.38

4.16 Comparison of Servomotors

4.16.1 Comparison between A.C. and D.C. Servomotor

Sr. No.	A.C. Servomotor	D.C. Servomotor
1)	Low power output of about $\frac{1}{2}$ W to 100 W.	Deliver high power output
2)	Efficiency is less about 5 to 20 %.	High efficiency.
3)	Due to absence of commutator maintenance is less.	Frequent maintenance required due to commutator.
4)	Stability problems are less.	More problems of stability.
5)	No radio frequency noise.	Brushes produce radio frequency noise.
6)	Relatively stable and smooth operation.	Noisy operation.
7)	A.C. amplifiers used have no drift.	Amplifiers used have a drift.

4.16.2 Comparison between Armature Controlled and Field Controlled D.C. Servomotors

Sr. No.	Field Controlled	Armature Controlled
1)	Due to low power requirement amplifiers are simple to design.	High power amplifiers are required to design.
2)	Control voltage is applied to the field.	Control voltage is applied to the armature
3)	Time constant is large.	Time constant is small.
4)	This is open loop system.	This is closed loop system.
5)	Armature current is kept constant.	Field current is kept constant.
6)	Poor efficiency.	Better efficiency.
7)	Suitable for small rated motors	Suitable for large rated motors.
8)	Costly as field coils are must	Permanent magnet can be used instead of field coils which makes the motor less expensive.

4.17 Models of Commonly used Electromechanical Systems

In this article the transfer functions of very commonly used systems are derived. This will help the reader to find out the transfer functions of the different practical systems.

4.17.1 Generators

Consider a separately excited generator which is many times used in various practical mechanical systems. Generators are required to drive the motors because vacuum diodes, transistor amplifiers are not suitable because of their low ratings. Consider a generator as shown in the figure.

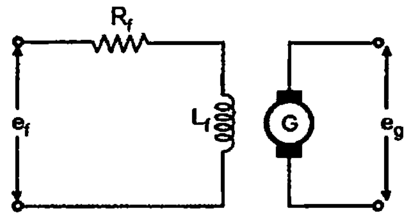


Fig. 4.39 Generator

R_f = Field resistance

L_f = Field inductance

e_f = Applied voltage (input)

e_g = Generated voltage (output)

Now for a generator,

$$e_g \propto \phi \qquad \text{where } \phi = \text{flux}$$

Flux is directly proportional to current passing through the field winding say i_f.

$$\therefore \quad e_g \propto i_f$$

Let K_a be the generator constant in V/A

$$\therefore \quad e_g = K_a \; i_f \qquad \text{... (1)}$$

Applying Kirchhoff's Law to field circuit.

$$e_f = i_f \; R_f + L_f \frac{di_f}{dt} \qquad \text{... (2)}$$

Taking Laplace of both the equations (1) and (2)

$$E_g(s) = K_a \; I_f(s)$$

$$E_f(s) = R_f \; I_f(s) + L_f(s) \; I_f(s) \qquad \text{neglecting } I_f(0)$$

$$\therefore \quad I_f(s) = \frac{E_f(s)}{R_f + s\,L_f}$$

$$\therefore \quad E_g(s) = \frac{K_a \; E_f(s)}{R_f + s\,L_f}$$

$$\therefore \quad \boxed{\frac{E_g(s)}{E_f(s)} = \frac{K_a}{R_f + s\,L_f}}$$

This is the T.F. of separately excited generator.

4.17.2 Generator Driving Motor

It is very common to find generator driving motor in practical mechanical systems. So let us discuss the T.F. of a system with generator driving motor.

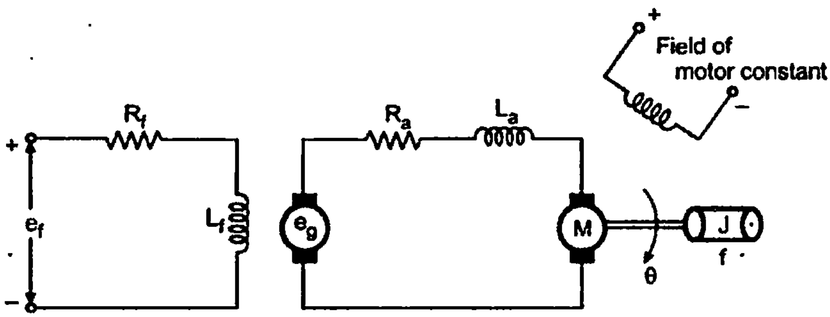


Fig. 4.40 Generator driving motor

R_f and L_f - Resistance and inductance of generator field

e_g - Generated voltage

K_g - Generator constant in V/A

R_a and L_a - Resistance and inductance of motor armature

J - M.I. of Load and f is frictional force

Now. T. F. of generator is,

$$\frac{E_g(s)}{E_f(s)} = \frac{K_g}{R_f + s\,L_f}$$

Now consider armature controlled motor,

Torque produced by motor is dependent on i_a and let K_T be torque constant.

$$T = K_T\, i_a$$

This torque is utilised to drive a load.

$$\therefore \qquad T = J\frac{d^2\theta}{dt^2} + f\,\frac{d\theta}{dt}$$

Now E_g is voltage applied to armature.

$$\therefore \qquad E_g = i_a R_a + L_a\frac{di_a}{dt} + e_b$$

where e_b is back e.m.f.

$$e_b \propto \omega \propto \frac{d\theta}{dt}$$

Let K_b be the back e.m.f. constant

$$\therefore \quad e_b = K_b \frac{d\theta}{dt}$$

$$\therefore \quad K_T i_a = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

Taking Laplace transform

$$K_T I_a(s) = J s^2 \theta(s) + f s \theta(s) \quad \text{... (3)}$$

$$E_g = i_a R_a + L_a \frac{di_a}{dt} + K_b \frac{d\theta}{dt}$$

Taking Laplace transform

$$E_g(s) = I_a(s) R_a + L_a s I_a(s) + K_b s \theta(s) \quad \text{... (4)}$$

finding $I_a(s)$ from equation (3)

$$I_a(s) = \frac{(J s^2 + f s) \theta(s)}{K_T}$$

Substituting in equation (4)

$$E_g(s) = \frac{s\theta(s)(J s + f)}{K_T}(R_a + s L_a) + K_b s \theta(s)$$

$$E_g(s) = \theta(s) s \left[\frac{(f + J s)(R_a + s L_a)}{K_T} + K_b\right]$$

$$E_g(s) - s K_b \theta(s) = \frac{s(R_a + s L_a)(s J + f)\theta(s)}{K_T}$$

$$\therefore \quad \frac{\theta(s)}{E_g(s) - s K_b \theta(s)} = \frac{K_T}{s(R_a + s L_a)(s J + f)}$$

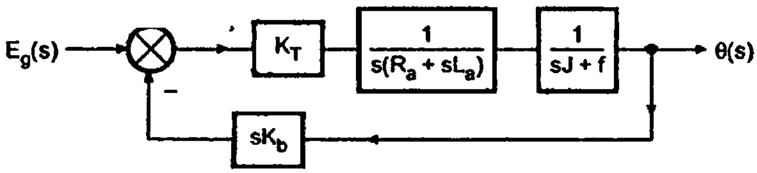


Fig. 4.41

Therefore using both the transfer functions of generator and armature controlled motor we can develop the combined block diagram as below :

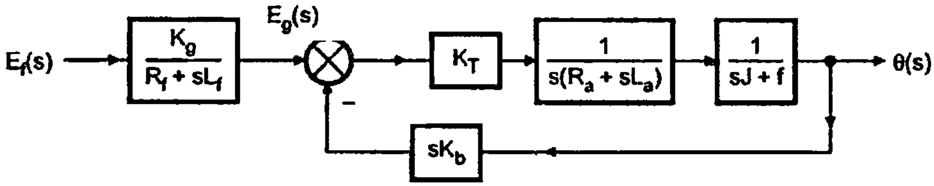


Fig. 4.42

Reducing the block diagram, solving feedback loop of motor which is

$$= \frac{\frac{K_T}{s(R_a + s L_a)(J s + f)}}{1 + \frac{s K_b \ K_T}{s(R_a + s L_a)(J s + f)}} = \frac{K_T}{s(R_a + s L_a)(J s + f) + s K_b K_T}$$

$$= \frac{K_T}{s[(R_a + s L_a)(J s + f) + K_b K_T]}$$

$$\therefore \quad \boxed{\frac{\theta(s)}{E_f(s)} = \frac{K_T \ K_g}{(R_f + s L_f)\{s[(R_a + s L_a)(J s + f) + K_b K_T]\}}}$$

4.17.3 Position Control System

Another very common system used in practice is position control system. This is used to control position of shaft, by use of potentiometer as error detector. The error is to be amplified by amplifier and then must be given to armature controlled motor whose shaft position will get controlled as per the controlled signal. The motor shaft is coupled to the load through gearing arrangement with ratio N_1/N_2.

Load has M.I. J and friction as f while θ_r is the reference position while θ_a is the actual position of shaft.

The circuit diagram can be drawn as below :

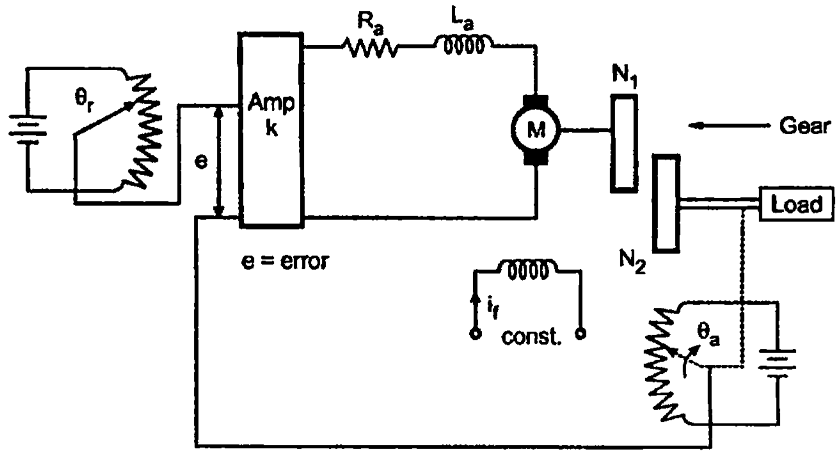


Fig. 4.43 Position control system

Let K_p be the potentiometer sensitivity, in V/rad

The corresponding block diagram can be drawn as shown in the Fig. 4.44.

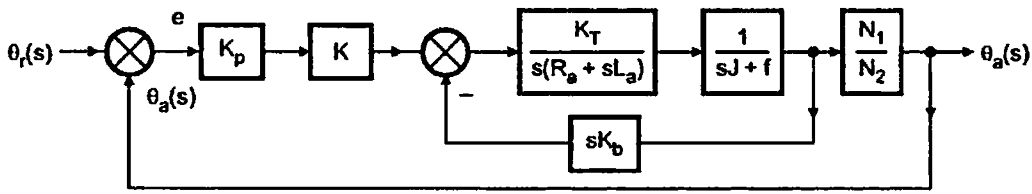


Fig. 4.44

Reducing the block diagram as shown in the Fig. 4.45.

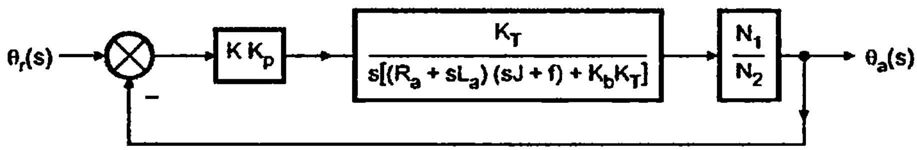


Fig. 4.45

$$G(s) = \frac{K K_P K_T (N_1/N_2)}{s[(R_a + s L_a)(J s + f) + K_b K_T]}$$

$$K K_P K_T (N_1/N_2) = K_s = \text{System gain constant}$$

$$H(s) = 1$$

∴ Over all T. F. can be calculated as below,

$$\frac{\theta_a(s)}{\theta_r(s)} = \frac{\dfrac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T]}}{1 + \dfrac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T]}}$$

∴ $$\boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_s}{s[(R_a + s L_a)(Js + f) + K_b K_T] + K_s}}$$

4.17.4 Position Control with Field Controlled Motor

In the above case if instead of armature controlled motor, field controlled motor is used then derive the T.F. of overall closed loop system.

Consider field controlled motor,

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

Now $T \propto \phi\, i_a$ and i_a is constant

∴ $T \propto \phi \propto i_f$

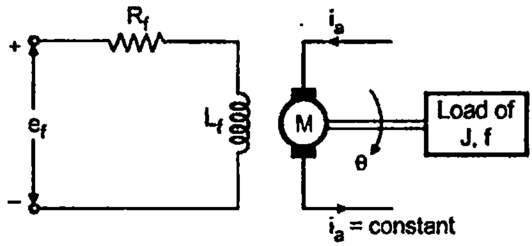


Fig. 4.46

Let K_f be constant in N-m/A

$$\therefore \quad T = K_f \, i_f$$

This torque is utilised to drive a load of moment of inertia J and friction f.

$$\therefore \quad T = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

$$\therefore \quad K_f \, i_f = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt}$$

Finding Laplace of all the equations,

$$E_f(s) = I_f(s) \, R_f + s \, L_f \, I_f(s)$$

and

$$K_f \, I_f(s) = J s^2 \, \theta(s) + s \, f \, \theta(s)$$

$$\therefore \quad I_f(s) = \frac{s \, \theta(s) \, [s J + f]}{K_f}$$

Substituting in $E_f(s)$

$$E_f(s) = \frac{s \, \theta(s) \, [s J + f] \, [R_f + s L_f]}{K_f}$$

$$\therefore \quad \frac{\theta(s)}{E_f(s)} = \frac{K_f}{s [s J + f][R_f + s L_f]}$$

Now using the same block diagram as derived in case (iii) replacing armature controlled motor T. F. by the field controlled we can get the new block diagram.

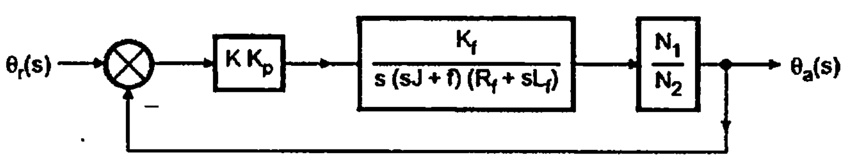


Fig. 4.47

Let $$K_s = K\, K_p\, K_f \left(\frac{N_1}{N_2}\right)$$

$$G(s) = \frac{K_s}{s(sJ+f)(R_f+sL_f)} \qquad H(s) = 1$$

$$\therefore \quad \frac{\theta_a(s)}{\theta_r(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K_s}{s(sJ+f)(R_f+sL_f)}}{1+\dfrac{K_s}{s(sJ+f)(R_f+sL_f)}}$$

$$\boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_s}{s(sJ+f)(R_f+sL_f)+K_s}}$$

4.17.5 Speed Control System

In some of the practical applications it is necessary to drive a load at a desired speed ω rad/sec. For this application an electromechanical system can be used which is shown as below.

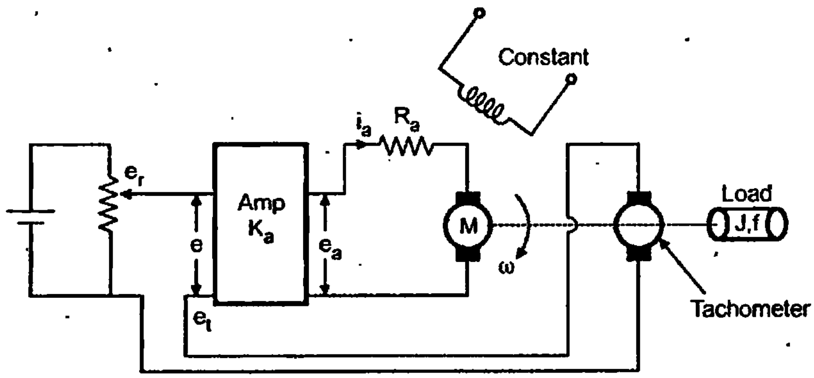


Fig. 4.48 Speed control system

System uses armature controlled motor and tachometer feedback. Let K_A be the amplifier gain. Let us derive the transfer function of this system. The objective of system is to move the load at a desired speed.

The d.c. tachometer output voltage is proportional to output speed ω.

e_r is reference voltage and e_t is tachometer voltage.

error $$e = e_r - e_t$$

$$e_a = K_a \times e = K_a\,(e_r - e_t) \qquad \text{... (5)}$$

$$e_t = K_t\,\omega \quad \text{where } K_t \text{ is constant.} \qquad \text{... (6)}$$

Neglecting inductance of armature

$$e_a = i_a\, R_a + e_b$$

$$e_b = K_b\,\omega$$

$$e_a = i_a R_a + K_b\,\omega \qquad \text{substituting from equation (5),}$$

$$\therefore \quad K_a\,(e_r - e_t) = i_a R_a + K_b\,\omega \qquad \text{substituting from equation (6),}$$

$$K_a e_r - K_a K_t\,\omega = i_a R_a + K_b\,\omega$$

Taking Laplace

$$K_a E_r(s) = (K_a K_t + K_b)\,\omega(s) + i_a(s) R_a \qquad \text{... (7)}$$

Now torque produced by motor

$$T \propto i_a$$

$$\therefore \quad T = K_T\, i_a \qquad \text{... (8)}$$

Now this drives a load

$$\therefore \quad K_T\, i_a = J\frac{d\omega}{dt} + f\,\omega \qquad \text{where } \omega = \frac{d\theta}{dt} \qquad \text{... (9)}$$

∴ Taking Laplace

$$K_T\, i_a(s) = Js\,\omega(s) + f\,\omega(s)$$

$$\therefore \quad i_a(s) = \frac{[J s + f]}{K_T} + f\,\omega(s) \qquad \text{... (10)}$$

Substituting in equation (7)

$$K_a\, E_r(s) = (K_a K_t + K_b)\,\omega(s) + \frac{[J s + f]\,\omega(s)}{K_T} R_a$$

$$\therefore \quad E_r(s) = \frac{(K_a K_t + K_b)\,\omega(s)}{K_a} + \frac{[J s + f]\,\omega(s) R_a}{K_a K_T}$$

$$\therefore E_r(s) - \left(K_t + \frac{K_b}{K_a}\right)\omega(s) = \frac{[J s + f] R_a}{K_a K_T}\,\omega(s)$$

$$\therefore \quad E_r(s) = \left\{K_t + \frac{K_b}{K_a} + \frac{[J s + f] R_a}{K_a K_T}\right\}\omega(s)$$

$$\therefore \quad \boxed{\frac{\omega(s)}{E_r(s)} = \frac{K_a K_T}{K_a K_T K_t + K_b K_T + [J s + f] R_a}}$$

This is the required transfer function of the speed control system.

4.17.6 Speed Control using Generator Driving Motor

A position control system described in Fig. 4.49 in which the armature of motor is applied with a control voltage through a generator. The field current of generator controls the voltage generated by the generator.

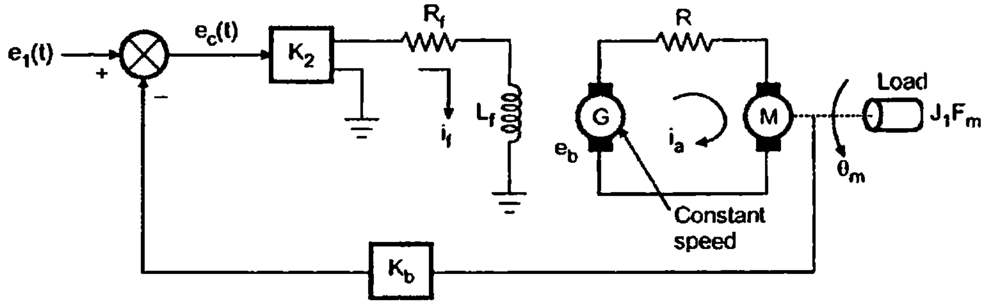


Fig. 4.49 Speed control system

Let us determine the transfer function of the system

Input is e_i (t).

Now $$e_c(t) = e_1(t) - K_b \frac{d(\theta)}{dt} \qquad \text{... (11)}$$

Taking Laplace transform

$$E_c(s) = E_1(s) - K_b\, s\,\theta(s)$$

Now $$E_c(t) \cdot K_2 = L_f \frac{di_f}{dt} + R_f\, i_f \qquad \text{... (12)}$$

applying Kirchhoff's law to the field circuit of generator.

Now $$e_a = K_a\, i_f$$ where K_a is constant.

Taking Laplace transform

$$E_a(s) = K_a\, I_f(s) \qquad \text{... (13)}$$

Taking Laplace transform of (12)

$$E_c(s)\, K_2 = I_f(s)\left[R_f + sL_f\right]$$

Eliminating I_f (s)

$$\frac{E_c(s)\, K_2}{(R_f + s\, L_f)} = I_f(s)$$

$$E_a(s) = \frac{K_a\, E_c(s)\, K_2}{(R_f + s\, L_f)}$$

$$\frac{E_a(s)}{E_c(s)} = \frac{K_a\, K_2}{(R_f + sL_f)}$$

Now consider field constant armature controlled motor. Armature is energized by output of the generator.

Applying Kirchhoff's law to armature circuit,

$$E_a(s) = E_b(s) + I_a(s)[R_a + s L_a]$$

Back e.m.f. is proportional to $\dot{\theta}_m$

$$\therefore \quad E_b(s) = K_b \frac{d\theta_m}{dt}$$

Taking Laplace Transform

$$E_b(s) = K_b \, s\theta_m(s)$$

$$\therefore \quad E_a(s) = K_b \, s\theta_m(s) + I_a(s)[R_a + s L_a]$$

In armature controlled d.c. motor as field current is constant so torque produced is directly proportional to the armature current.

$$T = K I_a(s) \text{ where K is motor constant.}$$

This torque is utilized to drive the load with moment of inertia J and friction F_m.

$$T = J\frac{d^2\theta_m}{dt^2} + F_m \frac{d\theta_m}{dt} \quad \text{Taking Laplace transform from}$$

$$\therefore \quad K I_a(s) = J s^2 \theta_m(s) + s F_m \theta_m(s)$$

$$I_a(s) = \frac{s}{K}(Js + F_m)\theta_m(s)$$

Substituting in equation for $E_a(s)$

$$E_a(s) = K_b \, s\theta_m(s) + \frac{s}{K}(Js + F_m)\theta_m(s)(R_a + s L_a)$$

$$\therefore \quad \frac{\theta_m(s)}{E_a(s)} = \frac{K}{s K K_b + s(R_a + s L_a)(Js + F_m)}$$

$$= \frac{K_m}{s[1 + T_m s]} \quad \text{... Neglecting } L_a \text{ of armature}$$

$$K_m = \frac{K}{[KK_b + R_a F_m]} = \text{Motor constant}$$

$$T_m = \frac{R_a J}{[KK_b + R_a F_m]} = \text{Time constant of motor}$$

Neglecting 'L_a' of armature motor.

$$\therefore \quad \frac{\theta_m(s)}{E_a(s)} = \frac{K_m}{s(1 + T_m s)}$$

$$\frac{E_a(s)}{E_c(s)} = \frac{K_a\, K_2}{[R_f + s\, L_f]}$$

$$\therefore \quad E_1(s) = \frac{K_a\, K_2\, E_c(s)}{(R_f + s\, L_f)} = \frac{\theta_m(s)\, s\,(1 + T_m\, s)}{K_m}$$

and

$$E_c(s) = E_1(s) - K_b\, s\,\theta_m$$

$$\therefore \quad \frac{K_a\, K_2\,[E_1(s) - K_b\, s\,\theta_m(s)]}{[R_f + s\, L_f]} = \frac{\theta_m(s)\, s\,(1 + T_m\, s)}{K_m}$$

$$\frac{K_a\, K_2\, K_m}{[R_f + s\, L_f]} E_1(s) = \theta_m(s)\left[s\,(1 + T_m\, s) + \frac{K_o\, K_a\, K_2\, K_m\, s}{(R_f + s\, L_f)}\right]$$

$$\therefore \quad \frac{K_a\, K_2\, K_m}{[1 + s\, T_f]} E_1(s) = \theta_m(s)\left[\frac{R_f\, s\,(1 + T_m\, s)(1 + T_f\, s) + K_b\, K_a\, K_2\, K_m\, s}{[1 + T_f(s)]}\right]$$

$$\frac{\theta_m(s)}{E_i(s)} = \frac{K_a\, K_2\, K_m}{s\,[K_b\, K_a\, K_2\, K_m + R_f\,(1 + T_m\, s)(1 + T_f\, s)]}$$

$$\boxed{\frac{\theta_m(s)}{E_i(s)} = \frac{K_2\, K_m\, K_g}{s\,[K_g\, K_b\, K_2\, K_m + (1 + T_m\, s)(1 + T_f\, s)]}}$$

where $K_g = \frac{K_a}{R_f}$ generator constant.

4.17.7 A Typical Position Control System used in Industry

A position control feedback system has a potentiometer bridge with a sensitivity of K_p V/radian as error detector. It feeds a d.c. amplifier with an open circuit gain K. This Supplies to a field of generator which has resistance R_f and inductance L_f. Generator constant is K_g V/A. This is connected to an armature controlled d.c. motor with motor torque constant K_T and back e.m.f constant K_b V/rad/sec. It drives a load of inertia J and friction f.

Draw the simplified block diagram and hence calculate its transfer function.

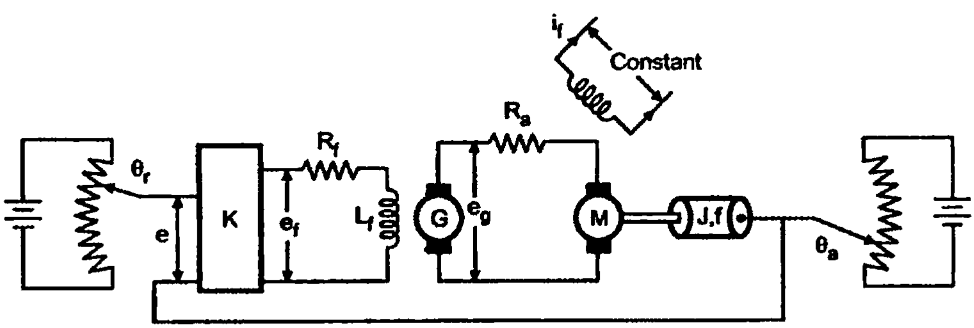


Fig. 4.50 Typical position control system

θ_r = Reference position , θ_a = Actual position

T. F of generator $\qquad \dfrac{E_g(s)}{E_f(s)} = \dfrac{K_g}{R_f + s\,L_f}$

Block diagram can be drawn from the analysis of the different cases discussed.

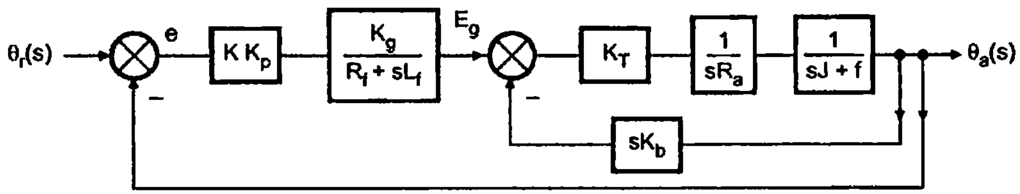


Fig. 4.51

Neglecting the inductance of armature winding of motor.

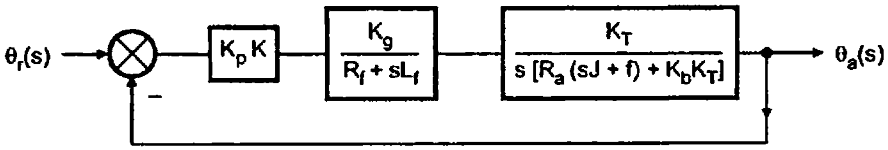


Fig. 4.52

$$G(s) = \frac{K_p\ K\ K_g\ K_T}{s\,(R_f + sL_f)\ [R_a\,(Js + f) + K_b\ K_T\,]} \qquad H(s) = 1$$

Let $\qquad K_R = K_p\ K\ K_g\ K_T$

$$\therefore \qquad T.\,F = \frac{\dfrac{K_R}{s\,(R_f + sL_f)\,[\,R_a\,(Js + f) + K_b\ K_T\,]}}{1 + \dfrac{K_R}{s\,[\,(R_a\,(Js + f) + K_b\ K_T\,]\ (R_f + sL_f)}}$$

$$\boxed{\frac{\theta_a(s)}{\theta_r(s)} = \frac{K_R}{s\,(R_f + sL_f)\,[R_a\,(Js + f) + K_b\ K_T\,] + K_R}}$$

4.18 D.C. Motor Position Control System

In industry to conrol the position of the shaft, a d.c. motor position control system is commonly used.

4.18.1 Transfer Function of D.C. Motor Position Control System

Consider the D.C. position control system which is controlling position of the shaft. Assume that the input and output of the system are the input shaft position and output shaft position respectively.

Assume following system constants,

r = Angular displacement of the reference input shaft

c = Angular displacement of the output shaft

θ = Angular displacement of the d.c. motor shaft used

K_1 = Gain of potentiometric error detector

K_p = Amplifier gain

e_a = Applied armature voltage

e_b = Back e.m.f.

R_a = Armature winding resistance

L_a = Armature winding inductance

i_a = Armature winding current

K_b = Back e.m.f. constant

K = Motor torque constant

J_m = Moment of inertia of motor.

b_m = Viscous friction coefficient of motor

J_L = Moment of inertia of load

b_L = Viscous friction coefficient of load

n = Gear ratio N_1/N_2

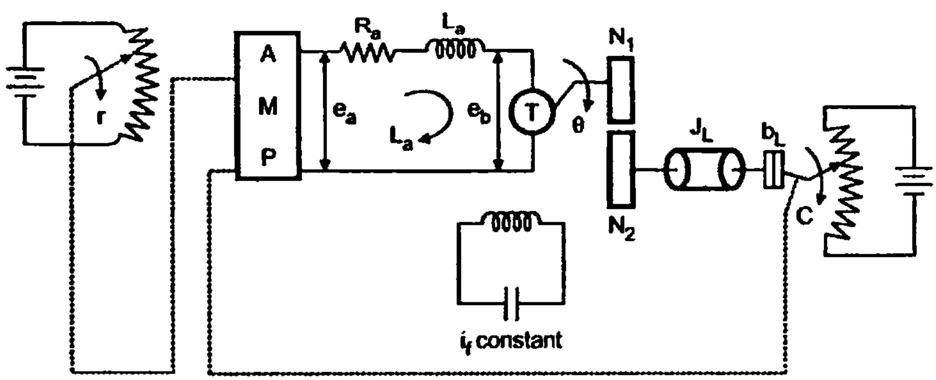


Fig. 4.53 D.C. Motor position control system

Equations describing above system can be written as follows :

Output shaft position is to be controlled so as to keep that at required position. Output is sensed by angular displacement 'c' and compared with input which is r. Error is amplified by amplifier with gain 'K_p' and given as input to the d.c. motor which in turn controls the angular position of the shaft of the motor 'θ' which in turn controls output position of shaft 'c' so as to modify the error.

For potentiometric error detector we can write.

$$E(s) = K_1 [R(s) - C(s)]$$

For amplifier

$$E_a(s) = K_p E(s)$$

For armature controlled d.c. motor

$$i_f = \text{constant so flux } \phi \text{ is constant}$$

$$T = K i_a \qquad \text{where } K = \text{motor torque constant}$$

$$e_b \propto \theta$$

$$\therefore \quad e_b = K_b \frac{d\theta}{dt}$$

For armature circuit

$$e_a = e_b + L_a \frac{di_a}{dt} + i_a R_a$$

Taking Laplace transforms

$$E_b(s) = K_b s \theta(s)$$

$$E_a(s) = E_b(s) + I_a(s)[R_a + s L_a]$$

$$T = K I_a(s)$$

Now torque is utilised to drive load + shaft of motor.

$$\therefore \quad J_{eq} = J_m + n^2 J_L = \text{Equivalent moment of inertia}$$

$$b_{eq} = b_m + n^2 b_L = \text{Equivalent frictional coefficient}$$

$$T = K I_a(s) = J_{eq} \frac{d^2\theta}{dt^2} + b_{eq} \frac{d\theta}{dt}$$

$$= [J_{eq} s^2 + b_{eq} s]\theta(s) \qquad \text{Laplace transform}$$

$$\therefore \quad I_a(s) = \frac{1}{K}[J_{eq} s^2 + b_{eq} s]\theta(s)$$

$$\therefore \quad E_a(s) = K_b s \theta(s) + \frac{\theta(s)}{K}[J_{eq} s^2 + b_{eq} s][R_a + s L_a]$$

$$\therefore \quad \frac{\theta(s)}{E_a(s)} = \frac{K}{s[J_{eq}s + b_{eq}][R_a + s L_a] + KK_b\, s}$$

$$= \frac{K}{s[KK_b + La\, J_{eq}s^2 + s(L_a\, b_{eq} + R_a\, J_{eq}) + b_{eq}R_a]}$$

'L_a' is generally small hence neglected.

$$= \frac{K}{s[KK_b + s\, R_a\, J_{eq} + b_{eq}R_a]}$$

$$= \frac{K_m}{s(1 + T_m\, s)}$$

$$K_m = \frac{K}{(b_{eq}R_a + KK_b)} = \text{Motor constant}$$

$$T_m = \frac{R_a\, J_{eq}}{(J_{eq}R_a + KK_b)} = \text{Motor time constant}$$

Now $\quad C(s) = n\,\theta(s)$

$$\therefore \quad \frac{C(s)}{E_a(s)} = n\,\frac{\theta(s)}{E_a(s)}$$

$$\therefore \quad \frac{C(s)}{E_a(s)} = \frac{n\, K_m}{s(1 + T_m\, s)}$$

$$\frac{C(s)}{K_p E(s)} = \frac{n\, K_m}{s(1 + T_m\, s)} \qquad \text{where } E_a(s) = K_p\, E(s)$$

$$C(s) = \frac{n\, K_m\, K_p\, E(s)}{s(1 + T_m\, s)}$$

$$C(s) = \frac{n\, K_m\, K_p\, K_l\, [R(s) - C(s)]}{s(1 + T_m\, s)}$$

$$\therefore \quad C(s) + \frac{n\, K_m\, K_p\, K_l}{s(1 + T_m\, s)}\, C(s) = \frac{n\, K_m\, K_p\, K_l\, R(s)}{s(1 + T_m\, s)}$$

$$\therefore C(s)\left[\frac{s(1 + T_m\, s) + n\, K_m\, K_p\, K_l}{s(1 + T_m\, s)}\right] = \frac{n\, K_m\, K_p\, K_l\, R(s)}{s(1 + T_m\, s)}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{\dfrac{nK_mK_pK_l}{s(1 + T_m s)}}{1 + \dfrac{nK_mK_pK_l}{s(1 + T_m s)}} = \frac{G(s)}{1 + G(s)H(s)}}$$

Block diagram :

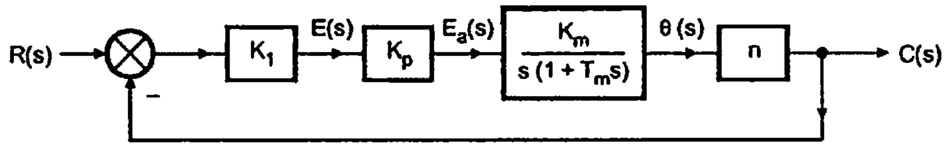
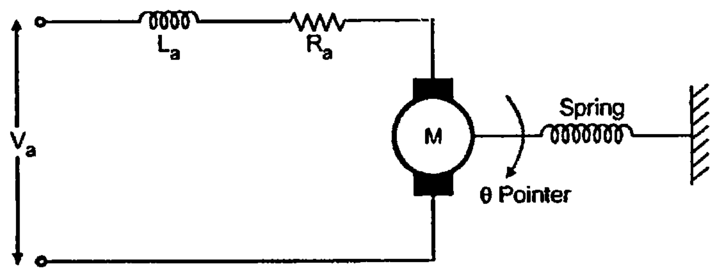


Fig. 4.54

Example 4.3 : *A d.c. motor drives a pointer which is spring loaded, to return to the reference position. If K_b = Back e.m.f. constant, K_T = Torque constant K_s = Spring constant and J = Moment of inertia. Find the transfer function.*



(M.U. : Nov.-93, Nov.-94)

Solution : Writing the system equations.

$$V_a(t) = I_a R_a + L_a \frac{dI_a}{dt} + E_b$$

Taking Laplace,

$$V_a(s) = E_b(s) + I_a(s)[R_a + s L_a] \quad \text{... (1)}$$

$$\therefore \quad I_a(s) = \frac{V_a(s) - E_b(s)}{(R_a + sL_a)} \quad \text{.... (2)}$$

Now $T_m = K_T I_a$, T_m = Motor torque ... (3)

and $E_b(s) = K_b s\theta(s)$ as $E_b \propto \frac{d\theta}{dt}$ ... (4)

$$\therefore \quad I_a(s) = \frac{V_a(s) - \theta(s)}{(R_a + s L_a)}$$

and

$$T_m(s) = K_T \left[\frac{V_a(s) - K_b s\,\theta(s)}{(R_a + s L_a)}\right] \quad \text{.. (5)}$$

This torque is used to drive a pointer with inertia J and spring load of constant K_s.

$$\therefore \quad T_m = J\frac{d^2\theta(t)}{dt^2} + K_s\,\theta(t)$$

Taking Laplace

$$T_m(s) = J\,s^2\,\theta(s) + K_s\,\theta(s) \qquad ...(6)$$

Equating (5) and (6)

$$\therefore \; J\,s^2\,\theta(s) + K_s\,\theta(s) = K_T\left[\frac{V_a(s) - K_b\,s\,\theta(s)}{R_a + sL_a}\right]$$

$$J\,s^2\,\theta(s) + K_s\,\theta(s) + \frac{K_T\,K_b\,s\,\theta(s)}{(R_a + sL_a)} = \frac{K_T}{(R_a + s\,L_a)}\,V_a(s)$$

$$\therefore \; \theta(s)\left[\frac{Js^2\,(R_a + s\,L_a) + K_s\,(R_a + sL_a) + K_T\,K_b s}{(R_a + s\,L_a)}\right] = \frac{K_T}{(R_a + s\,L_a)}\,V_a(s)$$

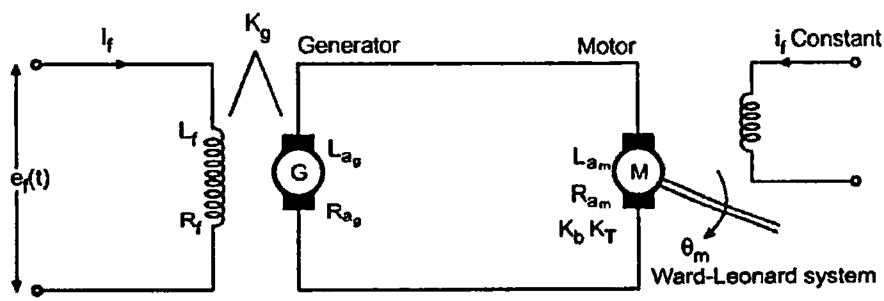
So transfer function is,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{s^2\,J\,(R_a + sL_a) + K_s\,(R_a + sL_a) + K_T\,K_b\,s}$$

$$\therefore \quad \frac{\theta(s)}{V_a(s)} = \frac{K_T}{(R_a + sL_a)\,[s^2\,J + K_s] + K_T\,K_b\,s}$$

Example 4.4 : *A d.c. generator supplies its output to a separately excited d.c. motor. The field current of the motor is constant. The voltage applied to the field circuit of d.c. generator is $e_f(t)$. Write the differential equations of this Ward-Leonard system relating the input $e_f(t)$ to the output $\theta_m(t)$. (Refer to figure shown below) Hence obtain expression for $\theta_m(s)$.*

(M.U.-May-96[PTDC] and April-96[Electrical])



Ward-Leonard system

Solution : Torque produced by motor,

$$T = K_T\,I_a(t) \qquad ...(1)$$

Voltage applied to the motor

$$E_g(t) = I_a\,R_a + L\,\frac{dI_a}{dt} + E_b(t) \qquad ...(2)$$

Back emf is proportional to the angular velocity

$$\therefore \quad E_b(t) = K_b \frac{d\theta_m(t)}{dt} \quad \text{... (3)}$$

Torque produced T is used to drive a load of inertia J and friction B

$$\therefore \quad T = J\frac{d^2\theta_m}{dt^2} + B\frac{d\theta_m}{dt} \quad \text{... (4)}$$

Equating (1) and (4)

$$K_T I_a(t) = J\frac{d^2\theta_m}{dt^2} + B\frac{d\theta_m}{dt}$$

$$\therefore \quad I_a(t) = \frac{J}{K_T}\frac{d^2\theta_m}{dt^2} + \frac{B}{K_T}\frac{d\theta_m}{dt} \quad \text{... (5)}$$

Substituting in equation (2) we can get resultant differential equation relating $E_g(t)$ and $\theta_m(t)$.

$$\text{Now} \quad E_f(t) = I_f R_f + L_f \frac{dI_f}{dt} \quad \text{... (6)}$$

$$\text{and} \quad E_g(t) = K_g I_f(t) \quad \text{... (7)}$$

$\therefore$ Substituting $I_f(t) = \frac{E_g(t)}{K_g}$ in equation (6) we can get, equation relating $E_f(t)$ and $E_g(t)$.

Substituting this in equation (2) we can obtain the final differential equation relating $E_f(t)$ and $\theta_m(t)$. It is very difficult to obtain in time domain so let us obtain it in Laplace domain.

Taking Laplace transform of all the equations

$$T(s) = K_T I_a(s) \quad \text{... (1)}$$

$$E_g(s) = I_a(s)\left[R_a + s L_a\right] + E_b(s) \quad \text{... (2)}$$

$$E_b(s) = K_b S\theta_m(s) \quad \text{... (3)}$$

$$T(s) = \theta_m(s)\left[Js^2 + Bs\right] \quad \text{... (4)}$$

$$\therefore \quad K_T I_a(s) = \theta_m(s)\left[Js + B\right]s$$

$$\therefore \quad I_a(s) = \frac{\theta_m(s)\, s\left[Js + B\right]}{K_T} \quad \text{... (5)}$$

$$\therefore \quad E_g(s) = \frac{\theta_m(s)\, s\left[Js + B\right]\left[R_a + s L_a\right]}{K_T} + K_b s\theta_m(s) \quad \text{... (6)}$$

$$\text{and} \quad E_f(s) = I_f(s)\left[R_f + s L_f\right] \quad \text{... (7)}$$

$$E_g(s) = K_g\, I_f(s)$$

So $$I_f(s) = \frac{E_g(s)}{K_g}$$

$$\therefore \quad E_f(s) = \frac{E_g(s)\,[R_f + sL_f]}{K_g} \qquad \text{... (8)}$$

Substituting $E_g(s)$ in equation (8) we get

$$\frac{E_f(s)\,K_g}{[R_f + s\,L_f]} = \theta_m(s)\left[\frac{s(Js+B)(R_a + s\,L_a)}{K_T} + K_b s\right]$$

$$\therefore \quad \frac{\theta_m(s)}{E_f(s)} = \frac{K_g\,K_T}{(R_f + s\,L_f)\,\{s(R_a + s\,L_a)(Js+B) + sK_bK_T\}}$$

$$\therefore \quad \theta_m(s) = \frac{K_g\,K_T\,E_f(s)}{(R_f + s\,L_f)\,\{s[(R_a + s\,L_a)(Js+B) + K_b\,K_T]\}}$$

4.19 Models of Thermal Systems

4.19.1 Heat Transfer System

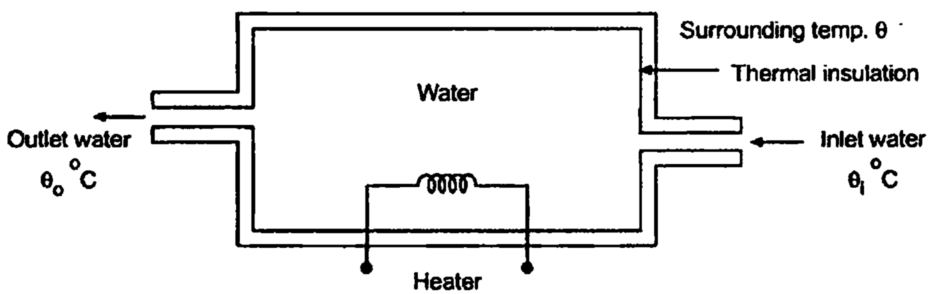


Fig. 4.55

A thermal system used for heating flow of water is shown below.

Electric heating element is provided in the tank to heat the water. The tank is insulated to reduce heat to the surroundings.

The necessary simplifying assumptions are :

1) There is no heat storage in the insulation.

2) All the water in the tank is perfectly mixed and hence at a uniform temperature.

θ_i = Inlet water temperature in °C.

θ_o = Outlet water temperature in °C.

θ = Surrounding temperature.

q = Rate of heat flow from heating element in J/sec.

q_i = Rate of heat flow to the water.

q_t = Rate of heat flow through tank insulation.

C = Thermal capacity in J/°C.

R = Resistance of thermal insulation.

So rate of heat flow for the water in tank is,

$$q_i = C\frac{d\theta_o}{dt} \quad \text{... (1)}$$

The rate of heat flow from the water to the surrounding atmosphere through insulation is,

$$q_t = \frac{\theta_o - \theta}{R} \quad \text{... (2)}$$

As per the heat transfer principles,

$$q = q_i + q_t \quad \text{... (3)}$$

Substituting equation (1) and (2)

$$q = C\frac{d\theta_o}{dt} + \frac{\theta_o - \theta}{R} \quad \text{... (4)}$$

Neglecting the term θ/R from the equation (4) this is because the variation of water temperature θ_o is over and above ambient temperature θ_ω.

$$\therefore \quad q = C\frac{d\theta_o}{dt} + \frac{\theta_o}{R}$$

Taking Laplace transform,

$$Q(s) = Cs\theta_o(s) + \frac{\theta_o(s)}{R}$$

∴ Transfer function is,

$$\boxed{\frac{\theta_o(s)}{Q(s)} = \frac{R}{1+sCR}}$$

The time constant of the system is RC.

4.19.2 Thermometer

Consider a thermometer placed in a water bath having temperature θ_i, as shown.

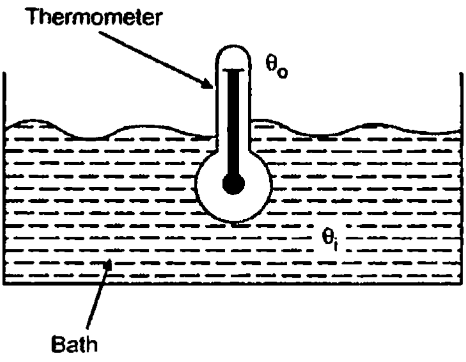


Fig. 4.56

θ_o is the temperature indicated by the thermometer. The rate of heat flow into the thermometer through its wall is,

$$\frac{dq}{dt} = \frac{\theta_i - \theta_o}{R}$$

Where R = Thermal resistance of the thermometer wall.

The indicated temperature, rises at a rate of

$$\frac{d\theta_o}{dt} = \frac{1}{C}\frac{dq}{dt}$$

where C is thermal capacity of the thermometer.

$$\therefore \quad \frac{d\theta_o}{dt} = \frac{1}{C}\cdot\left[\frac{\theta_i - \theta_o}{R}\right]$$

Taking Laplace of the equation,

$$\therefore \quad s\theta_o(s) = \frac{1}{RC}\left[\theta_i(s) - \theta_o(s)\right]$$

$$\therefore \quad \boxed{\frac{\theta_o(s)}{\theta_i(s)} = \frac{1}{1 + sRC}}$$

The time constant is RC.

4.20 Actuators

The actuator is a device which receives input signal from the controller and it produces the input signal to the plant according to control signal so that the output will approach the reference input signal to reduce the error to zero. Thus an actuator is generally after the controller and before the plant in the control system. An actuator can be of two types :

i) Hydraulic actuator 2) Pneumatic actuator

4.20.1 Hydraulic Actuator

The structure of an hydraulic actuator is shown in the Fig. 4.57.

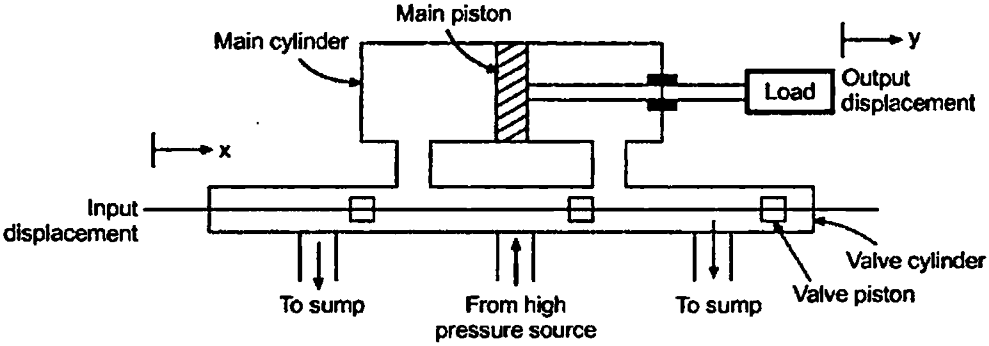


Fig. 4.57

They are piston devices in which motion of the spool regulates the oil flow to either side of the power cylinder. When the spool moves to the right, the high pressure oil enters the power cylinder to the left of the piston.

The differential pressure produced causes the power piston to move to the right, pushing the oil infront of it into the sump.

The load coupled rigidly to the piston moves a distance y from its reference position corresponding to the displacement x of the valve piston from its neutral position. The oil is pressurised by a pump and is recalculated in the system.

Equation of motion and transfer function

The rate of flow of oil into the piston is proportional to the rate of the movement of the piston.

$$Q = A\frac{dy}{dt}, \quad A = \text{area of piston} \qquad \text{... (1)}$$

If P is the differential pressure across the piston then the force on the piston is AP. This moves the load of mass M against friction B.

$$A \times P = M\frac{d^2 y}{dt^2} + B\frac{dy}{dt} \qquad \text{... (2)}$$

For small values of the displacement x, if P is the differential pressure on the piston and Q is the oil flow then,

$$K_2 P = K_1 x - Q \qquad \text{... (3)}$$

$$K_2\left[\frac{M}{A}\frac{d^2 y}{dt^2} + \frac{B}{A}\frac{dy}{dt}\right] = K_1 x - A\frac{dy}{dt}$$

Taking Laplace transform,

$$\frac{K_2 M}{A} s^2 Y(s) + \frac{K_2 B}{A} s Y(s) = K_1 X(s) - A Y(s) s$$

$$Y(s) \left[s^2 K_2 M + K_2 Bs + A^2 s\right] = K_1 A X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{K_1 A}{s^2 K_2 M + s K_2 B + A^2 s}$$

$$\therefore \quad \boxed{\frac{Y(s)}{X(s)} = \frac{\left(\frac{K_1 A}{K_2}\right)}{s\left[Ms + \left(B + \frac{A^2}{K_2}\right)\right]}}$$

This transfer function is similar to the electric motors.

4.20.2 Pneumatic Actuator

Pneumatic acting valve is used to obtain linear displacement of a plunger with pressurised air as input.

The air at pressure P is injected through inlet. Pressurised air pushes the diaphragm and plunger. The plunger has a mass M and friction on B with spring constant K.

Let A be the area of diaphragm then transfer function can be obtained as below.

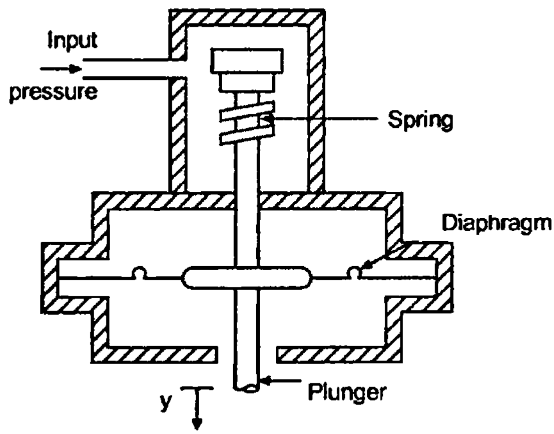


Fig. 4.58

Force exerted on the plunger is A × P. This force is opposed by mass, friction and spring.

$$A \times P = M \frac{d^2 y}{dt^2} + B \frac{dy}{dt} + Ky$$

Taking Laplace transform,

$$A\, P(s) = Ms^2 Y(s) + Bs\, Y(s) + K\, Y(s)$$

$$\therefore \quad \boxed{\frac{Y(s)}{P(s)} = \frac{A}{Ms^2 + Bs + K}}$$

The advantages of pneumatic systems are fire proof, explosion proof, simplicity and easy to maintain.

4.20.3 Comparison between Pneumatic and Hydraulic Systems

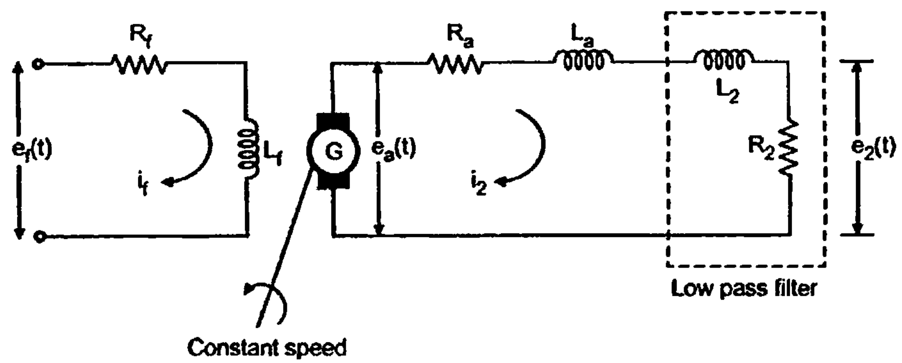
Sr.No.	Pneumatic systems	Hydraulic systems
1.	The fluid used is air.	The fluid used is oil.
2.	Air is compressible.	Oil is incompressible.
3.	Air does not have lubricating property.	Oil acts as a lubricator.
4.	The output power is much less compared to hydraulic.	The output power is much higher than pneumatic.
5.	At low velocities, the accuracy is poor.	At all velocities, the accuracy is satisfactory.
6.	No return pipes are required when air is used.	The return pipes are must.
7.	Can be operated for the temperature range of 0°C to 200°C. It is insensitive to temperature changes.	It is sensitive to the temperature changes and the range is 20°C to 70°C.
8.	These are fire proof and explosion proof.	These are not fire and explosion proof.
9.	Easy from maintenance point of view.	Difficult from maintenance point of view.

Note : The state space method of modeling the systems is separately covered in the chapter 15.

Examples with Solutions

➡ **Example 4.5 :** *The voltage generated by a d.c. generator is filtered by a low pass filter consisting R_2 and L_2 as shown below. This filtered voltage is controlled by the voltage applied to the field of generator.*
If $R_f = 40\ \Omega$, $L_f = 60\ H$, $R_a = 0.5\ \Omega$, $L_a = 1H$, $L_2 = 1H$, $R_2 = 2\Omega$ and generator constant $Kg = 120\ V$ / field Amp. Determine the transfer function $\frac{E_2(s)}{E_f(s)}$ of the system.

(M.U.: Dec.-97)



Solution : For field circuit, $e_f(t) = i_f(t) R_f + L_f \frac{d i_f}{dt}$ ... (1)

For armature circuit, $e_a(t) = i_2(t) R_a + i_2(t) R_2 + L_a \frac{di_2}{dt} + L_2 \frac{di_2}{dt}$ ... (2)

For generator, $e_a(t) = K_g i_f(t)$ ... (3)

For output, $e_2(t) = i_2(t) R_2$ ... (4)

Taking Laplace transform of all the equations,

$$E_f(s) = I_f(s) [R_f + s L_f] \quad ... (5)$$

$$E_a(s) = I_2(s)[(R_a + R_2) + s(L_a + L_2)] \quad ... (6)$$

$$E_a(s) = K_g I_f(s) \quad ... (7)$$

$$E_2(s) = I_2(s) R_2 \quad ... (8)$$

Equation (6) and (7),

$$K_g I_f(s) = I_2(s) [(R_a + R_2) + s(L_a + L_2)]$$

Substituting $I_a(s)$ from equation (8) and $I_f(s)$ from equation (5),

$$K_g \left[\frac{E_f(s)}{R_f + s L_f}\right] = \frac{E_2(s)}{R_2} [(R_a + R_2) + s(L_a + L_2)]$$

Substituting $I_2(s)$ from equation (8) and $I_f(s)$ from (5),

$$\frac{E_2(s)}{E_f(s)} = \frac{K_g R_2}{(R_f + s L_f) [(R_a + R_2) + s(L_a + L_2)]}$$

Substituting the values,

$$\therefore \quad \boxed{\frac{E_2(s)}{E_f(s)} = \frac{120 \times 2}{(40 + 60s)[(2.5 + 2s)]} = \frac{2.4}{(1 + 1.5s)(1 + 0.8s)}}$$

Example 4.6 : *A high gain speed control system uses a tachogenerator for speed sensing. The tachogenerator produces 5 V per 100 r.p.m. This voltage is compared with reference voltage to produce error signal. If the reference voltage is set to 10.8 volt. What is the value of expected speed?* **(M.U. : Nov.-94)**

Solution : The tachogenerator produces 5 V per 100 r.p.m.,

$\therefore$ Its constant is $\frac{5}{100} = 0.05$ V/r.p.m.

Now for the expected speed, error should be zero,

i.e $\quad V_r$ – tachogenerator output = error

$\therefore$ Tachogenerator output = V_r

Let N be expected speed in r.p.m.

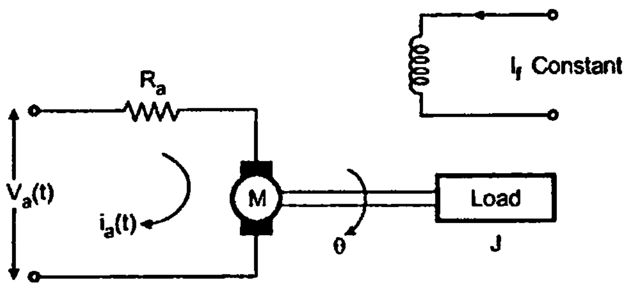
$$\therefore \quad N \times 0.05 = 10.8$$

$$\therefore \quad N = \mathbf{216\ r.p.m.} \qquad \text{... Expected speed}$$

Example 4.7 : *An armature controlled d.c. motor has an armature resistance of 0.37 Ω. The moment of inertia is 2.5 × 10^{-6} kg - m². A back emf of 2.09 V is generated per 100 r.p.m. of the motor speed. The torque constant of the motor is 0.2 N–m /A. Determine the transfer function of the motor relating the motor shaft shift and the input voltage.*

(M.U. : Dec.-96)

Solution : The motor can be shown as,



Now back emf is 2.09 V per 100 r.p.m. , N = 100 r.p.m. is,

$$\omega = \frac{2\pi N}{60} = 10.4719 \text{ rad/sec}$$

$$\therefore \text{Back emf constant } K_b = \frac{-2.09}{10.4719} = 0.1995 \text{ V/rad/sec}$$

$$K_T = 0.2 \text{ N-m/A}$$

For armature circuit $\quad V_a(t) = I_a R_a + E_b \quad$ ... (1)

Taking Laplace $\quad V_a(s) = I_a(s) R_a + E_b(s)$

$$E_b(t) = K_b \frac{d\theta}{dt} \qquad \text{... (2)}$$

Taking Laplace $\quad E_b(s) = K_b s\,\theta(s)$

$$T_m = K_T I_a(t) \qquad \text{... (3)}$$

Taking Laplace $T_m(s) = K_T I_a(s)$

This torque drives a load of inertia J.

$$\therefore \quad T_m(t) = J \frac{d^2 \theta}{dt^2} \qquad \text{... (4)}$$

Taking Laplace $T_m(s) = J s^2 \theta(s)$

$$\therefore \quad K_T I_a(s) = J s^2 \theta(s)$$

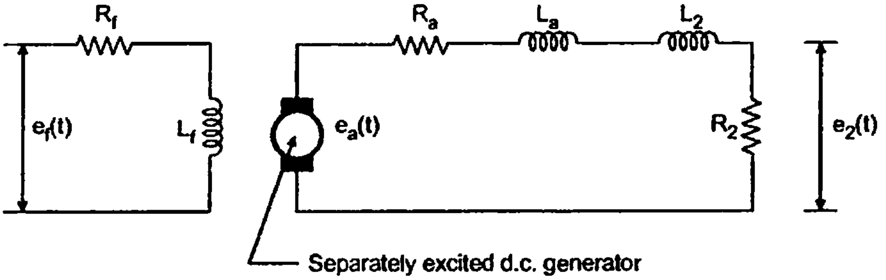
$$\therefore \quad I_a(s) = \frac{2.5 \times 10^{-6}}{0.2} s^2 \theta(s) = 1.25 \times 10^{-5} s^2 \theta(s)$$

$$\therefore \quad V_a(s) = 1.25 \times s^2 10^{-5} \theta(s) \times 0.37 + K_b s \theta(s)$$

$$\therefore \quad V_a(s) = 4.625 \times 10^{-6} s^2 \theta(s) + 0.1995 s\theta(s)$$

$$\therefore \quad \boxed{\frac{\theta(s)}{V_a(s)} = \frac{1}{4.625 \times 10^{-6} s^2 + 0.1995 s} = \frac{216216.22}{s(s + 43135.135)}}$$

Example 4.8 : *For the system shown below determine* $\frac{E_2(s)}{E_f(s)}$. **(M.U. : Dec.-97)**



Solution :

For field circuit, $\quad e_f(t) = i_f(t) R_f + L_f \frac{di_f}{dt}$

For armature circuit, $\quad e_a(t) = i_2(t) R_a + i_2(t) R_2 + L_a \frac{di_2}{dt} + L_2 \frac{di_2}{dt}$

For generator, $\quad e_a(t) = K_g i_f(t)$

For output, $\quad e_2(t) = i_2(t) R_2$

Taking Laplace of all the equations we get,

$$E_f(s) = I_f(s)[R_f + s L_f]$$

$$E_a(s) = I_2(s)\left[(R_a + R_2) + s(L_a + L_2)\right] = K_g I_f(s)$$

$$E_2(s) = I_2(s) R_2$$

Hence equating $E_a(s)$ equations,

$$K_g I_f(s) = I_2(s) \left[I_2(s)(R_a + R_2) + s(L_a + L_2)\right]$$

and hence using values of $I_f(s)$ and $I_2(s)$ from remaining equations we get,

$$K_g \times \frac{E_f(s)}{[R_f + s L_f]} = \frac{E_2(s)}{R_2} \times [(R_a + R_2) + d(L_a + L_2)]$$

$$\therefore \quad \boxed{\frac{E_2(s)}{E_f(s)} = \frac{K_g R_2}{(R_f + sL_f)[(R_a + R_2) + s(L_a + L_2)]}}$$

Example 4.9 : *A 50 Hz, 2 phase a.c. servomotor has the following parameters :*

Starting torque = 0.186 Nm

Rotor inertia = 1×10^{-5} kg − m²

Supply voltage = 120 V

No load angular velocity = 304 rad/s

Assuming straight line torque-speed characteristics of the motor and zero friction, obtain its transfer function. **(Gate)**

Solution : The starting torque is nothing but locked rotor torque.

$$\therefore \quad K_{tm} = \frac{\text{Locked rotor torque}}{\text{Rated voltage}} = \frac{0.186}{120}$$

$$= 1.55 \times 10^{-3}$$

Let m be the slope of linearised torque-speed characteristics.

$$\therefore \quad m = \frac{\text{Locked rotor torque}}{\text{No load angular speed}} = \frac{0.186}{304}$$

$$= -6.118 \times 10^{-4}$$

The torque at any angular speed ω is given by,

$$T_m = K_{tm} E_{2t} + m\frac{d\theta}{dt} \quad \text{... (1)}$$

where E_{2t} = rotor voltage

Taking Laplace, $T_m(s) = K_{tm}\, E_{2t}(s) + m\, s\theta\,(s)$ ... (2)

This torque is used to drive load of inertia J_m. The friction is given zero.

$$\therefore \quad T_m = J_m \frac{d^2\theta}{dt^2} \qquad ... (3)$$

$$\therefore \quad T_m(s) = s^2 J_m \theta(s) \qquad ... (4)$$

Equating (2) and (4),

$$K_{tm}E_{2t}(s) + m\, s\theta(s) = s^2 J_m \theta(s)$$

$$\therefore \quad K_{tm}\, E_{2t}(s) = s\theta(s)\,[sJ_m - m]$$

Hence the transfer function of the motor is,

$$\frac{\theta(s)}{E_{2t}(s)} = \frac{K_{tm}}{s[s\,J_m - m]} = \frac{1.55\times10^{-3}}{s[1\times10^{-5}\, s-(-6.118\times10^{-4})]}$$

$$\therefore \quad \boxed{\frac{\theta(s)}{E_{2t}(s)} = \frac{155}{s(s+61.18)}}$$

This is the required transfer function.

Example 4.10 : *A two phase a.c. servomotor having a torque constant of 0.045 Nm/V controls a position load through a gear ratio of 10:1. The effective moment of inertia and coefficient of viscous friction referred to load side are 0.25 kg-m² and 1.0 N-m/(rad/sec). The synchro error detector produces an error signal of 0.1V per degree error in misalignment. Develop the block diagram representation of the control system and there from obtain the transfer function.* **(M.U. : May-99)**

Solution : For the error detector, K_1 = 0.1 V/degree error

$$\therefore \quad K_1 = 0.1\times\frac{180}{\pi}$$

$$= 5.7295 \text{ V/radian}$$

$$K_{tm} = \text{Torque constant}$$

$$= 0.045 \text{ Nm/V}$$

$$\frac{N_1}{N_2} = \frac{1}{10}$$

$$J_L = 0.25 \text{ kg - m}^2 \text{ on load side}$$

$$\therefore \quad J_{eq} = \text{Motor side} = \left(\frac{N_1}{N_2}\right)^2 \times J_L = \frac{1}{100}\times 0.25$$

$$= 2.5\times 10^{-3}\ \text{kg-}m^2$$

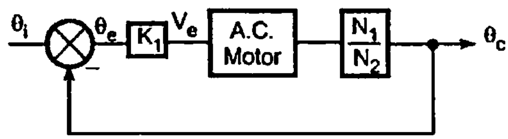
$$B_L = 1\ \text{N-m/(rad/sec)}$$

$$\therefore \quad B_{eq} = \text{Motor side} = \left(\frac{N_1}{N_2}\right)^2 \times B_l = \frac{1}{100}\times 1$$

$$= 0.01\ \text{N-m/ (rad/sec)}$$

The block diagram of the system is,

$$G(s) = K_1 \times \text{motor T.F.} \times \frac{N_1}{N_2}$$



For motor, $T_m = K_{tm}\,E_2$

$$\therefore \quad T_m(s) = K_{tm}\,E_2(s) \quad \ldots (1)$$

and $$T_m(t) = J_{eq}\frac{d^2\theta}{dt^2} + B_{eq}\frac{d\theta}{dt}$$

$$\therefore \quad T_m(s) = \left(s^2 J_{eq} + s B_{eq}\right)\theta(s) \quad \ldots (2)$$

Equating equations (1) and (2),

$$K_{tm}\,E_2(s) = \left(s^2 J_{eq} + s B_{eq}\right)\theta(s)$$

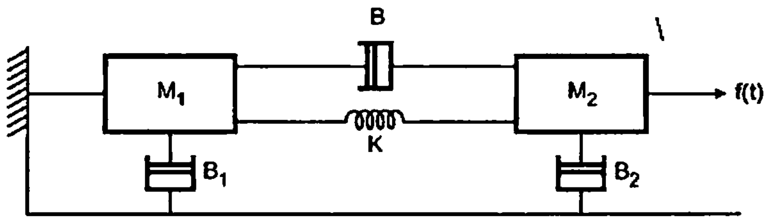
$$\therefore \quad \frac{\theta(s)}{E_2(s)} = \frac{K_{tm}}{s\left(s J_{eq} + B_{eq}\right)} = \frac{0.045}{s\left(s\times 2.5\times 10^{-3} + 0.01\right)} = \frac{18}{s(s+4)}$$

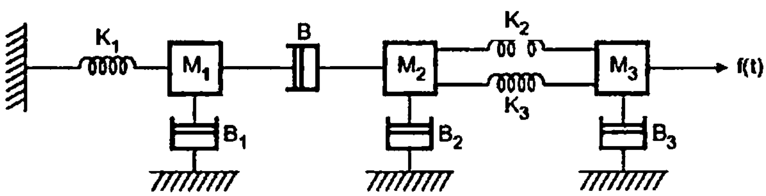
$$\therefore \quad G(s) = 5.7295\times\frac{18}{s(s+4)}\times\frac{1}{10} = \frac{10.3131}{s(s+4)}$$

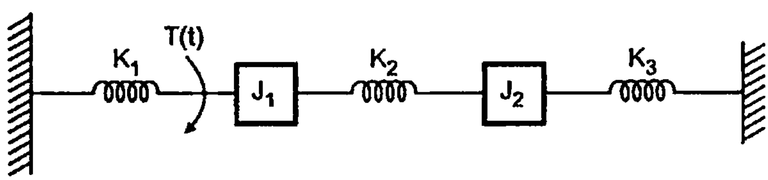
$$\therefore \quad \boxed{\frac{\theta_c(s)}{\theta_i(s)} = \frac{\dfrac{10.3131}{s(s+4)}}{1+\dfrac{10.3131}{s(s+4)}} = \frac{10.3131}{s^2+4s+10.3131}}$$

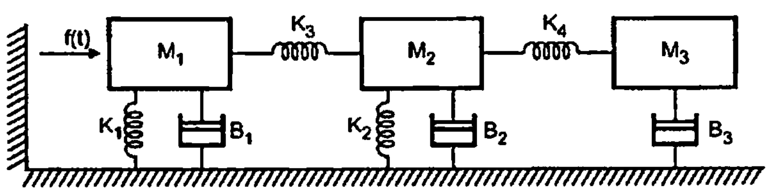
Review Questions

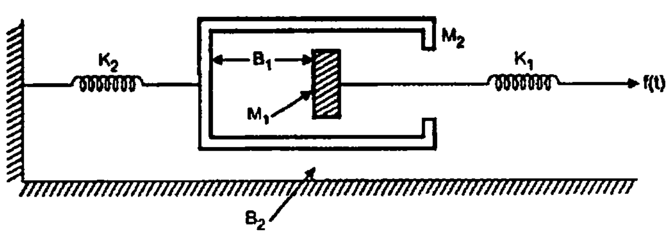
1. *Explain the derivation of analogous networks using*
   i) Force-voltage *ii) Force-current analogy*
2. *Write a short note on direct and inverse analogous networks.*
3. *Draw the analogous electrical networks based on*
   a) F - V analogy *b) F - I analogy of the following mechanical systems*











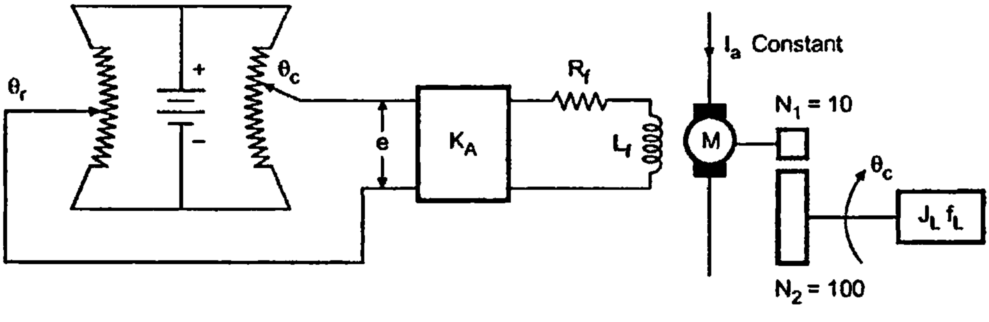
4. *Distinguish between A.C. servomotor and D.C. servomotor.*
5. *Derive transfer function of a.c. servomotor stating the assumption made.*
6. *State the applications of a.c. servomotor.*
7. *Derive the transfer function of field controlled d.c. servomotor.*
8. *Answer the following giving reasons :*

   a) A.C. servomotor has a smaller diameter and more length.

   b) Field controlled D.C. servomotor is preferred than armature controlled.
9. *Develop block diagram for armature controlled D.C. servomotor and find its transfer function.*
10. *An armature controlled d.c. motor is supplied in series with a resistance from a 24 V d.c. supply. The motor takes current of 5A on stalling and the stalling torque being 0.915 N-m. The motor runs at 1000 r.p.m. taking a current of 1 A. The value of* $J = 4 \times 10^{-3}$ *Kg -* m^2 *and friction constant as* 1.5×10^{-3} *Nm/ (rad/sec)*

    Determine the transfer function of the motor. **(Ans. :** $\frac{\theta(s)}{V(s)} = \frac{2.9}{s(1+0.3\,s)}$**)**
11. *For the closed loop system shown below, draw the block diagram and determine transfer function* $\theta_c(s) / \theta_r(s)$. *The given values are Error detector gain* K_e *= 8 V/rad, Amplifier gain* K_A *= 10 V/A* R_f *= 5 Ω,* L_f *= 0.25 H,* K_f *= 0.05 Nm/A,* J_{motor} *= 0.02 Kg –* m^2*,* J_L *= 3 Kg -* m^2*.* f_{motor} *= 0.03 Nm/(rad /sec).*

    f_L *= 5.5 Nm/ (rad/sec)*

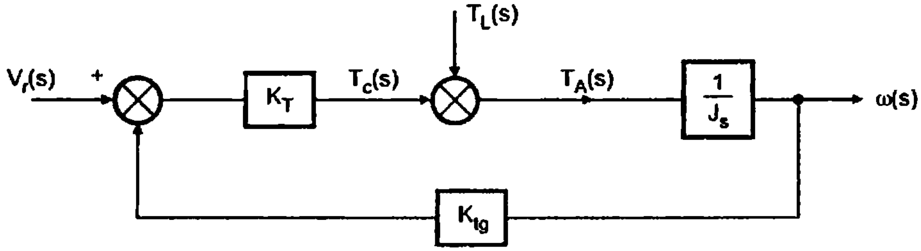


(Ans. : $\frac{\theta_c(s)}{\theta_r(s)} = \frac{32.12}{(s^3 + 21.5s^2 + 30.35 + 32.12)}$**)**

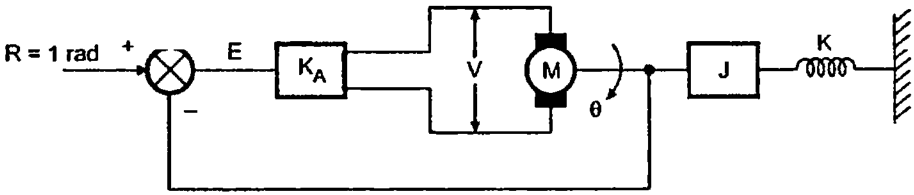
12. *The moment of inertia* J_m *and the coefficient of viscous friction* f_m *for a field controlled d.c. motor are motor respectively* 5×10^{-4} kg - m^2 and 12.5×10^{-4}*Nm (rad/sec). The motor torque constant* K_f *being 2.5 Nm/A. Determine the transfer function relating the angular speed of the shaft and the field current.* **(Ans. :** $\frac{\omega(s)}{I_f(s)} = \frac{2000}{1+0.4s}$**)**

13. *Figure given below shows a block diagram of speed regulator system. The accelerating torque is the difference between the torque developed by the controller and load T_L. The controller gain is 0.00102 Nm/rad and the tachogenerator constant is 0.191 V/rad/sec. The load speed is adjusted to 1000 r.p.m. The moment of inertia is J and friction is negligible.*

Calculate a) The reference voltage. b) The speed if a constant load torque of 0.001 Nm is suddenly applied. **(Ans. : a) 20 V b) 951 r.p.m.)**



14. *An instrument servo consisting of motor, spring loaded shaft, etc is shown below*



Where V = *Voltage in volts*

R = *Motor resistance 1 Ω*

L = *Motor inductance 0.1 H*

K_A = *Amplifier gain = 10 V/V*

K = *Spring constant = 0.001 Nm/rad*

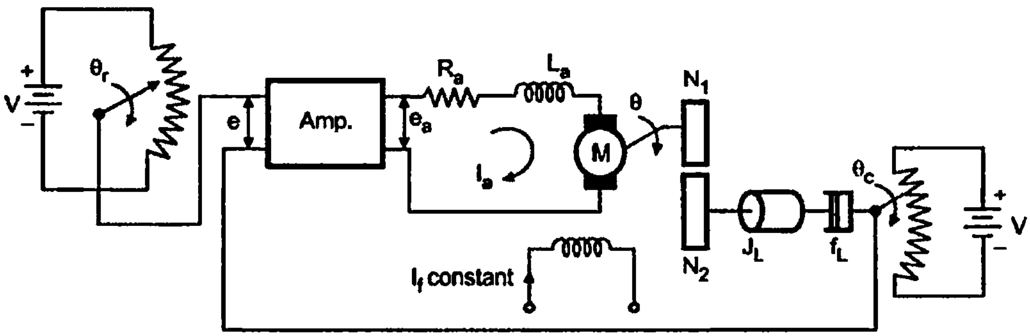
K_b = *Back e.m.f. constant = 0.01 V (rad/sec)*

K_T = *Torque content = 0.01 N-m/A*

J = *Moment of inertia = 0.005 N-m-s²*

If the input is 1 radians, what is the steady state error? **(Ans. : $e_{ss} = \frac{1}{\left[1+\frac{K_T\,K_A}{(K\times R)}\right]} = \frac{1}{101}$)**

15. *The positions servomechanism is shown in figure below. θ_r is reference position and θ_c actual angular displacement of shaft in radians. Obtain its closed loop transfer function* $\frac{\theta_c(s)}{\theta_r(s)}$



Given	θ	=	*angular displacement of the motor shaft*
	K_f	=	*gain f error detector = 7.64* V/rad
	R_a	=	$2\ \Omega$, L_a = *negligible*
	K_b	=	5.5×10^{-2} V/rad/sec
	K_T	=	6×10^{-5} *Nm/A*
	J_m	=	1×10^{-5} Kg - m^2
	f_m	=	*negligible*
	J_L	=	4.4×10^{-3} Kg - m^2
	f_L	=	4×10^{-2} Nm/ (rad/sec)
	N_1	=	*1*
	N_2	=	*10*

(Ans. : $\frac{\theta_c(s)}{\theta_r(s)} = \frac{42.3}{s^2 + 7.7\,s + 42.3}$**)**

16. *Derive the transfer function of a typical d.c. position control system.*
17. *Derive the transfer function of a typical d.c. speed control system.*
18. *Obtain the mathematical model of heat transfer system.*
19. *Obtain the mathematical model of thermometer.*
20. *What is actuator ? Explain hydraulic and pneumatic actuator systems.*
21. *Compare hydraulic and pneumatic actuators.*

□□□

5 Block Diagram Representation

5.1 Background

If a given system is complicated, it is very difficult to analyse it as a whole. With the help of transfer function approach, we can find transfer function of each and every element of the complicated system. And by showing connection between the elements, complete system can be splitted into different blocks and can be analysed conveniently. This is the **basic concept of block diagram representation.**

Basically block diagram is a pictorial representation of the given system. It is very simple way of representing the given complicated practical system. In block diagram, the interconnection of system components to form a system can be conveniently shown by the blocks arranged in proper sequence. It explains the cause and effect relationship existing between input and output of the system, through the blocks.

To draw the block diagram of a practical system, each element of practical system is represented by a block. The block is called **functional block.** It means, block explains mathematical operation on the input by the element to produce the corresponding output. The actual mathematical function is indicated by inserting corresponding transfer function of the element inside the block.

Key Point: *For a closed loop systems, the function of comparing the different signals is indicated by the **summing point** while a point from which signal is taken for the feedback purpose is indicated by **takeoff point** in block diagrams.*

All these summing points, blocks and takeoff points then must be connected exactly as per actual elements connected in the practical system. The connection between the blocks is shown by lines called **branches** of the block diagram. An arrow is associated with each and every branch which indicates the direction of flow of signal along the branch.

Key Point: *The signal can travel along the direction of an arrow only.*

It cannot pass against the direction of an arrow. Hence **block diagram is an unilateral property of the system.**

In short any block diagram has following five basic elements associated with it :

1) Blocks 2) Transfer functions of elements shown inside the blocks.

3) Summing points 4) Takeoff points 5) Arrows.

5.1.1 Illustrating Concept of Block Diagram Representation

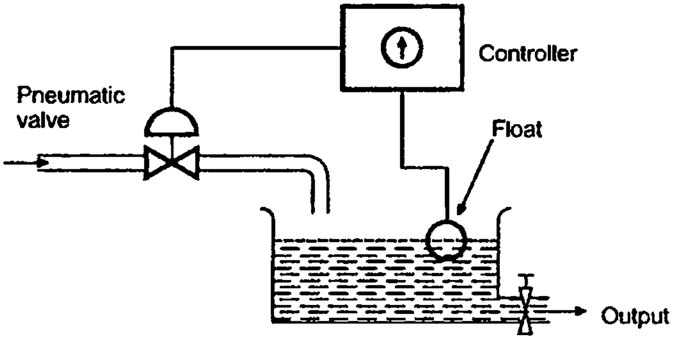


Fig. 5.1 Liquid level control system

For example : Consider the liquid level system as shown in the Fig. 5.1. So to represent this by block diagram, identify the elements which are,
i) Controller (ii) Pneumatic valve (iii) Tank (iv) Float.

In this system, the level of water is sensed by the float. Hence the float position acts as the feedback. According to the float position, with respect to desired level of water, the controller operates the pneumatic valve controlling the flow of water in the tank. When the required level is reached, controller operates the pneumatic valve in such a way that the flow of water in the tank, stops. If the output from the tank is taken i.e. the water from the tank is drained then the float position changes from the desired position and accordingly the controller operates the pneumatic valve to start the flow of water in the tank.

Hence indicating all the elements by blocks, the block diagram of the system can be developed as shown in the Fig. 5.2.

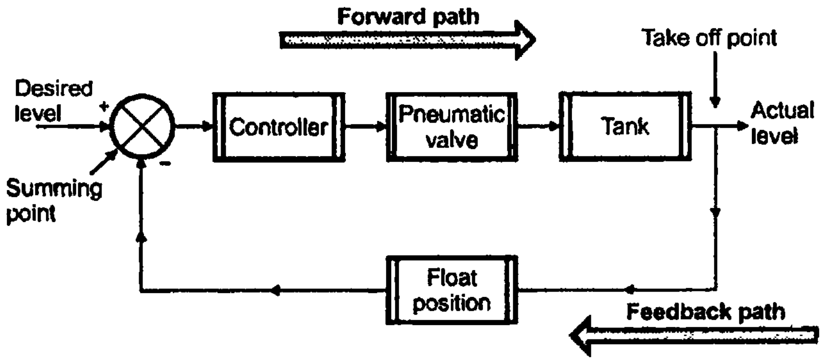


Fig. 5.2 Block diagram of liquid level control

Consider another example of bottle filling mechanism. When bottle gets filled by the contents upto the required level it should get replaced by an empty bottle. This system can be made closed loop and hence can be as shown in the Fig. 5.3.

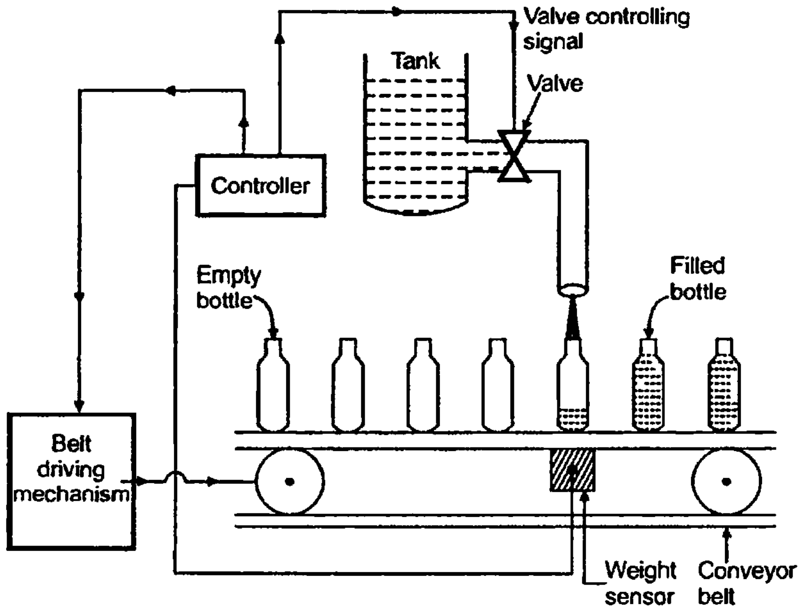


Fig. 5.3 Automatic bottle filling mechanism

In the system shown, conveyor belt is driven by the controller as well as valve position is also controlled by the controller.

When empty bottle comes at the specific position, weight sensor senses the weight and gives signal to controller. Controller stops conveyor movement and opens the valve so bottle starts getting filled. When required level is achieved, again weight sensor sensing the proper weight sends a signal to controller which sends signals to start movement of belt and also closing the valve position with proper time delay till next empty bottle comes at the proper position.

This system can be represented by a block diagram as shown in the Fig. 5.4.

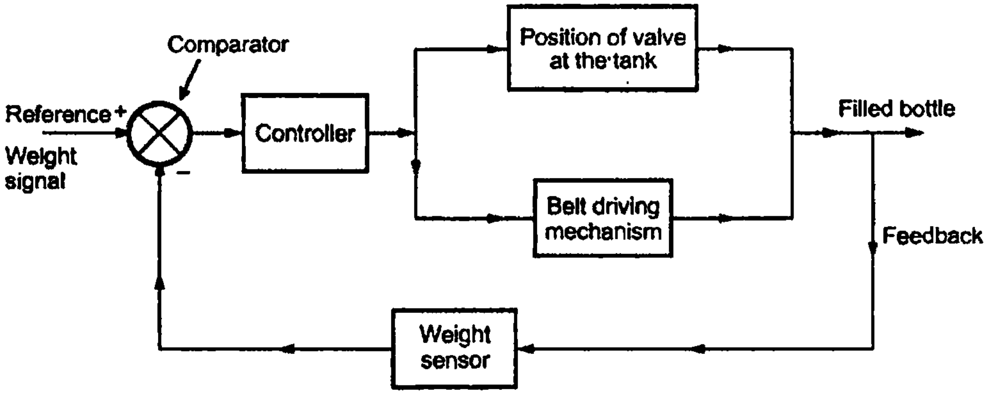


Fig. 5.4 Block diagram

5.1.2 Advantages of Block Diagram

The various advantages of block diagram representation are,

1) Very simple to construct the block diagram for complicated systems.
2) The function of individual element can be visualised from block diagram.
3) Individual as well as overall performance of the system can be studied by using transfer functions shown in the block diagram.
4) Overall closed loop T.F. can be easily calculated by using block diagram reduction rules.

5.1.3 Disadvantages

The various disadvantages of block diagram representation are,

1) Block diagram does not include any information about the physical construction of the system.
2) Source of energy is generally not shown in the block diagram. So number of different block diagrams can be drawn depending upon the point of view of analysis. So block diagram for given system is not unique.

5.2 Simple or Canonical Form of Closed Loop System

Key Point: *A block diagram in which, forward path contains only one block, feedback path contains only one block, one summing point and one takeoff point represents **simple** or **canonical form** of a closed loop system.*

This can be achieved by using block diagram reduction rules without disturbing output of the system. This form is very useful as its closed loop transfer function can be easily calculated by using standard result. This result is derived in this section.

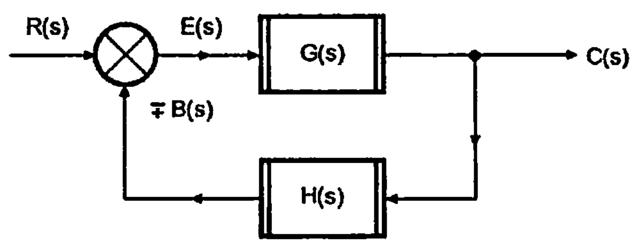


Fig. 5.5

The simple form can be as shown in the Fig. 5.5.

where , R(s) → Laplace of reference input r(t)

C(s) → Laplace of controlled output c(t)

E(s) → Laplace of error signal e(t)

B(s) → Laplace of feedback signal b(t)

G(s). → Equivalent forward path transfer function

H(s) → Equivalent feedback path transfer function.

Key Point: *G(s) and H(s) can be obtained by reducing complicated block diagram by using block diagram reduction rules.*

5.2.1 Derivation of T.F. of Simple Closed Loop System

Referring to the Fig. 5.5, we can write following equations as,

$$E(s) = R(s) \mp B(s) \quad \text{... (1)}$$

$$B(s) = C(s)H(s) \quad \text{... (2)}$$

$$C(s) = E(s)G(s) \quad \text{... (3)}$$

$$B(s) = C(s)H(s) \text{ and substituting in equation (1)}$$

$$E(s) = R(s) \mp C(s)H(s)$$

$$E(s) = \frac{C(s)}{G(s)}$$

$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

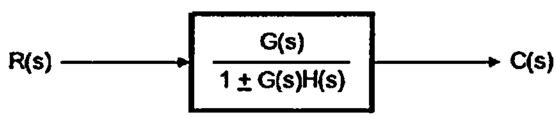
$$C(s) = R(s)G(s) \mp C(s)G(s)H(s)$$

$$\therefore C(s)\,[1 \pm G(s)H(s)] = R(s)\,G(s)$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

Use **+ sign for negative feedback** and use **– sign for positive feedback.**

This can be represented as shown in the Fig. 5.6.



Closed Loop T.F.

Fig. 5.6

Key Point: *This can be used as a standard result to eliminate such simple loops in a complicated system reduction procedure.*

5.3 Rules for Block Diagram Reduction

Any complicated system if brought into its simple form as shown in the Fig. 5.5, its T.F. can be calculated by using the result derived eariler. To bring it into simple form it is necessary to reduce the block diagram but using proper logic such that output of that system and the value of any feedback signal should not get disturbed. This can be achieved by using following mathematical rules while block diagram reduction.

Rule 1 : **Associative law :** Consider two summing points as shown in the Fig. 5.7.

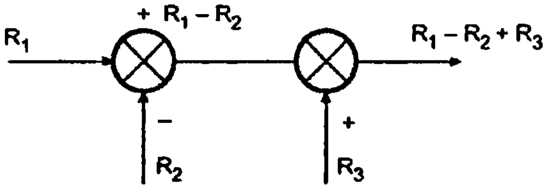


Fig. 5.7

Now change the position of two summing points. Output remains same.

So associative law holds good for summing points which are directly connected to each other (i.e. **there is no intermediate block between two summing points or there is no takeoff point in between the summing points).**

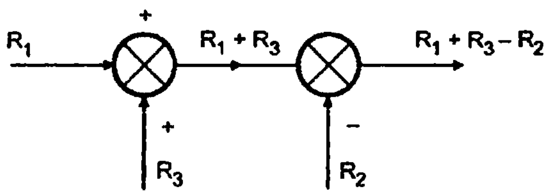


Fig. 5.8

Consider summing points with a block in between as shown in the Fig. 5.9.

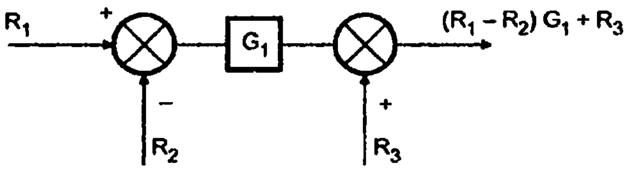


Fig. 5.9

Now interchange two summing points.

Now the output does not remain same.

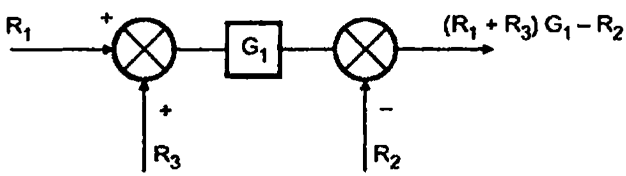


Fig. 5.10

Key Point: *So associative law is applicable to summing points which are directly connected to each other.*

Rule 2 : For blocks in series :

The transfer functions of the blocks which are connected in series get multiplied with each other.

Consider system as shown in the Fig. 5.11.

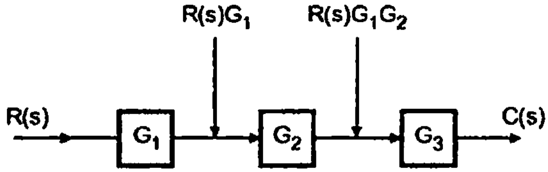


Fig. 5.11

$$C(s) = R(s)\left[G_1 G_2 G_3\right]$$

So instead of three different blocks, only one block with T.F. $\left[G_1 G_2 G_3\right]$ can be used as shown in the Fig. 5.12.

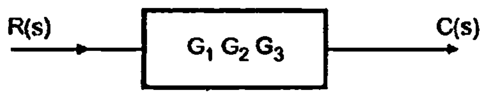


Fig. 5.12

Output in both cases is same.

Key Point: *If there is a takeoff or a summing point in between the blocks the blocks cannot be said to be in series.*

Consider the combination of the blocks as shown in the Fig. 5.13.

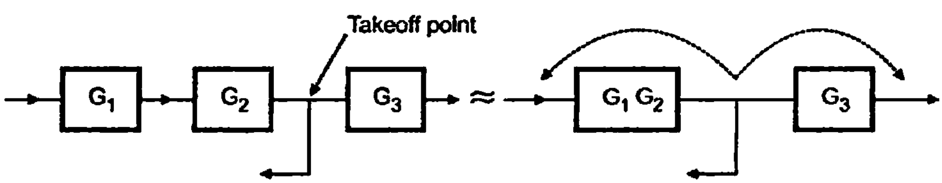


Fig. 5.13

Key Point: *In this combination* G_1G_2 *are in series and can be combined as* G_1G_2 *but* G_3 *is now not in series with* G_1G_2 *as there is takeoff point in between.*

To call G_3 to be in series with G_1G_2 it is necessary to shift the takeoff point before G_1G_2 or after G_3. The rules for such shifting are discussed later.

Rule 3 : For blocks in parallel :

The transfer functions of the blocks which are connected in parallel get added algebraically (considering the sign).

Consider system as shown in the Fig. 5.14.

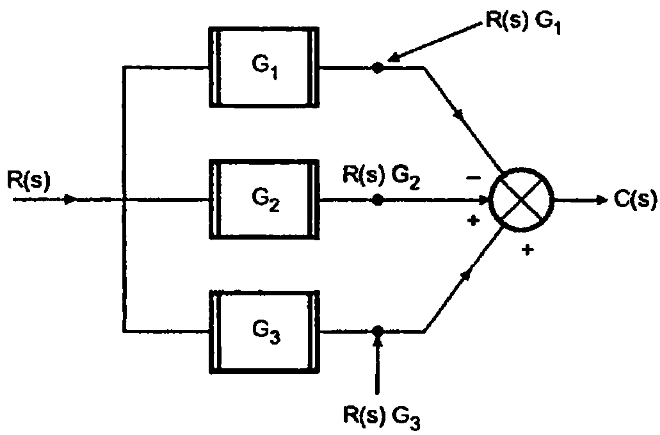


Fig. 5.14

$$C(s) = -R(s)G_1 + R(s)G_2 + R(s)G_3$$
$$= R(s)\left[G_2 + G_3 - G_1\right]$$

Now replace three blocks with only one block with T.F. $G_2 + G_3 - G_1$ (Fig. 5.15)

$$C(s) = R(s)\left[G_2 + G_3 - G_1\right]$$

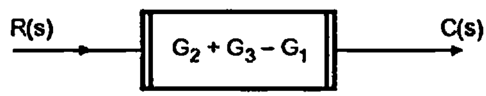


Fig. 5.15

Output is same. So blocks which are in parallel get added algebraically.

Identify the blocks in parallel correctly. The confusing cases are discussed here. Let there exists a takeoff point as shown in the Fig. 5.16 along with blocks G_1, G_2 which appear to be in parallel.

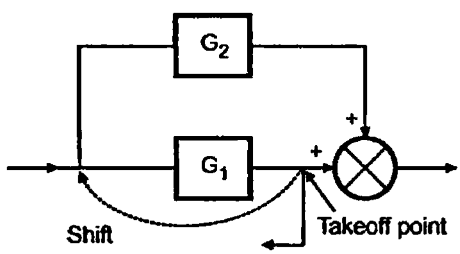
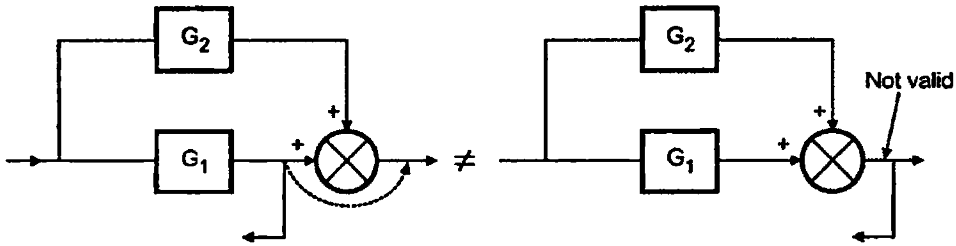


Fig. 5.16

Key Point: ***But unless and until this takeoff point is shifted before the block, blocks cannot be said to be in parallel.***

Shifting of takeoff point is discussed next. Secondly shifting a takeoff point after a summing point needs some adjustments to keep output same. In above case the takeoff point cannot be shown after summing point without any alteration. This type of shifting is discussed as critical rules later as such shifting makes the block diagram complicated and should be avoided as far as possible.



(a) Avoid such shifting as far as possible **Fig. 5.17** **(b) Without any alteration such shifting is invalid**

Similarly consider a configuraion as shown in the Fig. 5.18.

This combination is not the parallel combination of G_1 and H_1.

Key Point: *For a parallel combination the direction of signals through the blocks in parallel must be same.*

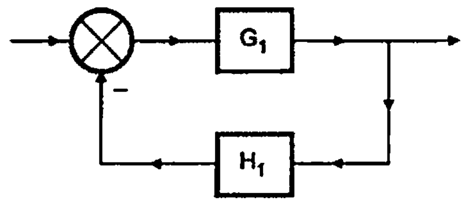


Fig. 5.18

In this case direction of signal through G_1 and H_1 is opposite. Such a combination is called minor **feedback** loop and reduction rule for this is discussed later.

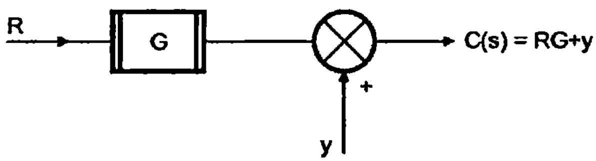


Fig. 5.19

Rule 4 : Shifting a summing point behind the block :

$$C(s) = RG + y$$

Now we have to shift summing point behind the block.

Key Point: *The output must remain same, while shifting a summing point.*

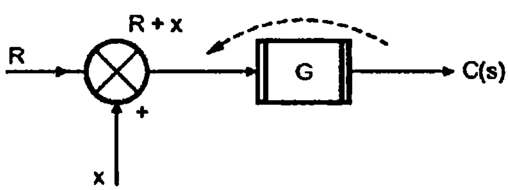


Fig. 5.20

$$\therefore \quad (R + x)G = C(s)$$

$$RG + xG = RG + y$$

$$\therefore \quad xG = y$$

$\therefore x = \frac{y}{G}$ so signal y must be multiplied with $\frac{1}{G}$ to keep output same.

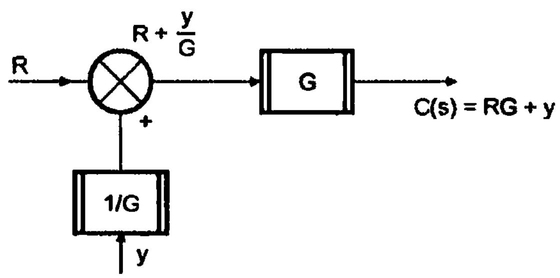


Fig. 5.21

Thus while shifting a summing point behind the block i.e. before the block, add a block having T.F. as reciprocal of the T.F. of the block before which summing point is to be shifted, **in series with all the signals** at that summing point.

Rule 5 : Shifting a summing point beyond the block :

Consider the combination shown in the Fig. 5.22.

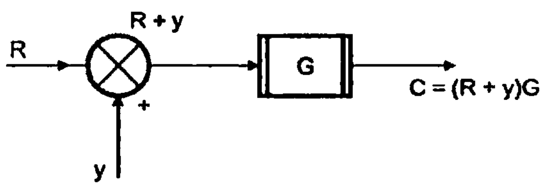


Fig. 5.22

Now to shift summing point after the block keeping output same, consider the shifted summing point without any change.

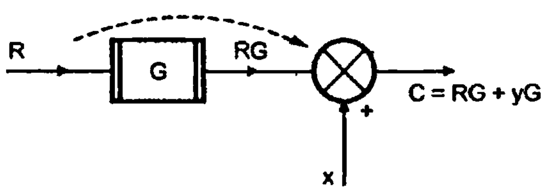


Fig. 5.23

$$\therefore \quad RG + x = RG + yG$$

$$\therefore \quad x = yG$$

i.e. signal y must get multiplied with T.F. of the block beyond which summing point is to be shifted.

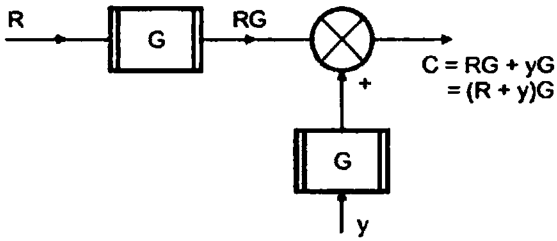


Fig. 5.24

Thus while shifting a summing point after a block, add a block having T.F. same as that of block after which summing point is to be shifted, in series with all the signals at that summing point.

Rule 6 : Shifting a takeoff point behind the block :

Consider the combination shown in the Fig. 5.25.

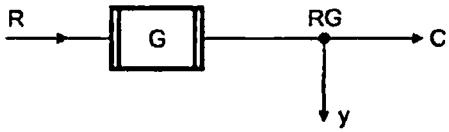


Fig. 5.25

$$C = RG$$

$$y = RG$$

Key Point: ***To shift takeoff point behind the block value of signal takingoff must remain same.***

Though shifting of takeoff point without any change does not affect output directly, the value of feedback signal which is changed affects the output indirectly which must be kept same. But without any change it is just R as shown in the Fig. 5.26.

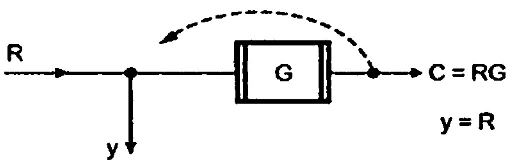


Fig. 5.26

But it must be equal to RG. So a block with T.F. G must be introduced in series with signal takingoff after the block.

Thus while shifting a takeoff point behind the block, add a block having T.F. same as that of the block behind which takeoff point is to be shifted, in series with all the signals takingoff from that takeoff point.

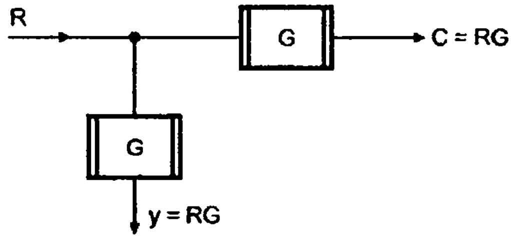


Fig. 5.27

Rule 7 : Shifting a takeoff point beyond the block :

Consider the combination shown in the Fig. 5.28.

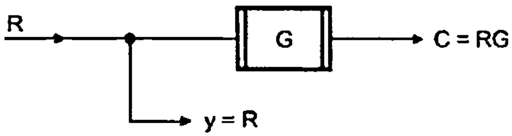


Fig. 5.28

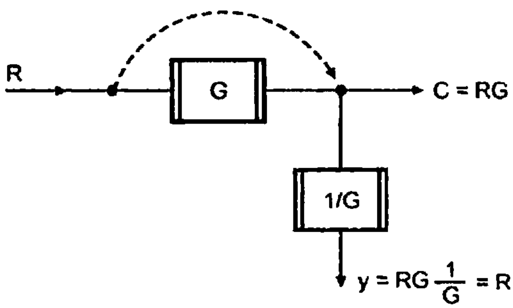


Fig. 5.29

To shift takeoff point beyond the block, value of 'y' must remain same. To keep value of 'y' constant it must be multiplied by '1/G'.

While shifting a takeoff point beyond the block, add a block in series with all the signals which are takingoff from that point, having T.F. as reciprocal of the T.F. of the block beyond which takeoff point is to be shifted.

Rule 8 : Removing minor feedback loop :

This includes the removal of internal simple forms of the loops by using standard result derived earlier in section 5.2.

Key Point: *After eliminating such a minor loop if summing point carries only one signal input and one signal output, it should be removed from the block diagram to avoid further confusion.*

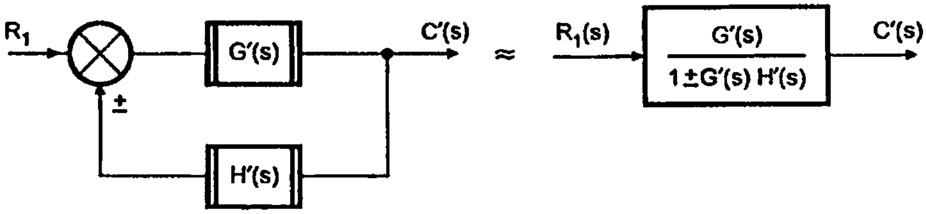


Fig. 5.30

Rule 9 : For multiple input system use superposition theorem :

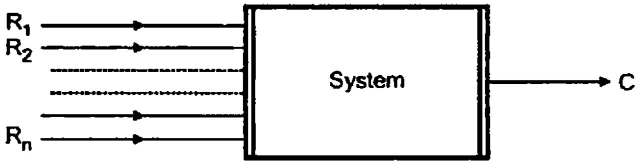


Fig. 5.31

Consider only one input at a time treating all other as zero.

Consider R_1, $R_2 = R_3 = \ldots\ldots R_n = 0$ and find output C_1.

Then consider R_2, $R_1 = R_3 = \ldots\ldots R_n = 0$ and find output C_2.

At the end when all inputs are covered, take algebraic sum of all the outputs.

Total output $C = C_1 + C_2 + \ldots\ldots C_n$

Same logic can be extended to find the outputs if system is multiple input multiple output type. Separate ratio of each output with each input is to be calculated, assuming all other inputs and outputs zero. Then such components of outputs can be added to get resultant outputs of the system. In very few cases, it is not possible to reduce the block diagram to its simple form by use of above discussed nine rules. In such case there is a requirement to shift a summing point before or after a takeoff point to solve the problem. These rules are discussed below but reader should avoid to use these rules unless and until it is the requirement of the problem. Use of these rules in simple problems may complicate the block diagram.

5.3.1 Critical Rules

Rule 10 : Shifting takeoff point after a summing point. Consider a situation as shown in Fig. 5.32.

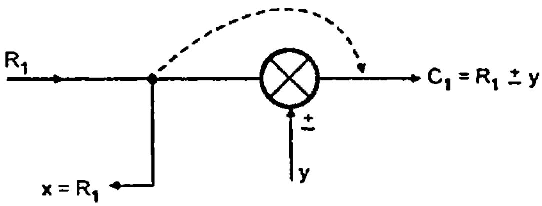


Fig. 5.32

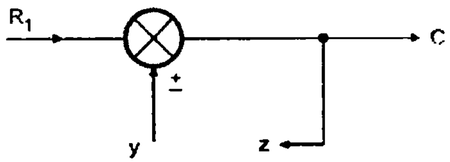


Fig. 5.33

Now after shifting the takeoff point, let signal takingoff be 'z' as shown in the Fig. 5.33.

Now $z = R_1 \pm y$

But we want feedback signal as $x = R_1$ only.

So signal 'y' must be inverted and added to C_1 to keep feedback signal value same. And to add the signal, summing point must be introduced in series with takeoff signal. So modified configuration becomes as shown in the Fig. 5.34.

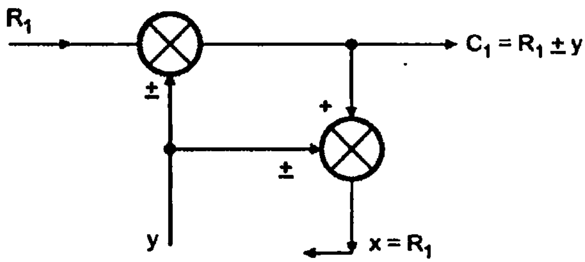


Fig. 5.34

Rule 11 : Shifting takeoff point before a summing point :

Consider a situation as shown in the Fig. 5.35.

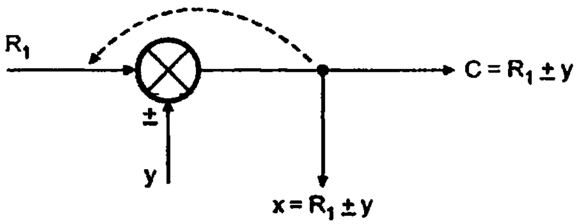


Fig. 5.35

Now after shifting the takeoff point, let signal takingoff be 'z' as shown in the Fig. 5.36.

Now $z = R_1$ only because nothing is changed.

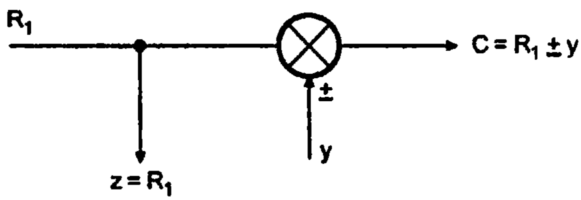


Fig. 5.36

But we want feedback signal x which is $R_1 \pm y$. Hence to z, signal 'y' must be added with same sign as it is present at summing point, which can be achieved by using summing point in series with takeoff signal as shown in the Fig. 5.37.

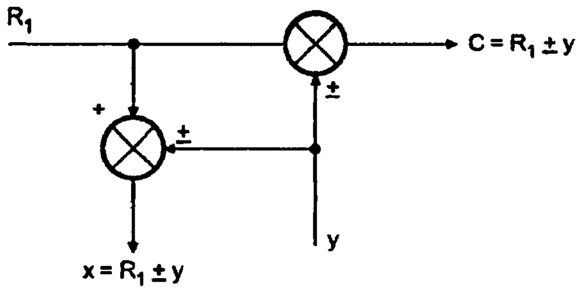


Fig. 5.37

Thus it can be noticed that shifting of takeoff point before or after a summing point adds an additional summing point in the block diagram and this complicates the block diagram. No doubt, in some rare cases, it is not possible to reduce the block diagram without such shifting of takeoff point before or after a summing point. Apart from such cases, do not use such shifting, which will complicate the simple block diagrams.

Key Point: *Do not use these rules unless and until problem really needs them.*

5.3.2 Converting Nonunity Feedback to Unity Feedback

Consider a nonunity feedback system, shown in the Fig. 5.38.

It is minor feedback loop and hence,

$$\frac{C}{R} = \frac{G}{1+GH} \quad ...(1)$$

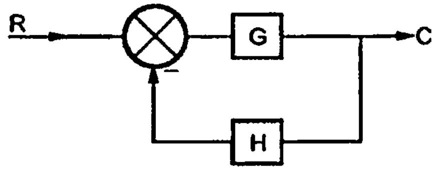


Fig. 5.38 Nonunity feedback system

It is to be converted to unity feedback system. **But the denominator of closed loop transfer function must remain same as before.**

$\therefore \quad \frac{C}{R} = \frac{G'}{1+G'}$ with unity feedback i.e. H = 1

$\therefore \quad 1 + G' = 1 + GH$ i.e. $G' = GH$

Thus, $$\frac{C}{R} = \frac{GH}{1+GH} \quad ...(2)$$

But transfer function must remain same hence it is necessary to introduce block of $\frac{1}{H}$ in series with R. Thus converted unity feedback system is as shown in the Fig. 5.39.

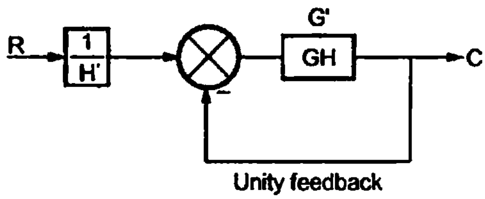


Fig. 5.39 Equivalent unity feedback system

$$\therefore \quad \frac{C}{R} = \frac{1}{H} \times \frac{GH}{1+GH} = \frac{G}{1+GH} \qquad \text{...same as before}$$

5.3.3 Procedure to Solve Block Diagram Reduction Problems

Step 1 : Reduce the blocks connected in series.

Step 2 : Reduce the blocks connected in parallel.

Step 3 : Reduce the minor internal feedback loops.

Step 4 : As far as possible **try to shift takeoff points towards right and summing points to the left.** Unless and until it is the requirement of problem do not use rule 10 and 11.

Step 5 : Repeat steps 1 to 4 till simple form is obtained.

Step 6 : Using standard T.F. of simple closed loop system, obtain the closed loop T.F. $\frac{C(s)}{R(s)}$ of the overall system.

Example 5.1 : *Determine the transfer function C(s) / R(s) of the system shown in the Fig. 5.40.*

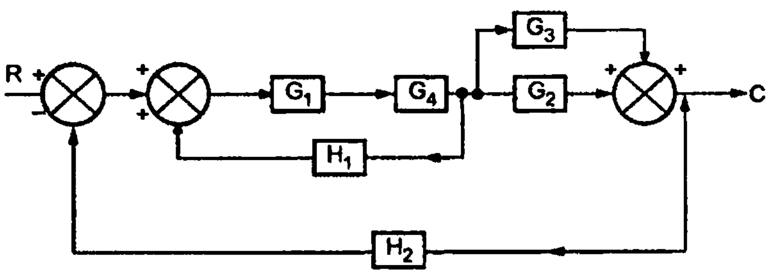
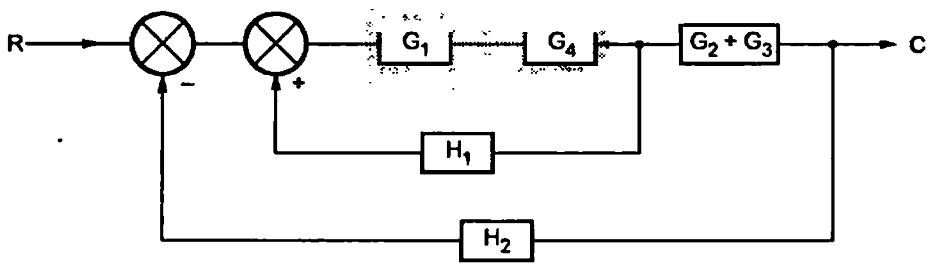


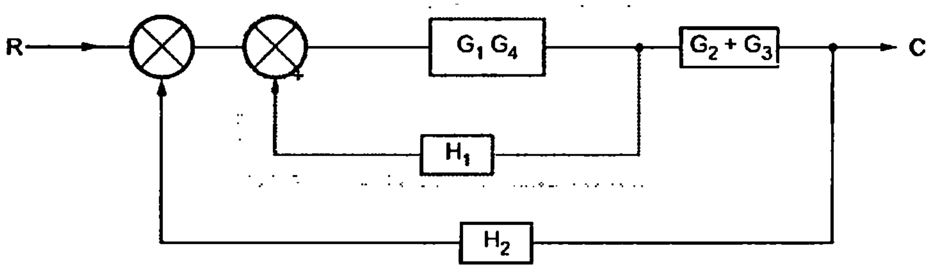
Fig. 5.40

Solution : The blocks G_2 and G_3 are in parallel so combining them as (G_2+G_3) we get,

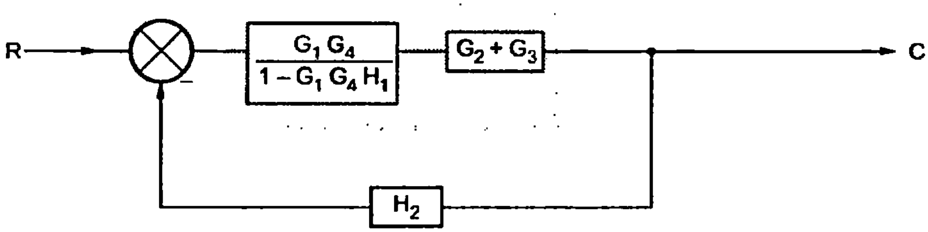
Blocks in series

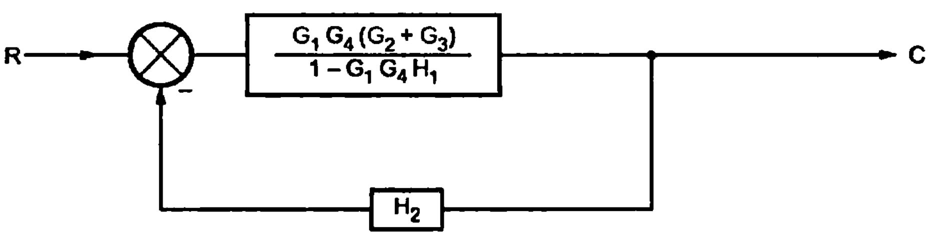


Minor feedback loop



Blocks in series





Feedback loop

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{G_1 G_4 (G_2+G_3)}{1-G_1 G_4 H_1}}{1+\dfrac{G_1 G_4 (G_2+G_3) H_2}{1-G_1 G_4 H_1}}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_4 (G_2+G_3)}{1-G_1 G_4 H_1+G_1 G_4 (G_2+G_3) H_2}} \quad \text{... Ans.}$$

5.4 Analysis of Multiple Input Multiple Output Systems

In these problems, the law of superposition is to be used, considering each input separately. While assuming the other inputs as zero, most of the times if only input is applied to the summing point, summing point is to be removed if not necessary.

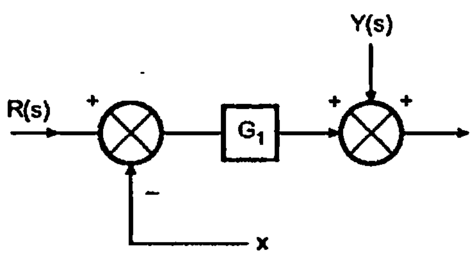


Fig. 5.41

Key Point: *While removing summing point if sign of the signal present, at that summing point which is to be removed is negative then it must be carried forward in the further analysis.*

This can be achieved by introducing a **block of transfer function –1 in series with that signal. This is the important step to be remembered while solving problems on multiple input multiple output systems.**

e.g. consider a part of system showing two inputs R(s) and Y(s).

Other details are not shown for simplicity.

When R(s) is considered alone, Y(s) must be assumed zero and summing point at Y(s) can be removed as with Y(s) = 0 there remains only a single signal present at that point so system gets modified as shown in the Fig. 5.42.

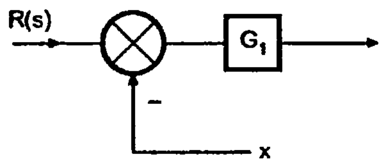


Fig. 5.42

Now sign of signal from block G_1 is positive at the summing point which is removed, hence there is no need of adding any other block.

Now when R(s) = 0 with Y(s) active, the summing point at R(s) also can be removed. But now **sign of the signal 'x' at that summing point is negative which must be considered and carried forward for further analysis.** This is possible by adding a block of –1 in series with x without altering any other sign. This avoids the confusion and problem can be solved without any error.

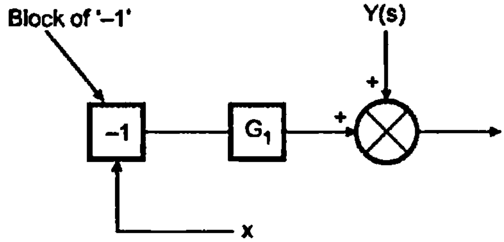


Fig. 5.43

Example 5.2 : *Using block diagram reduction technique find the transfer function from each input to the output C for the system shown in the Fig. 5.44.*

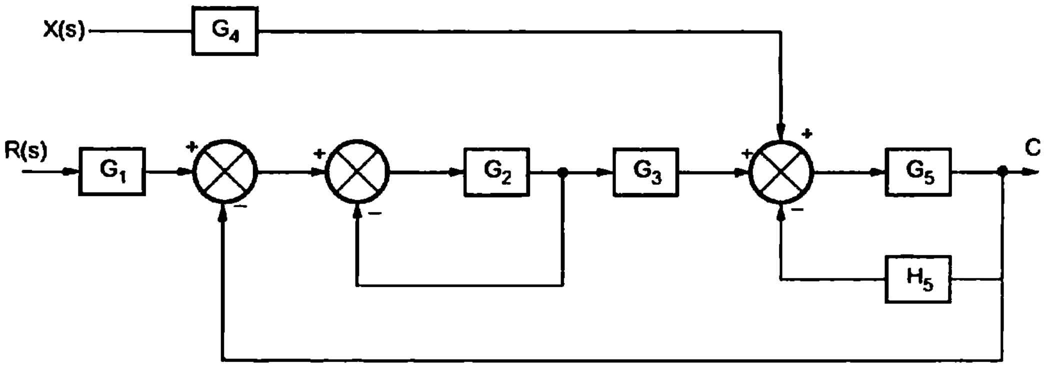


Fig. 5.44

Solution : With X(s) = 0, block diagram reduces as,

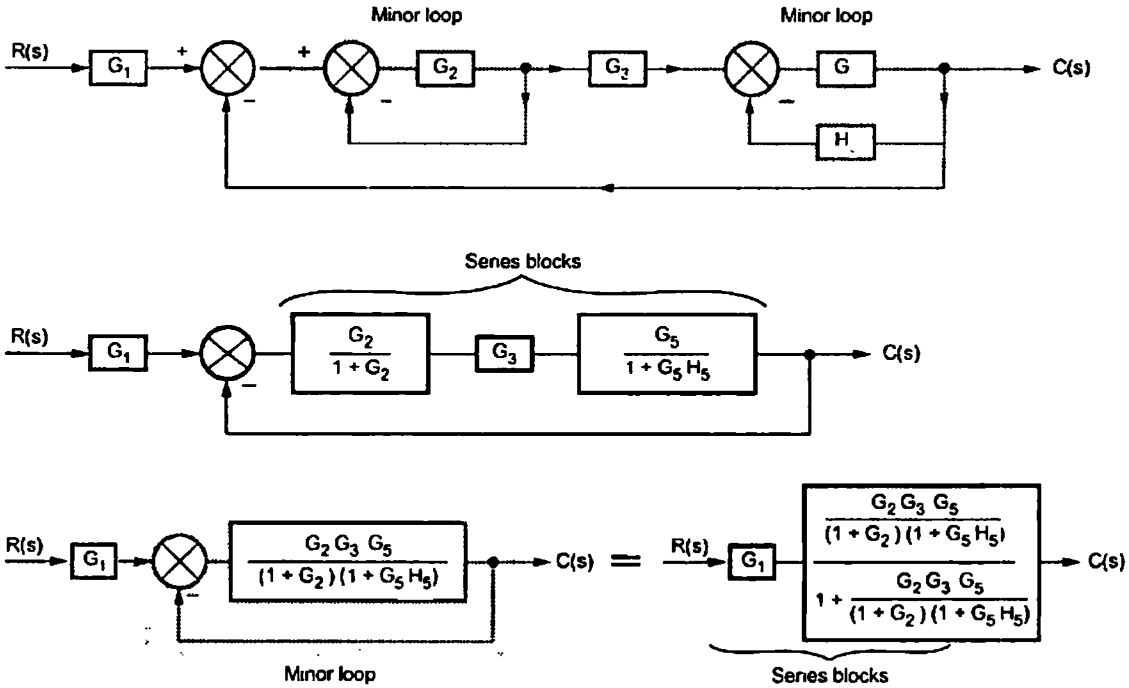


Fig. 5.44(a)

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_5}{1+G_2+G_5H_5+G_2G_5H_5+G_2G_3G_5}}$$

With R(s) = 0, G_1 vanishes but minus sign at summing point must be considered by introducing block of – 1, as shown.

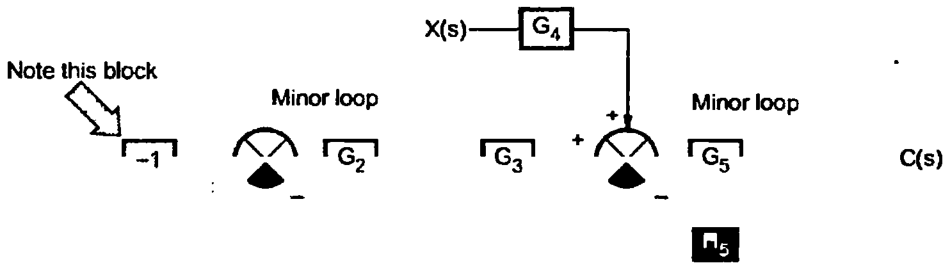


Fig. 5.44(b)

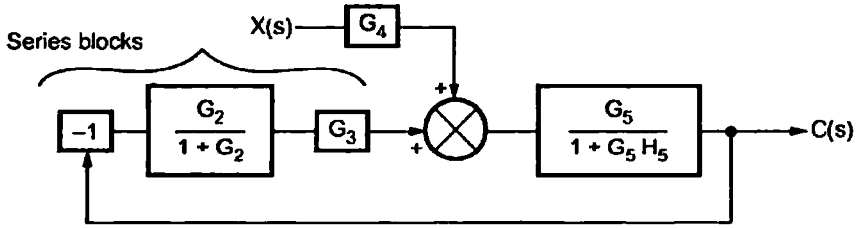
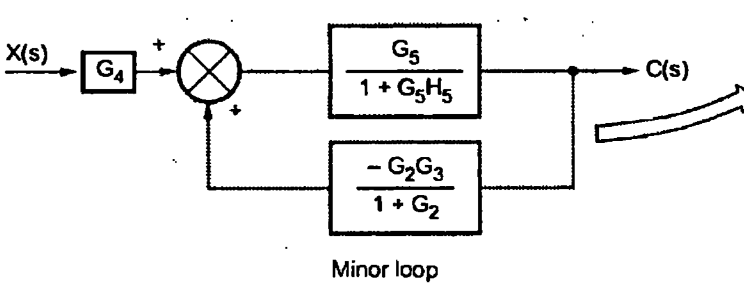


Fig. 5.44(c)

Rearranging the input - output we get,



$$\frac{\dfrac{G_5}{1+G_5H_5}}{1-\left(\dfrac{G_5}{1+G_5H_5}\right)\cdot\left(\dfrac{-G_2G_3}{1+G_2}\right)}$$

Fig. 5.44(d)

$$\therefore \quad \frac{C(s)}{X(s)} = \frac{G_4G_5(1+G_2)}{1+G_5H_5+G_2+G_2G_5H_5+G_2G_3G_5}$$

5.5 Block Diagram from System Equations

The block diagram can be constructed from the set of equations representing the system. **The addition and subtraction of the various terms is represented by the summing points while multiplication factors are represented by the blocks of the respective transfer functions.**

Consider an equation,

$$V_2 = 4V_1 - 5V_3$$

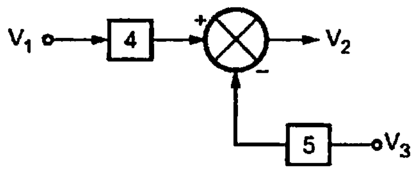


Fig. 5.45

The corresponding block diagram has blocks of transfer functions 4 and 5 along with a summing point to represent the subtraction. It is as shown in the Fig. 5.45.

The signals V_1 and V_3 are taken from their generating points and available V_2 can be used further for generating other variables and output.

Combining the block diagrams of all the system equations, the complete block diagram representations of the system can be obtained.

Key Point: *The block diagrams of electrical systems can be easily obtained by this method. As methods of writing equations for electrical system may vary, the block diagram is not unique for the electrical systems.*

➡ **Example 5.3 :** *Obtain the block diagram for the given electrical network.*

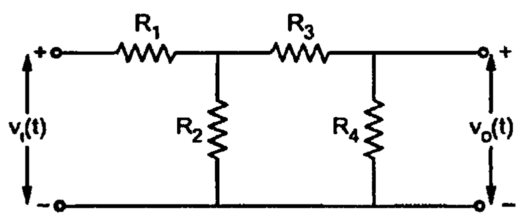


Fig. 5.46

Solution : Convert the given network into Laplace domain and assume the currents as shown in the Fig. 5.46(a).

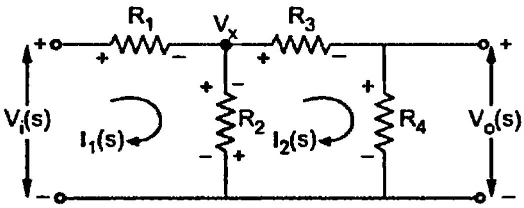


Fig. 5.46(a)

The KVL equations for the two loops are,

$$-I_1R_1 - I_1R_2 + I_2R_2 + V_i = 0 \qquad \text{i.e} \quad I_1 = V_i\left(\frac{1}{R_1+R_2}\right) + I_2\left(\frac{R_2}{R_1+R_2}\right) \qquad ...(1)$$

$$-I_2R_3 - I_2R_4 - I_2R_2 + I_1R_2 = 0 \qquad \text{i.e} \quad I_2 = I_1\left[\frac{R_2}{R_2+R_3+R_4}\right] \qquad ...(2)$$

And $V_o = I_2 R_4$

The block diagrams for the three equations are,

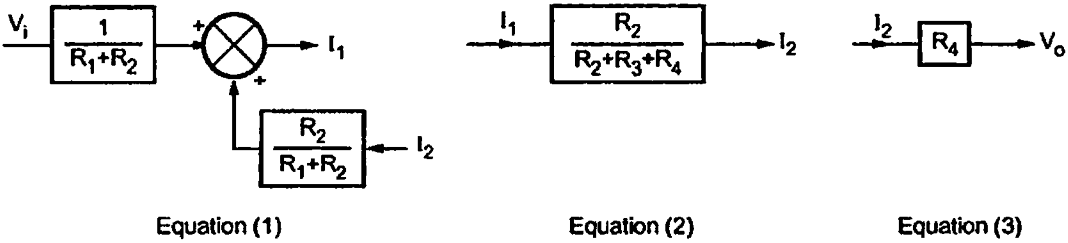


Fig. 5.46(b)

Thus the overall block diagram is as shown in the Fig. 5.46(c).

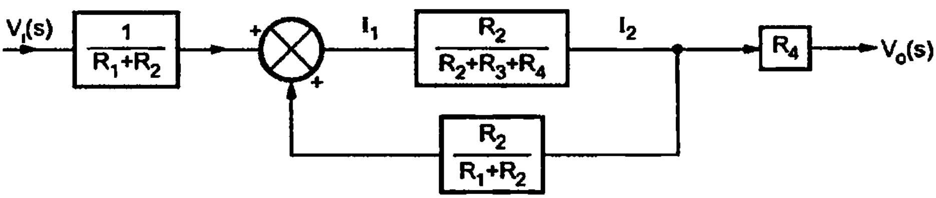
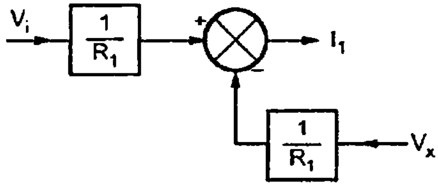


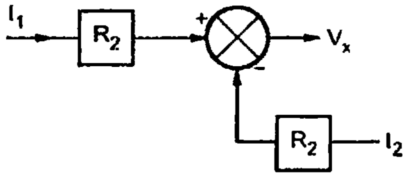
Fig. 5.46(c)

If the equations are written with different method, the different block diagram can be obtained but the transfer function of the system remains same. For the above network we can write equations as,

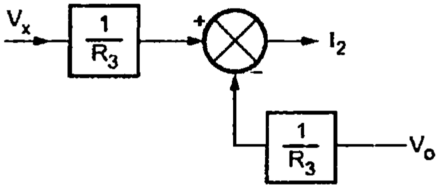
$$I_1 = \frac{V_i - V_x}{R_1} = \frac{V_i}{R_1} - \frac{V_x}{R_1}$$



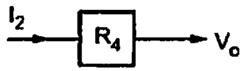
$$V_x = (I_1 - I_2)R_2 = I_1R_2 - I_2R_2$$



$$I_2 = \frac{V_x - V_o}{R_3} = \frac{V_x}{R_3} - \frac{V_o}{R_3}$$



$$V_o = I_2 R_4$$



The overall block diagram is shown in the Fig. 5.46(d).

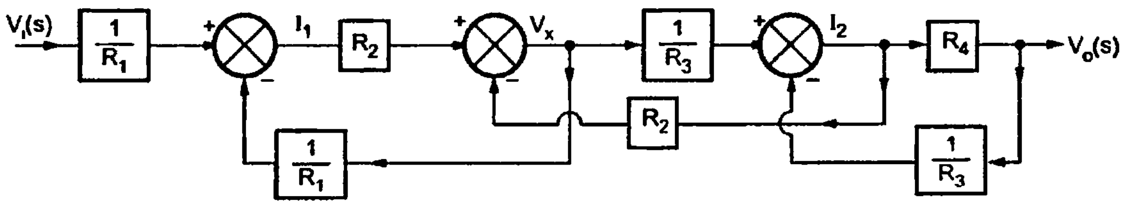
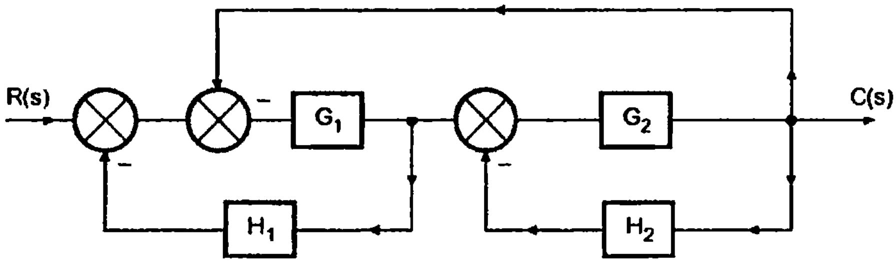


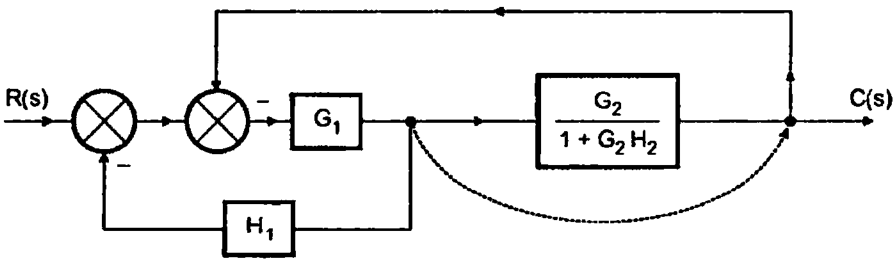
Fig. 5.46(d)

Examples with Solutions

Example 5.4 : *Reduce the block diagram and obtain its closed loop T.F. C(s)/R(s).*

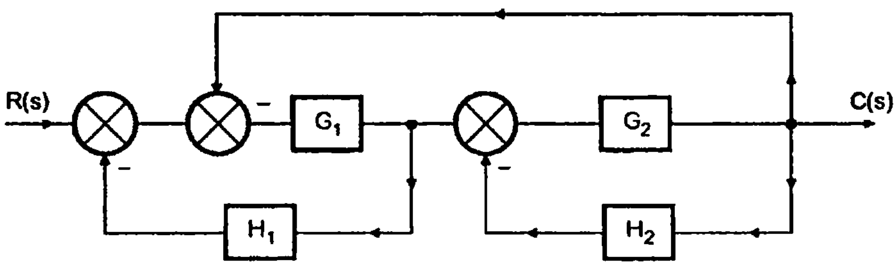


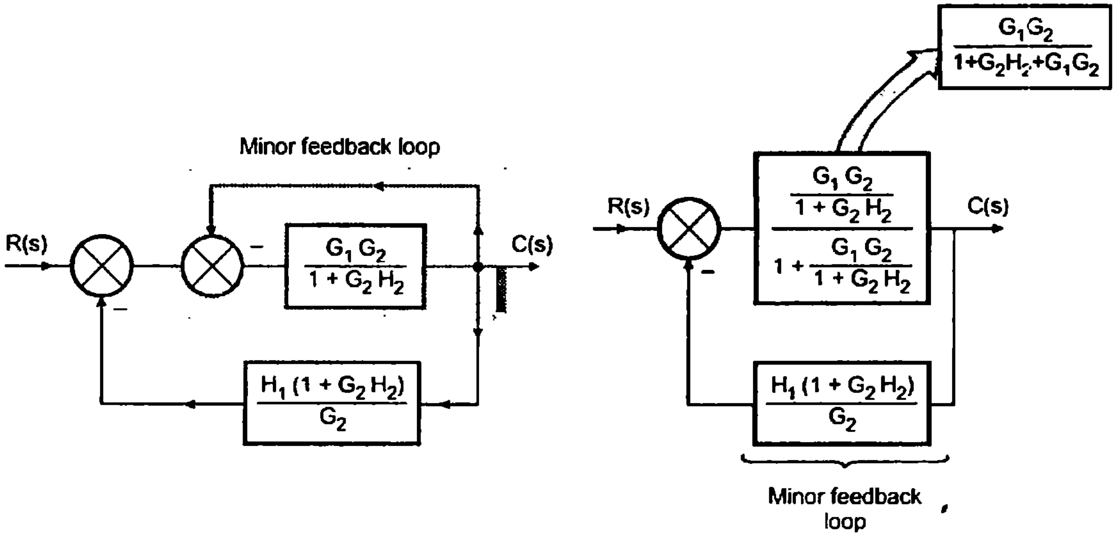
Solution : No blocks are connected in series or parallel. Blocks having transfer functions G_2 and H_2 form minor feedback loop so eliminating that loop we get,

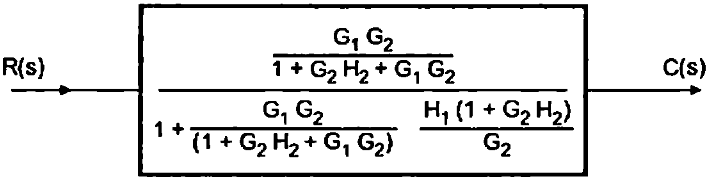


Key Point: *Always try to shift takeoff point towards right i.e. output side and summing point towards left i.e. input side.*

So shift takeoff point after G_1 to the right. While doing so, it is necessary to add a block having T.F. equal to reciprocal of the T.F. of the block after which takeoff point is to be shifted, in series with signal at that takeoff point. So in series with H_1 we get a block of $1 / \left(\frac{G_2}{1+G_2H_2}\right)$ i.e $\frac{1+G_2H_2}{G_2}$ after shifting takeoff point.



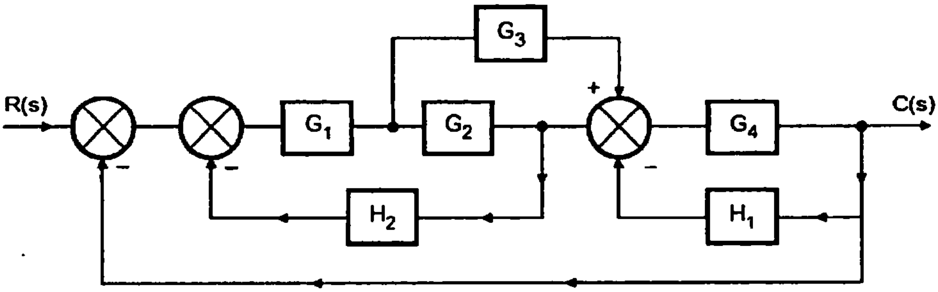




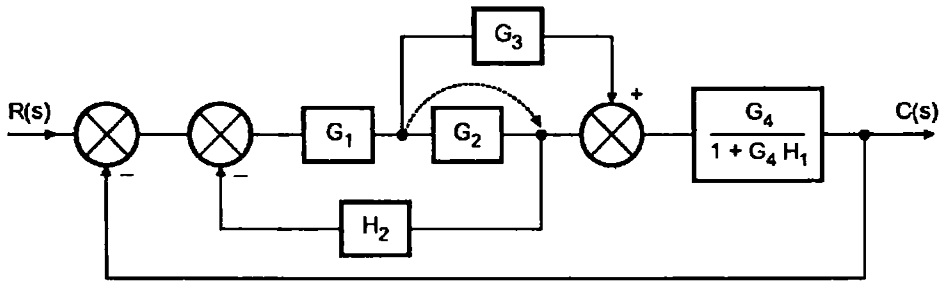
Simplifying,

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2 + G_2 H_2 + G_1 H_1 + G_1 G_2 H_1 H_2}$$

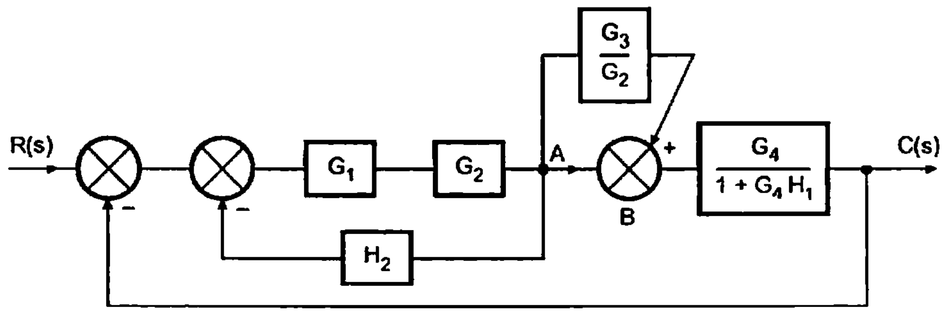
Example 5.5 : *Reduce the block diagram to its simple form and hence obtain C(s)/R(s).*



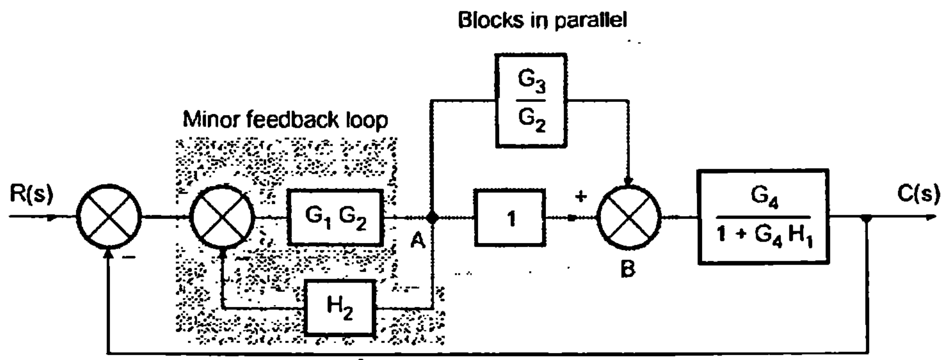
Solution : The blocks G_4 and H_1 form minor feedback loop,

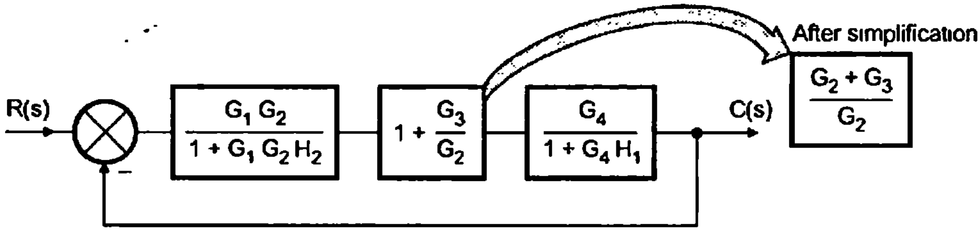


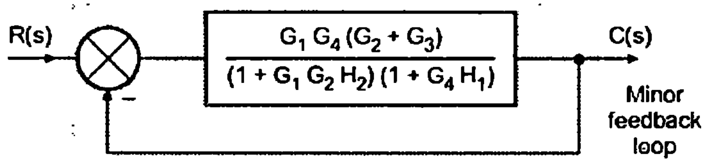
Shifting takeoff point as shown ,

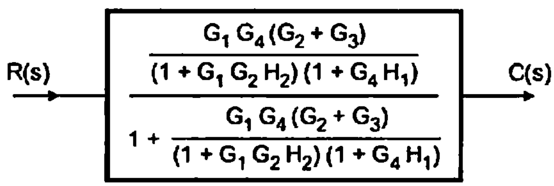


The signal from takeoff point A reaching to summing point B is without any block i.e. gain of that branch joining A to B is one. So introduce block of T.F. '1' in between A and B to avoid further confusion as shown below.



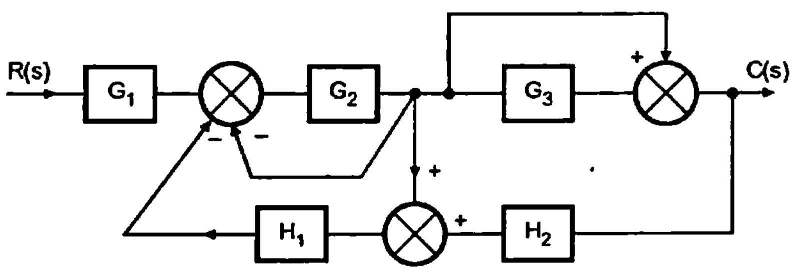




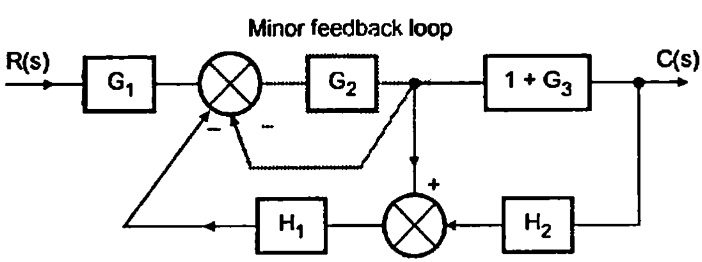


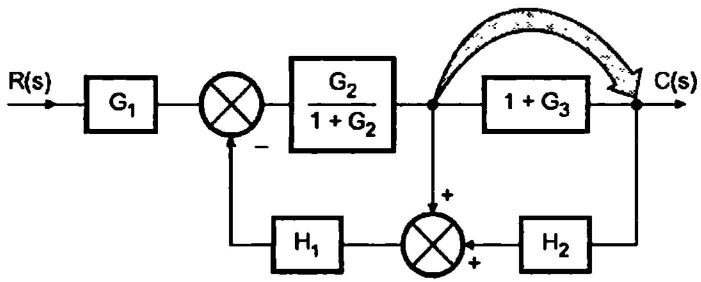
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_4 (G_2 + G_3)}{1 + G_1 G_2 H_2 + G_4 H_1 + G_1 G_2 G_4 H_1 H_2 + G_1 G_4 (G_2 + G_3)}$$

Example 5.6 : *Reduce the block diagram using reduction rules and obtain C(s)/R(s).*

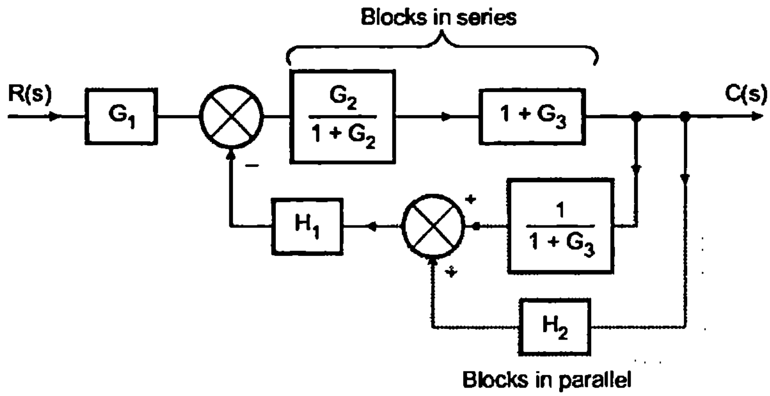


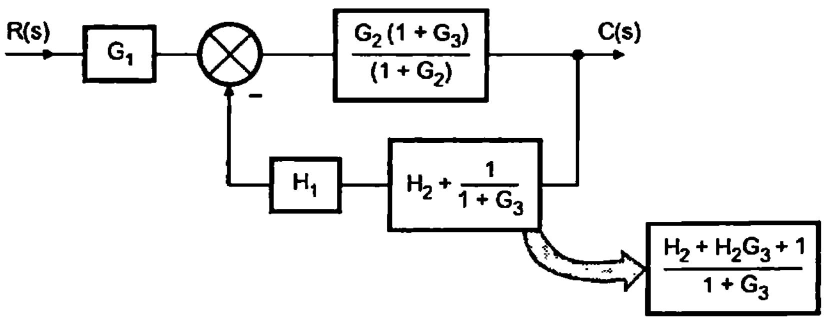
Solution : Block with T.F. G_3 and unity gain block are in parallel so combining them we get,

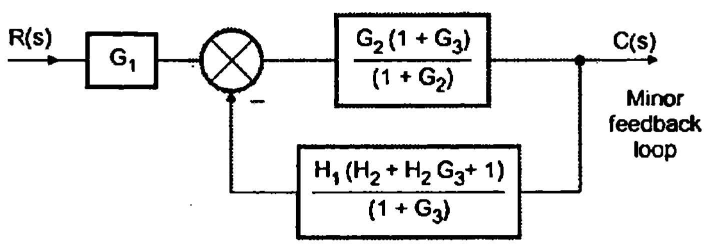


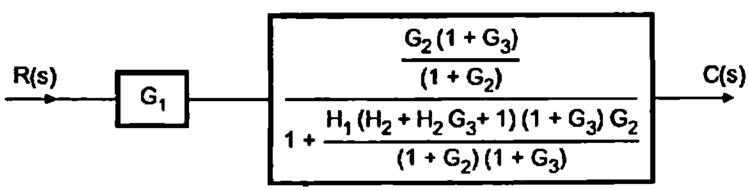


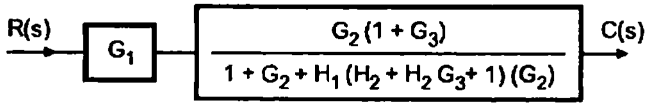
Shifting takeoff point,





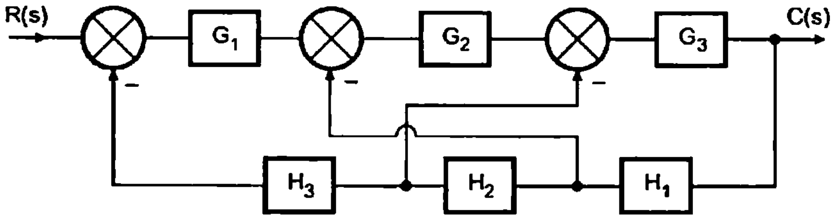




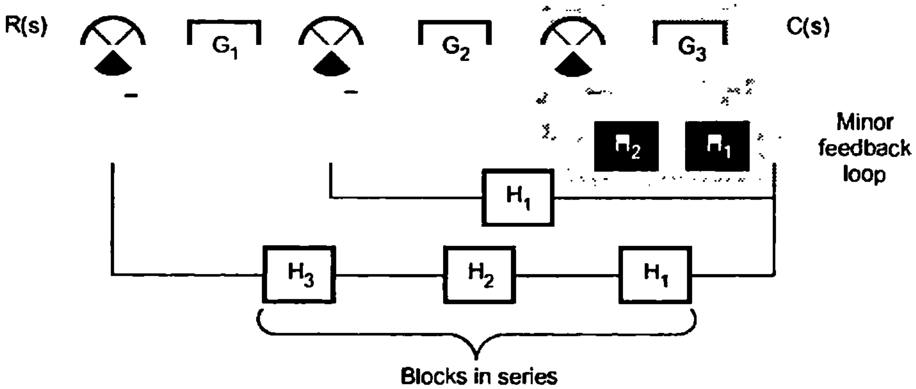


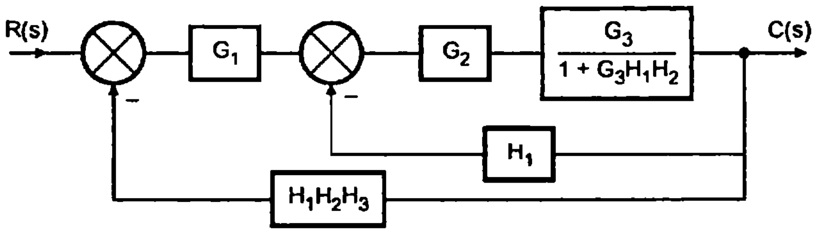
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_2 (1+G_3)}{1+G_2+H_1 G_2+H_1 H_2 G_2+H_1 H_2 G_2 G_3}$$

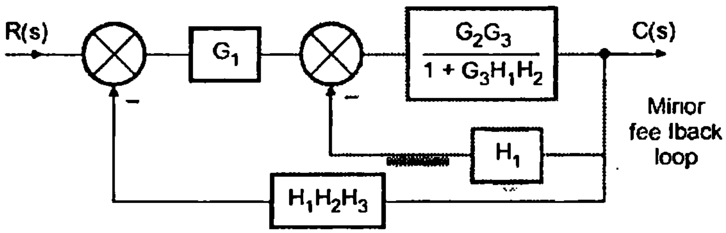
Example 5.7 : *Obtain C(s)/R(s) using block diagram reduction rules.*

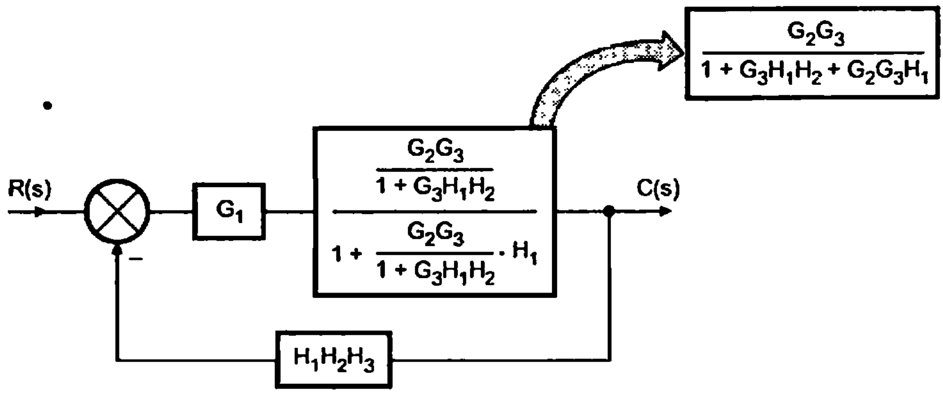


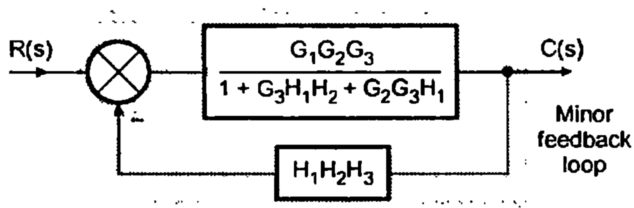
Solution : Separating out the feedbacks at different summing points we can rearrange the above block diagram as below.

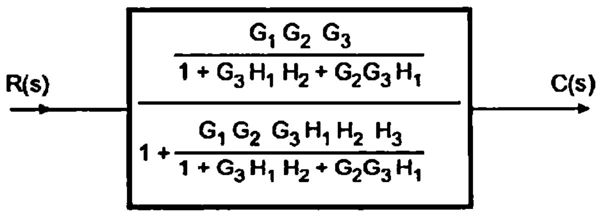






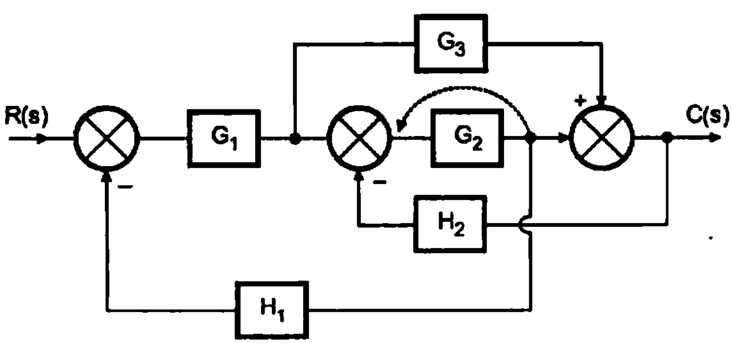




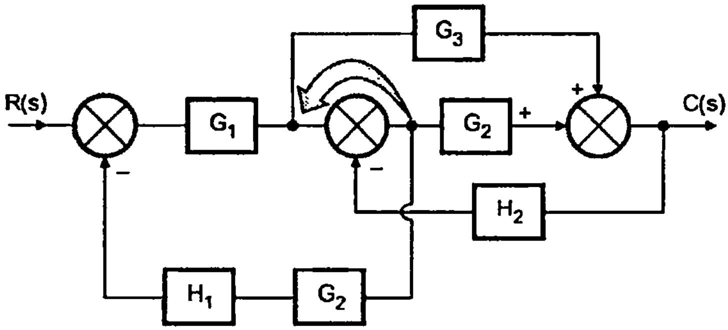


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1+G_3 H_1 H_2+G_2 G_3 H_1+G_1 G_2 G_3 H_1 H_2 H_3}$$

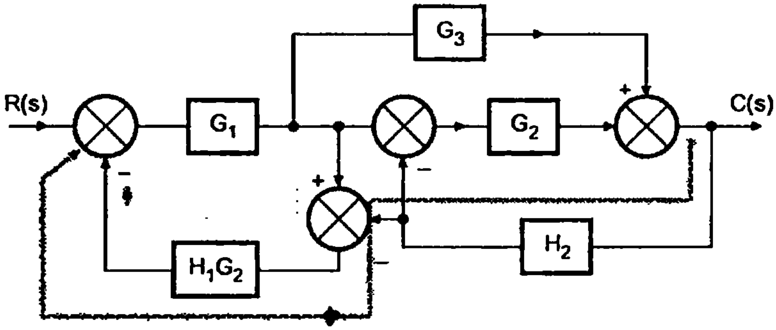
Example 5.8 : *Use of Rule No. 10, critical rule illustration. Reduce the block diagram and obtain* $\frac{C(s)}{R(s)}$.



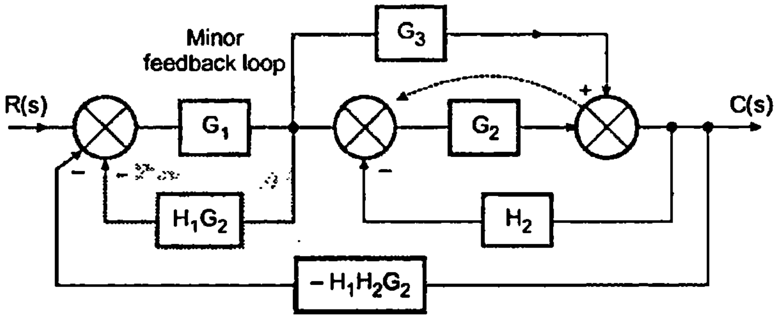
Solution : Shifting takeoff point before G_2.



Shifing takeoff point before summing point using critical rule No. 10,



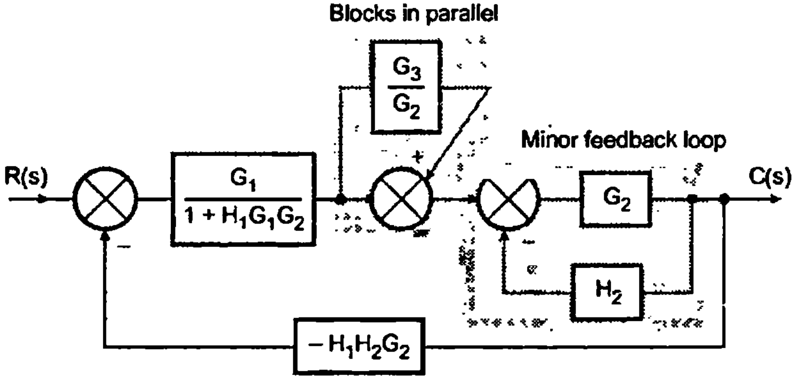
Separating the paths in the feedback path shown dotted,

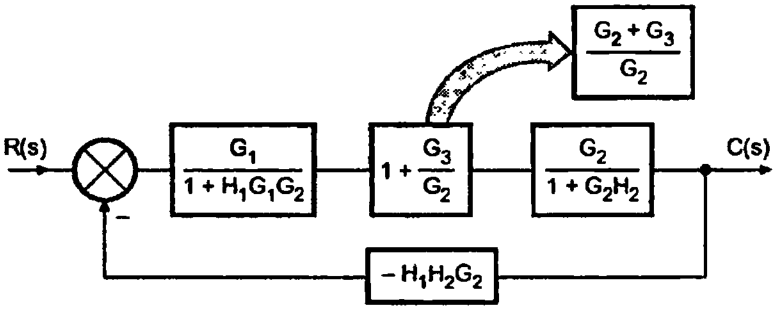


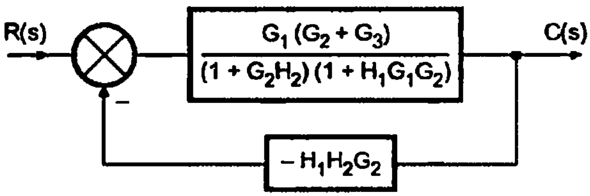
Key Point: *Remember that though the paths through summing point are separated, signs at the summing points to those paths must be carried as it is.*

Hence after H_2, carry the negative sign and then $H_1\ G_2$ to get $-H_1\ H_2\ G_2$.

Shifting summing point as shown and then interchanging the two summing points using Associative law we get,



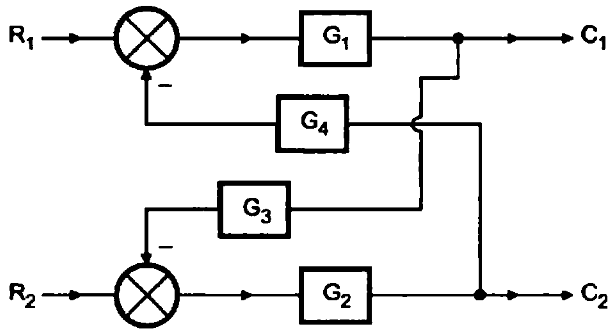




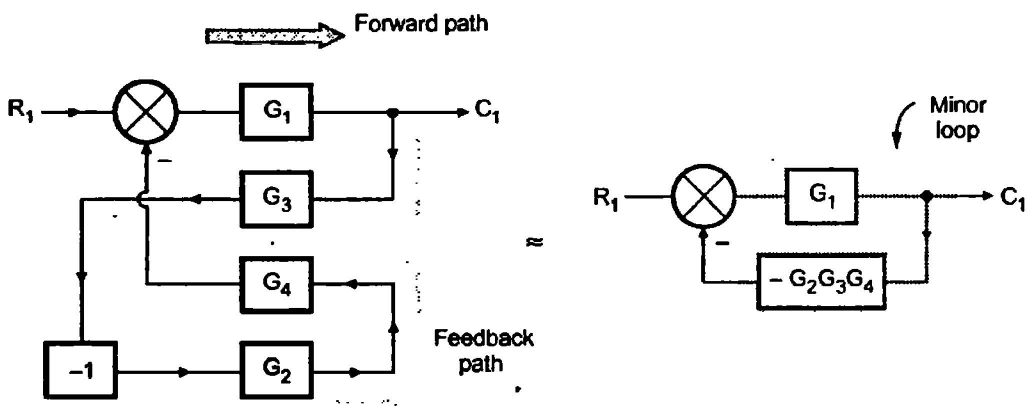
$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1(G_2+G_3)}{(1+G_2H_2)(1+H_1G_1G_2)}}{1+\dfrac{G_1(G_2+G_3)(-H_1H_2G_2)}{(1+G_2H_2)(1+H_1G_1G_2)}}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1(G_2+G_3)}{1+G_2H_2+H_1G_1G_2-G_1G_2G_3H_1H_2}}$$

Example 5.9 : *Obtain the expression for C_1 and C_2 for the given multiple input multiple output system.*

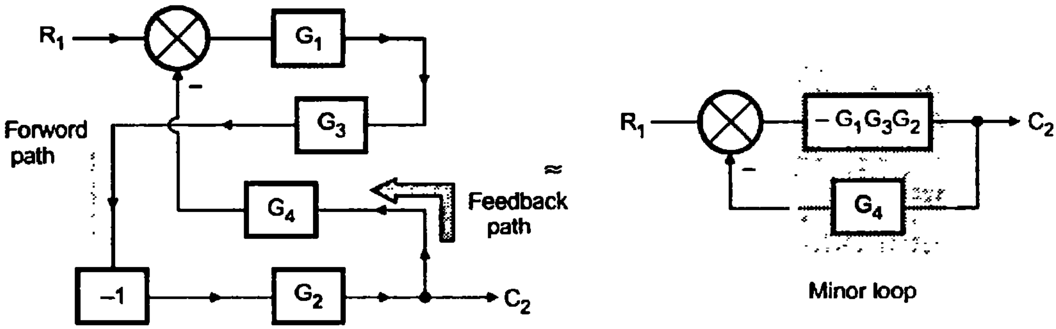


Solution : In this case there are two inputs and two outputs. Consider one input at a time assuming other zero and one output at a time. Consider R_1 acting, $R_2 = 0$ and C_2 not considered $R_2 = 0$ and C_2 is suppressed (not considered). C_2 suppressed does not mean that $C_2 = 0$. Only it is not the focus of interest while C_1 is considered. As $R_2 = 0$, summing point at R_2 can be removed but block of '−1' must be introduced in series with the signal which is shown negative at that summing point.



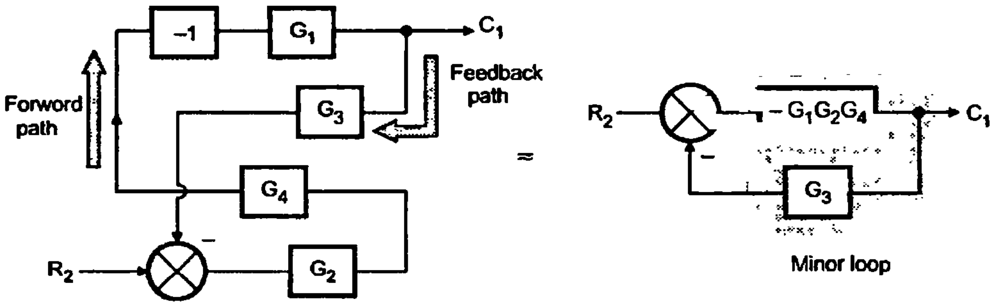
$$\frac{C_1}{R_1} = \frac{G_1}{1+[G_1][-G_2 G_3 G_4]} = \frac{G_1}{1-G_1 G_2 G_3 G_4}$$

For $\frac{C_2}{R_1}$, assume C_1 suppressed.



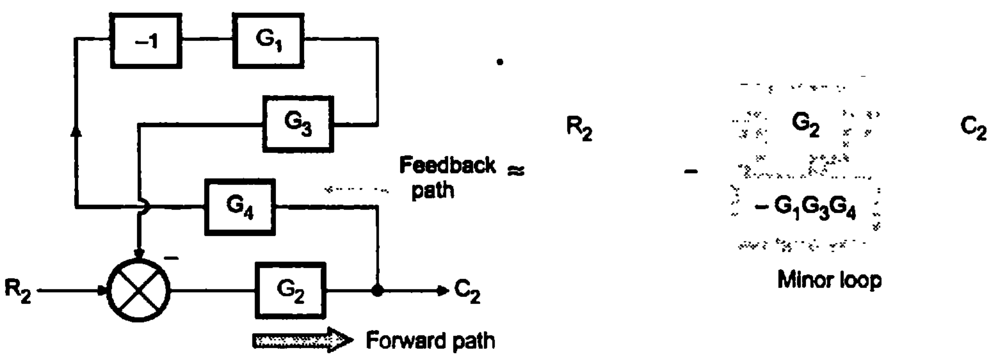
$$\therefore \quad \frac{C_2}{R_1} = \frac{-G_1\,G_3\,G_2}{1+[-G_1\,G_3\,G_2][G_4]} = \frac{-G_1\,G_2\,G_3}{1-G_1\,G_2\,G_3\,G_4}$$

For $\frac{C_1}{R_2}$, $R_1 = 0$ and C_2 is suppressed.



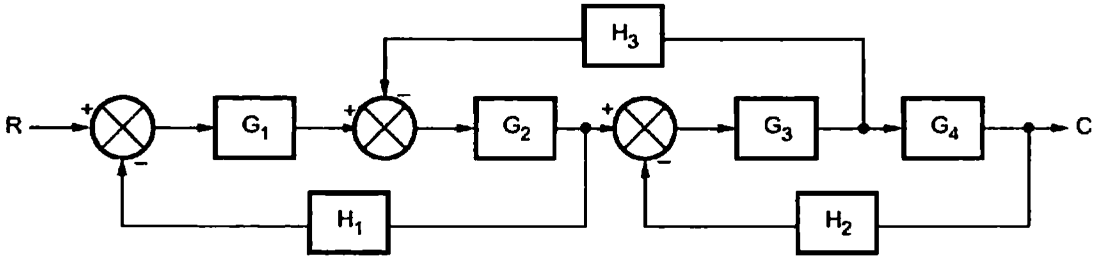
$$\therefore \quad \frac{C_1}{R_2} = \frac{-G_1\,G_2\,G_4}{1+[-G_1\,G_2\,G_4][G_3]} = \frac{-G_1\,G_2\,G_4}{1-G_1\,G_2\,G_3\,G_4}$$

For $\frac{C_2}{R_2}$, $R_1 = 0$ and C_1 is suppressed.

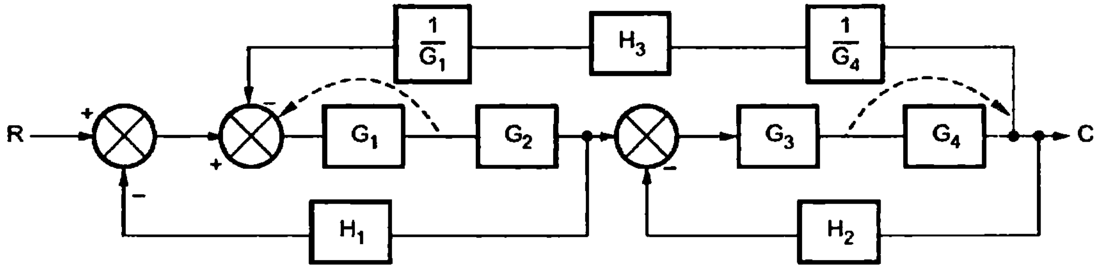


$$\frac{C_2}{R_2} = \frac{G_2}{1+[G_2][-G_1\,G_3\,G_4]} = \frac{G_2}{1-G_1\,G_2\,G_3\,G_4}$$

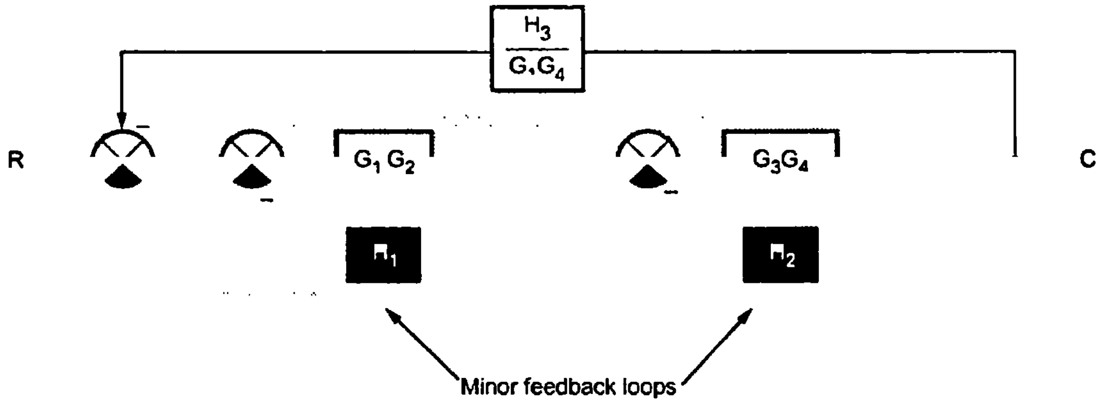
Example 5.10 : *Determine the transfer function C(s)/R(s) of the system shown below by block diagram reduction method.* **(M.U. : Nov.-94, Dec.-98)**



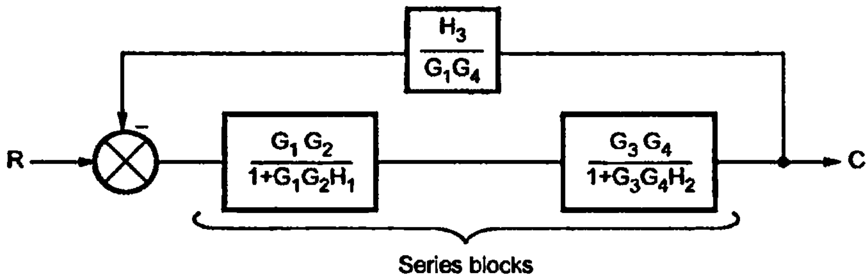
Solution : Shifting summing point before G_1 and takeoff point after G_4 we get,



Exchanging summing points and takeoff points using associative law and combining series blocks we get,



Eliminating minor feedback loops we get,

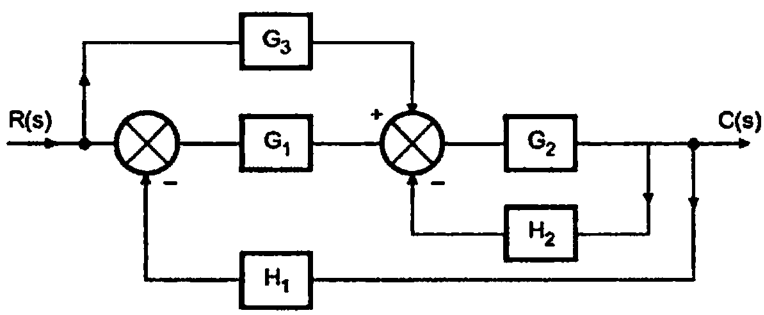


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\left[\frac{G_1G_2}{1+G_1G_2H_1}\right]\left[\frac{G_3G_4}{1+G_3G_4H_2}\right]}{1+\left[\frac{G_1G_2}{1+G_1G_2H_1}\right]\left[\frac{G_3G_4}{1+G_3G_4H_2}\right]\left[\frac{H_3}{G_1G_4}\right]}$$

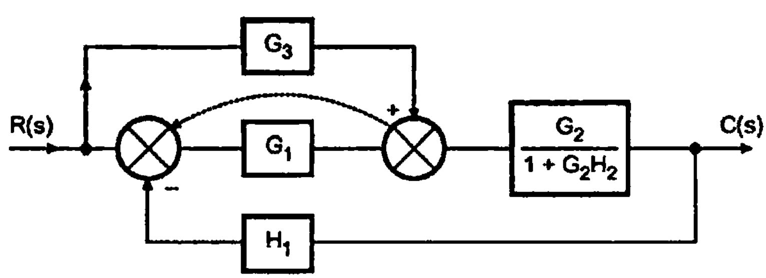
$$= \frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)+G_2G_3H_3}$$

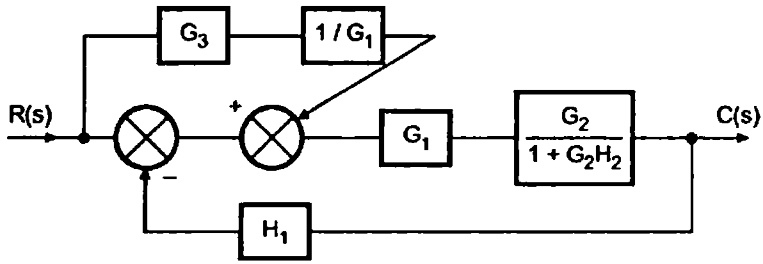
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_1G_2H_1+G_3G_4H_2+G_1G_2G_3G_4H_1H_2+G_2G_3H_3}}$$

Example 5.11 : *Obtain the closed loop transfer function.*

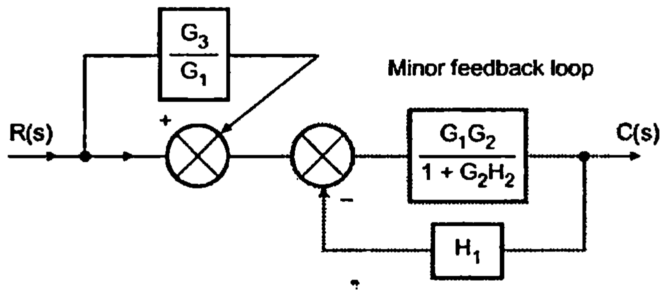


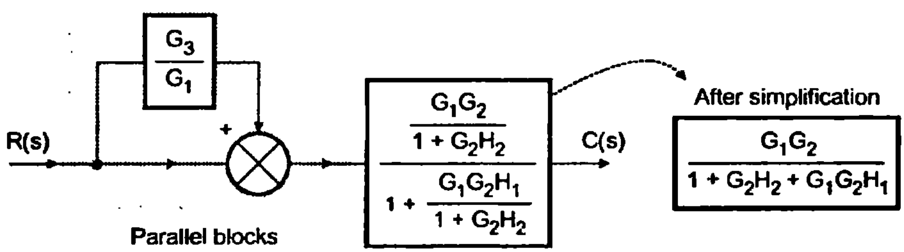
Solution : No blocks are in series or parallel so shifting summing point towards left i.e. before the block having transfer function G_1 as shown in Figure.



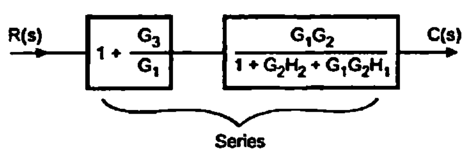


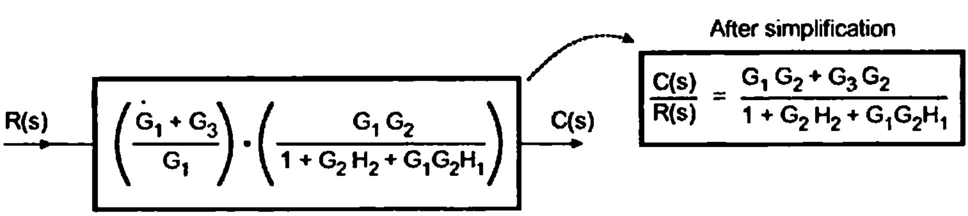
Using Associative law for two summing points we get,



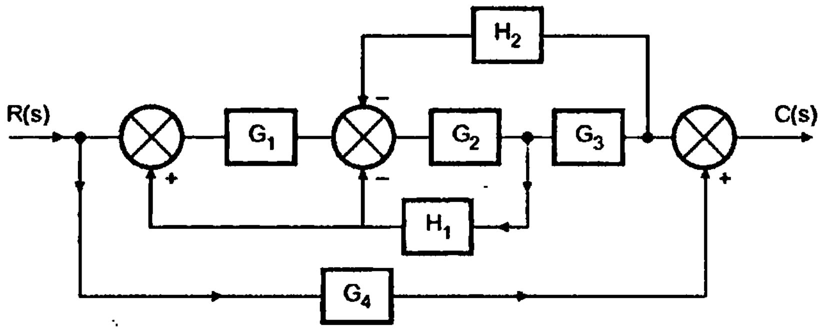


The two blocks with transfer function 1 and with transfer function $\frac{G_3}{G_1}$ are in parallel. So they will add to each other so we have,



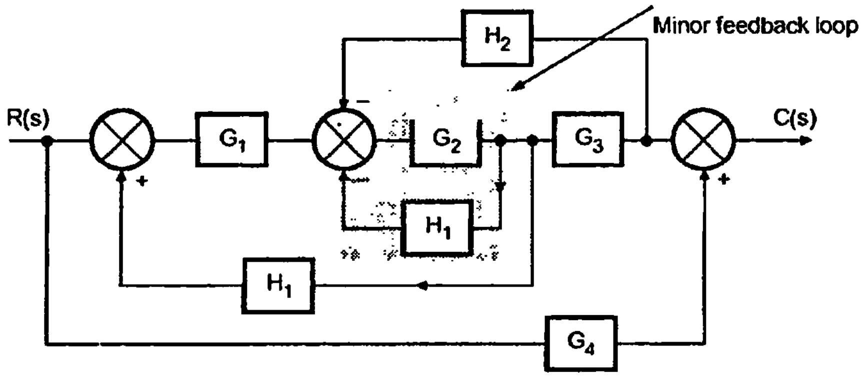


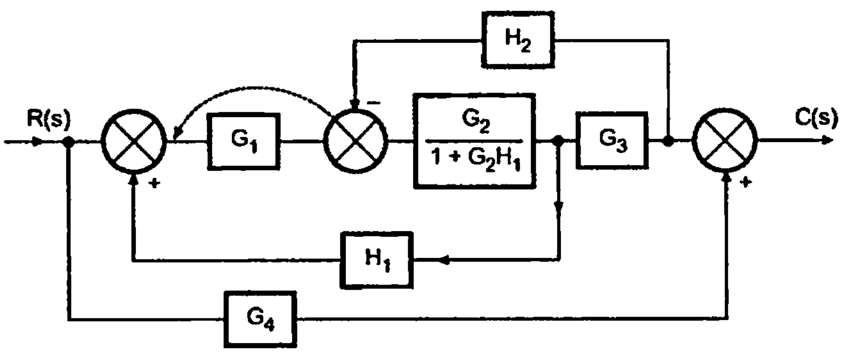
Example 5.12 : *Obtain the closed loop transfer function C(s)/R(s)*



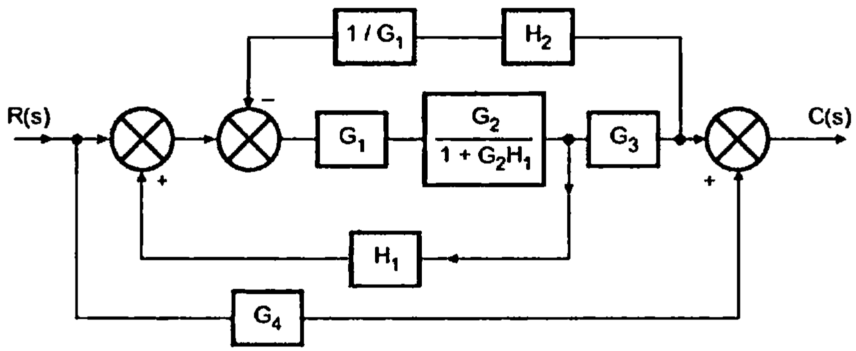
(M.U. : Dec.-97, Dec.-2005, Dec.-2007)

Solution : Separating two feedback from second takeoff point which is after block having transfer function G_2 as shown, we get,

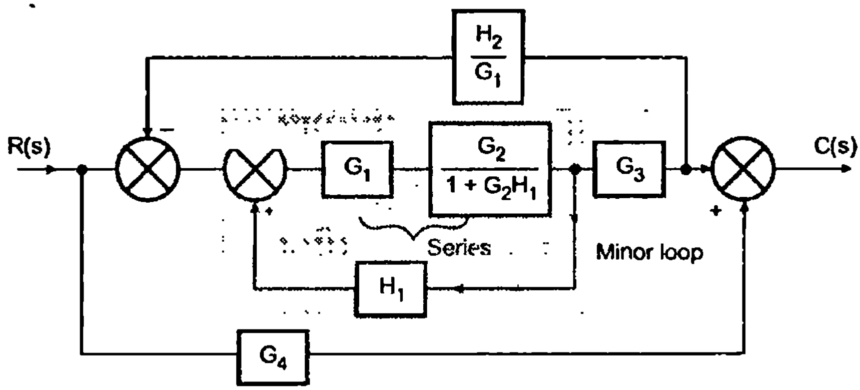


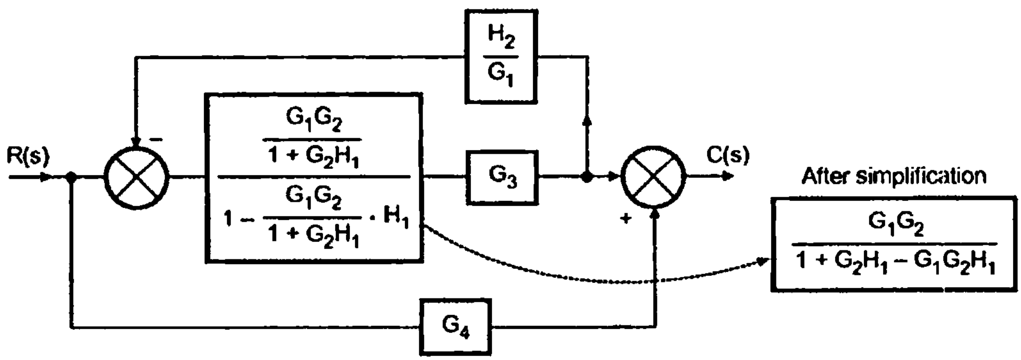


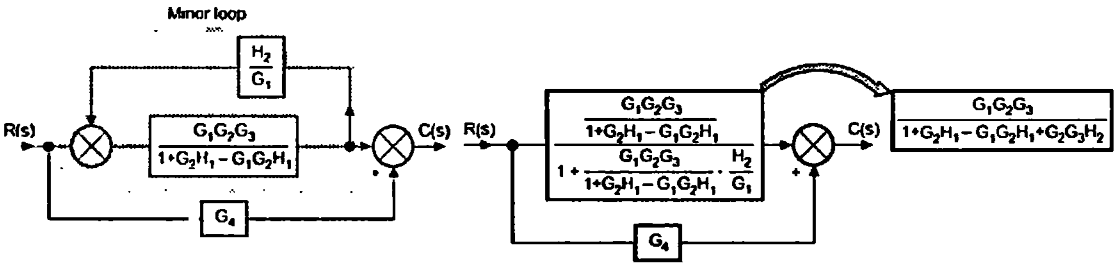
Shifting summing point behind the block having transfer function 'G_1' as shown we get,

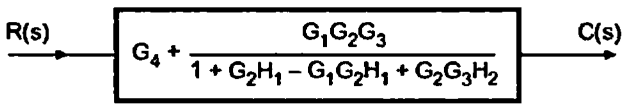


Use Associative law for the two summing points and interchange their positions, we get,



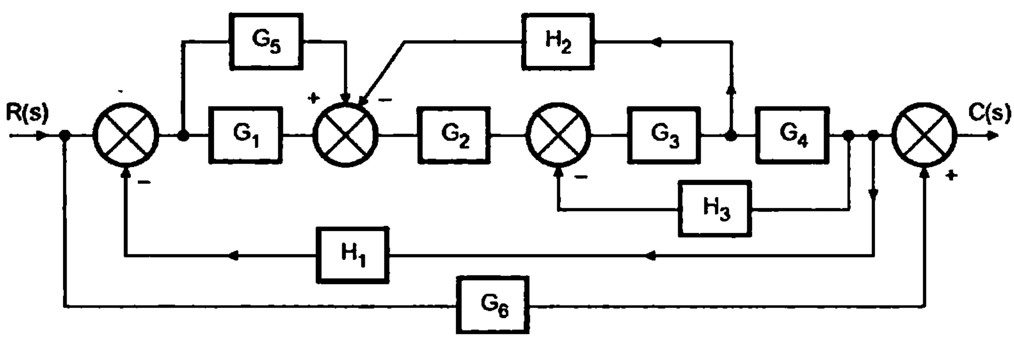




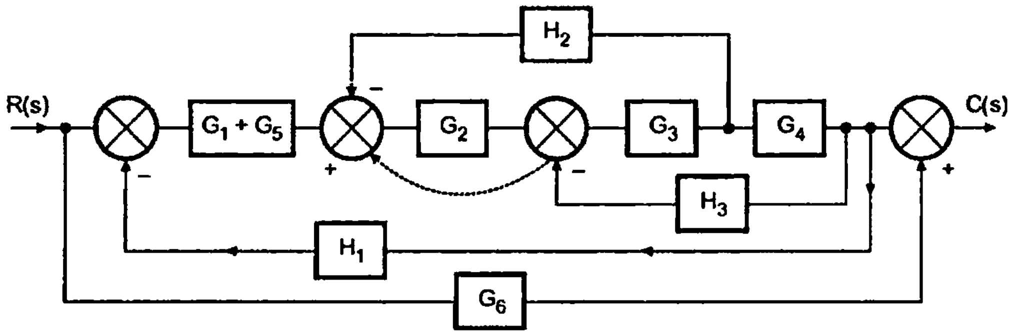


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_4+G_4G_2H_1-G_4G_1G_2H_1+G_2G_3G_4H_2+G_1G_2G_3}{1+G_2H_1-G_1G_2H_1+G_2G_3H_2}$$

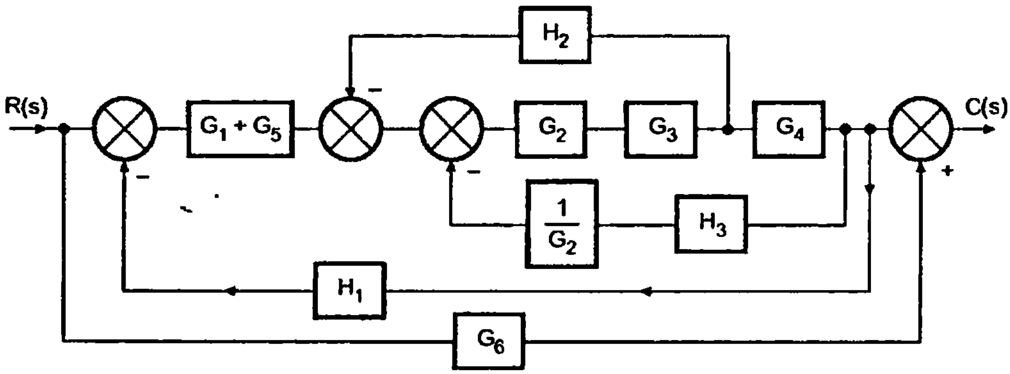
Example 5.13 : *Obtain the closed loop transfer function C(s)/R(s).*



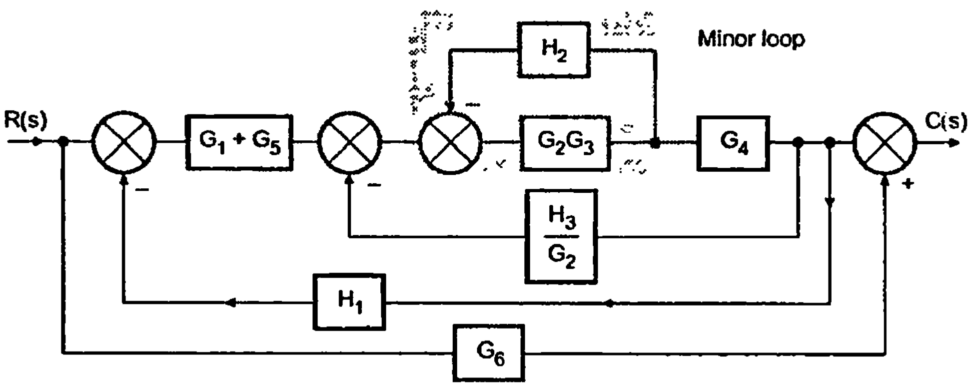
Solution : The blocks G_1 and G_5 are in parallel, so add them.

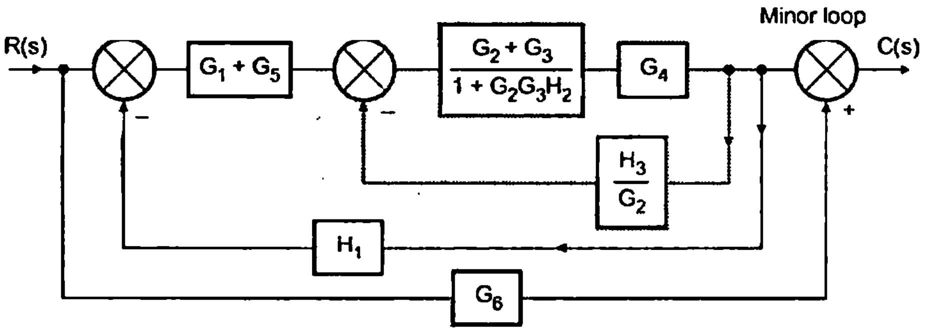


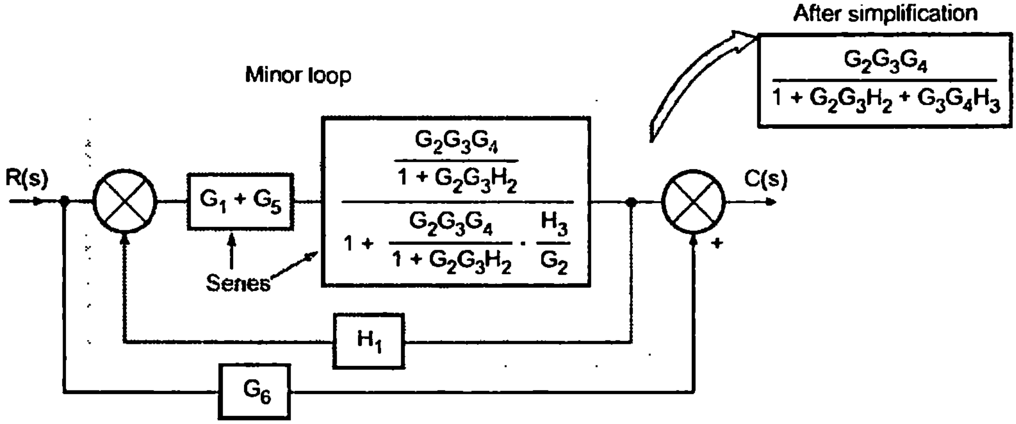
Shifting summing point behind the block 'G_2', towards left as shown we get,

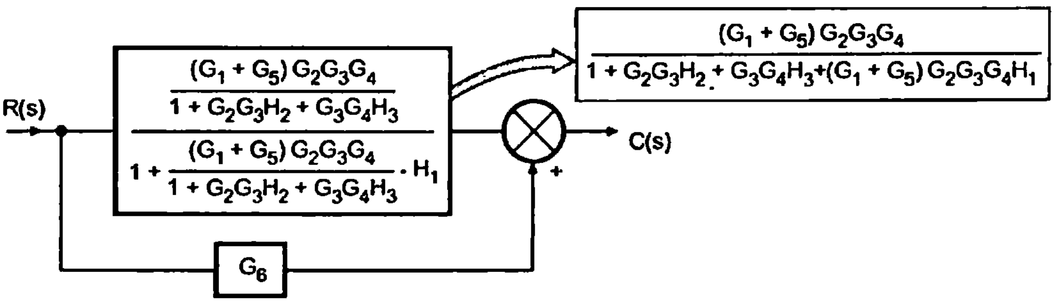


Using Associative law for the two summing points in between and interchanging their position we get,

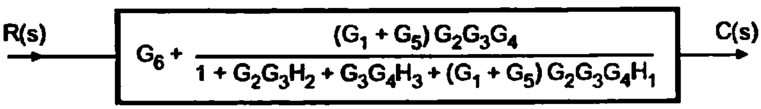






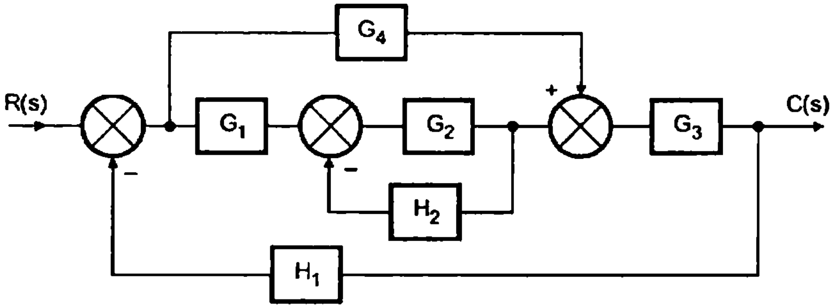


The block G_6 and equivalent block obtained are in parallel,



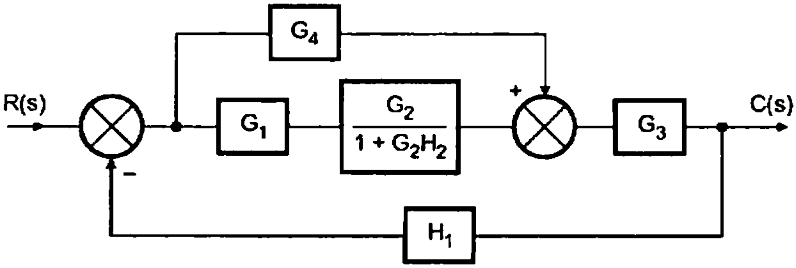
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_6 + G_2 G_3 G_6 H_2 + G_3 G_4 G_6 H_3 + (G_1 + G_5) G_2 G_3 G_4 G_6 H_1 + (G_1 + G_5) G_2 G_3 G_4}{1 + G_2 G_3 H_2 + G_3 G_4 H_3 + (G_1 + G_5) G_2 G_3 G_4 H_1}$$

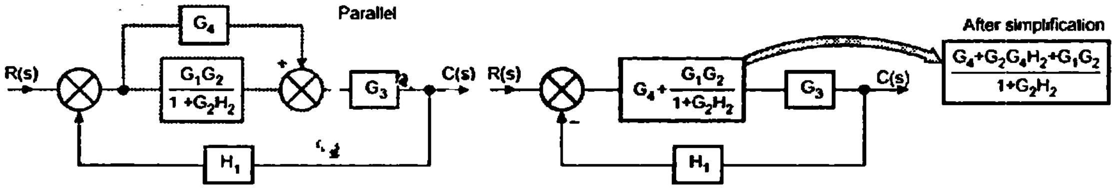
Example 5.14 : *Obtain the closed loop transfer function C(s)/R(s).*

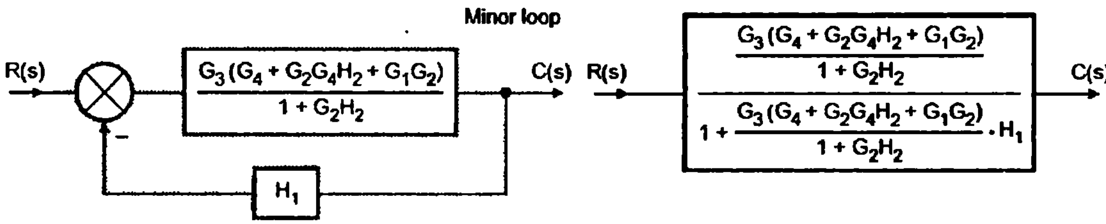


(M.U. : May-97)

Solution : The blocks G_2 and H_2 form minor feedback loop.



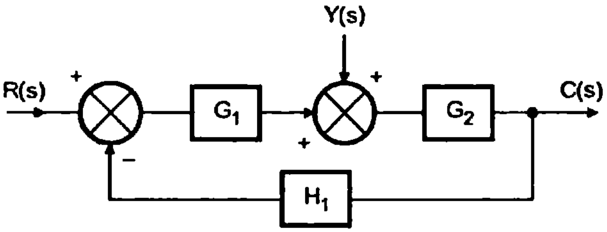




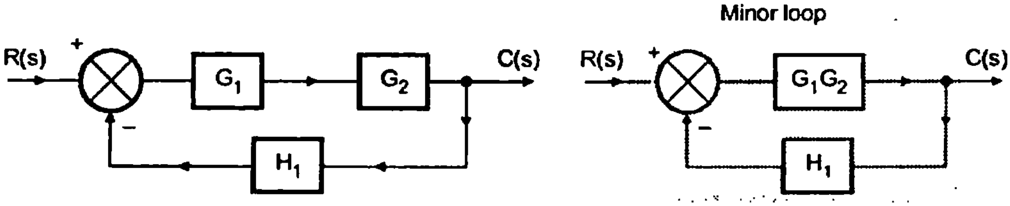
$$\frac{\dfrac{G_3(G_4 + G_2G_4H_2 + G_1G_2)}{1 + G_2H_2}}{1 + \dfrac{G_3(G_4 + G_2G_4H_2 + G_1G_2)}{1 + G_2H_2} \cdot H_1}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_3 [G_4 + G_2 G_4 H_2 + G_1 G_2]}{1 + G_2 H_2 + G_3 H_1 [G_4 + G_2 G_4 H_2 + G_1 G_2]}$$

Example 5.15 : *Obtain the resultant output C(s) interms of the inputs R(s) and Y(s).*



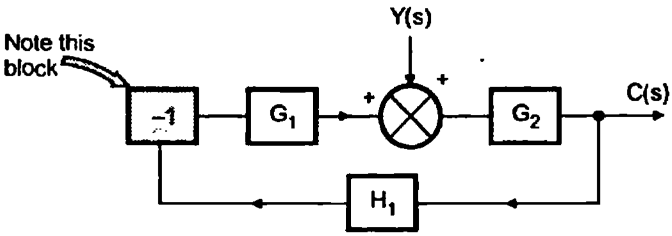
Solution : As there are two inputs, consider each input separately. Consider R(s), assuming Y(s) = 0.



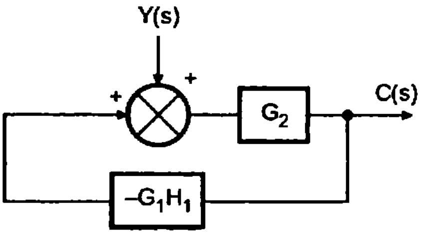
$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2 H_1} \quad \text{i.e.} \quad C(s) = R(s)\left[\frac{G_1 G_2}{1 + G_1 G_2 H_1}\right]$$

Now consider Y(s) acting with R(s) = 0.

Now sign of signal obtained from H_1 is negative which must be carried forward, though summing point at R(s) is removed, as R(s) = 0, so we get,



Combining the blocks $G_1 H_1$ and –1 as in series,



$$G_{eq} = G_2$$

$$H_{eq} = -G_1 H_1$$

Feedback sign positive at input summing point.

$$\therefore \frac{C(s)}{Y(s)} = \frac{G_{eq}}{1 - G_{eq} H_{eq}}$$

$$= \frac{G_2}{1 - G_2(-G_1H_1)} = \frac{G_2}{1 + G_1G_2H_1}$$

So part of C(s) due to Y(s) alone is,

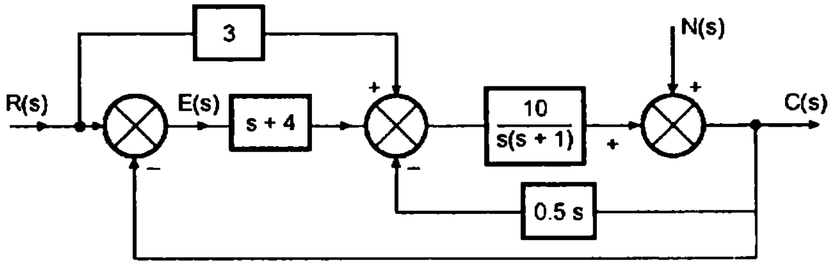
$$C(s) = Y(s)\left[\frac{G_2}{1 + G_1G_2H_1}\right]$$

Hence the net output C(s) is given by algebraically adding its two components,

$$\boxed{C(s) = \frac{G_1G_2R(s) + G_2Y(s)}{1 + G_1G_2H_1}}$$

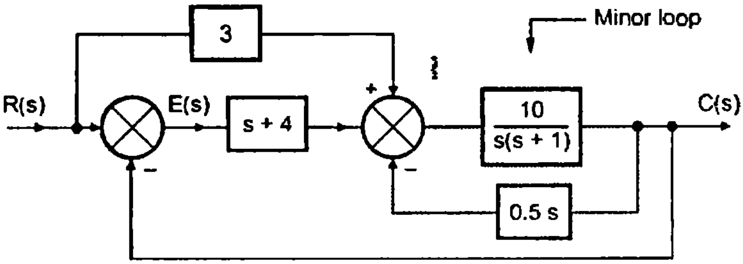
Example 5.16 : *The system block diagram is given below.*

Find i) $\frac{C(s)}{E(s)}$ *if* $N(s) = 0$ *ii)* $\frac{C(s)}{R(s)}$ *if* $N(s) = 0$ *iii)* $\frac{C(s)}{N(s)}$ *if* $R(s) = 0$

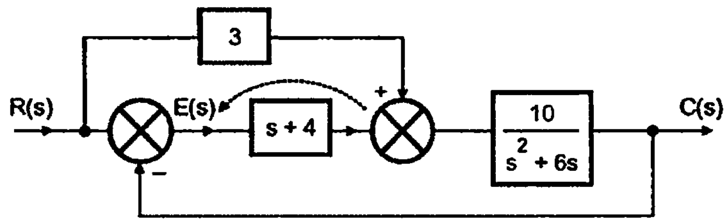


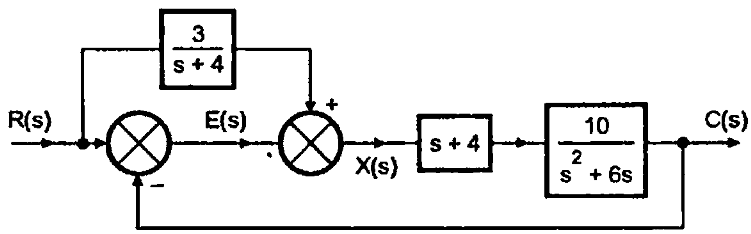
(M.U. : May-1995)

Solution : i) With N(s) = 0 block diagram becomes



$$\text{Minor feedback loop} = \frac{\frac{10}{s(s+1)}}{1+\frac{10}{s(s+1)}0.5s} = \frac{10}{s^2+s+5s} = \frac{10}{s^2+6s}$$





Assume output of second summing points as X(s).

Hence
$$E(s) = R(s) - C(s) \quad \text{... (i)}$$

$$C(s) = X(s)\frac{10(s+4)}{s^2+6s} \quad \text{... (ii)}$$

$$X(s) = E(s) + \frac{3}{s+4}R(s) \quad \text{... (iii)}$$

Substituting value of X(s) and R(s) from (i) and (ii) in (iii) we get,

$$\frac{s^2+6s}{10(s+4)}C(s) = E(s) + \frac{3}{s+4}E(s) + \frac{3}{s+4}C(s)$$

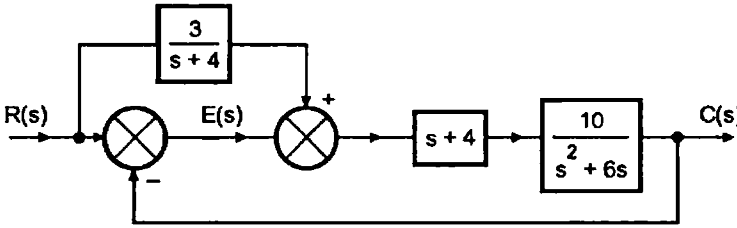
$$\left[\frac{s^2+6s}{10(s+4)} - \frac{3}{(s+4)}\right]C(s) = \left(1+\frac{3}{s+4}\right)E(s)$$

$$\frac{(s^2+6s-30)}{10(s+4)}C(s) = \frac{(s+7)}{(s+4)}E(s)$$

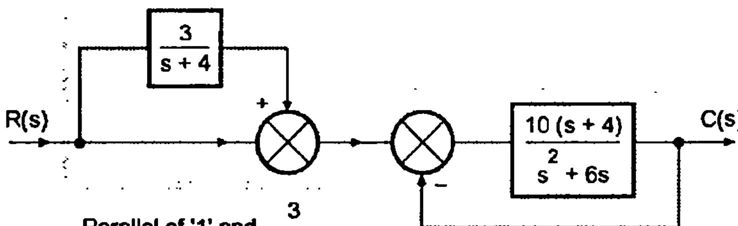
$$\therefore \quad \boxed{\frac{C(s)}{E(s)} = \frac{10(s+7)}{s^2+6s-30} \quad \text{when } N(s) = 0}$$

ii) To find $\frac{C(s)}{R(s)}$, we have to reduce block diagram solving minor feedback loop and shifting summing point to the left as shown earlier in (i).

So referring to block diagram after these two steps i.e.



Exchanging two summing points using associative law,

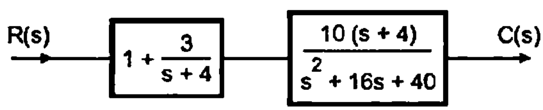


Parallel of '1' and $\frac{3}{s+4}$

$= 1 + \frac{3}{s+4}$

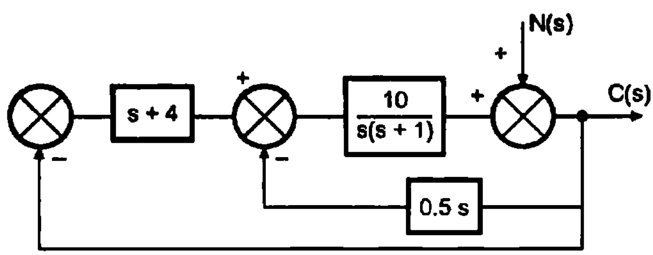
Minor loop $\frac{G}{1+GH} = \frac{\frac{10(s+4)}{s^2+6s}}{1+\frac{10(s+4)}{s^2+6s}}$

∴ Block diagram becomes,

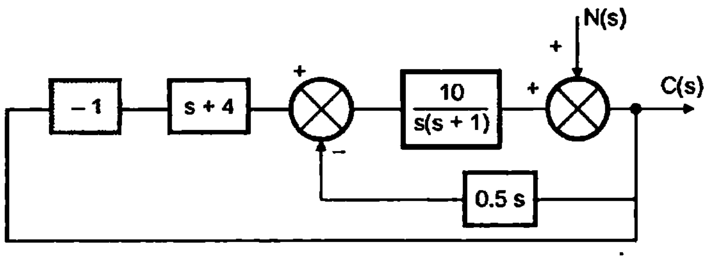


$$\therefore \quad \frac{C(s)}{R(s)} = \left(\frac{s+7}{s+4}\right) \times \left(\frac{10(s+4)}{s^2+16s+40}\right) = \frac{10(s+7)}{s^2+16s+40}$$

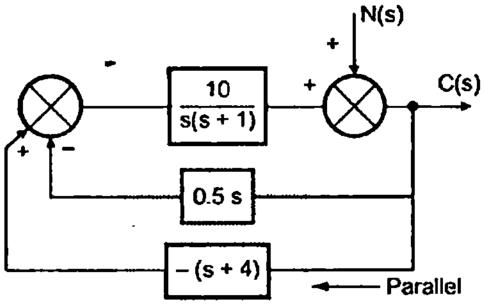
iii) With R(s) = 0 block diagram becomes,



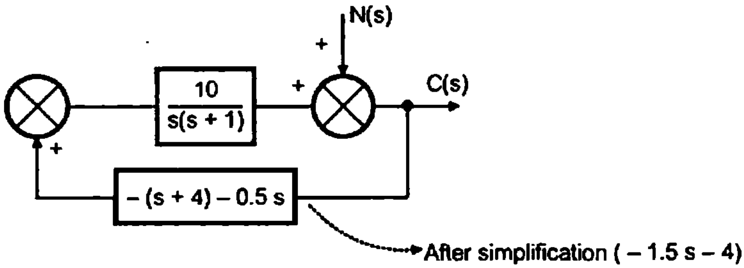
The block of '3' will not exist as R(s) = 0. Similarly first summing point will also vanish but student should note that negative sign of feedback must be considered as it is, though summing point gets deleted.



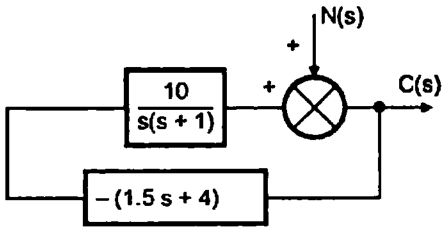
In general while deleting summing point, it is necessary to consider the signs of the different signals at that summing points and should not be disturbed. So introducing block of '–1' to consider negative sign.

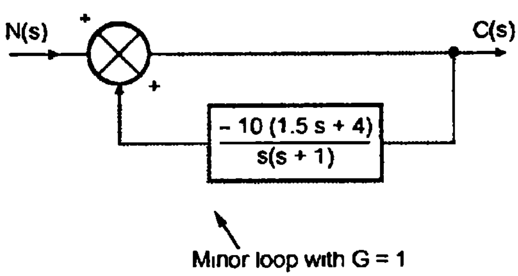


Two blocks are in parallel, adding them with signs.



Removing summing point, as sign is positive no need of adding a block.



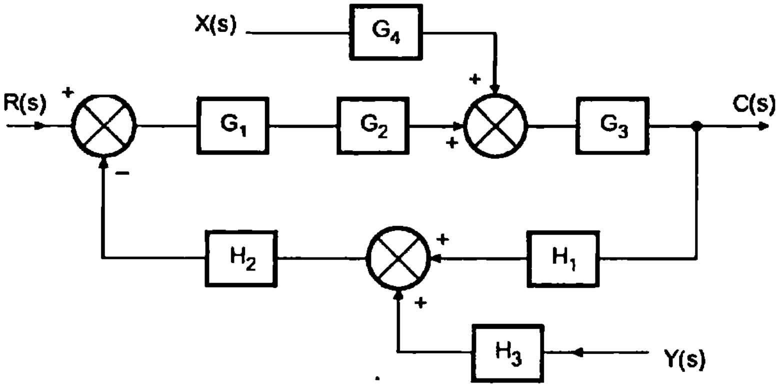


$$\therefore \quad \frac{C(s)}{N(s)} = \frac{1}{1-\left[\frac{-10\,(1.5s+4)}{s(s+1)}\right]} = \frac{1}{1+\frac{15s+40}{s(s+1)}}$$

$$\therefore \quad \boxed{\frac{C(s)}{N(s)} = \frac{s(s+1)}{s^2+16s+40}}$$

Example 5.17 ; *Use block diagram reduction technique and obtain the transfer functions i) $\frac{C(s)}{R(s)}$, ii) $\frac{C(s)}{X(s)}$, iiii) $\frac{C(s)}{Y(s)}$*

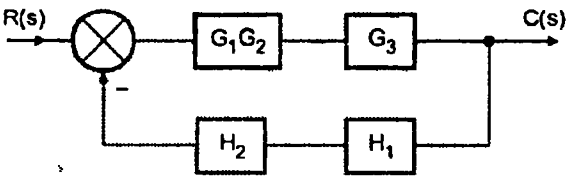
Also find total output of the system. **(M.U. : Nov.-95)**



Solution : In such multiple input systems, it is necessary to use superposition principle And it must be noted that while removing summing point, it is necessary to consider signs of various signals at that summing point.

i) Consider R(s) acting alone, X(s) = Y(s) = 0

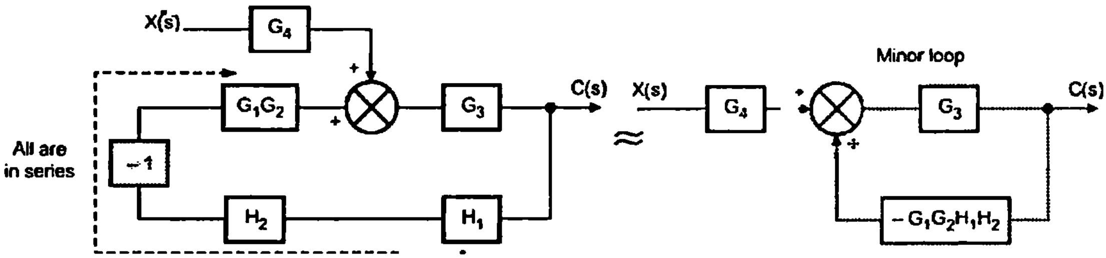
So blocks G_4 and H_3 will vanish along with the summing points and signs of all signals are positive.

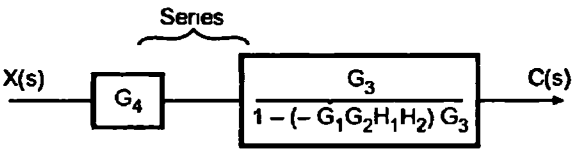


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_1G_2G_3\,H_1H_2} \qquad \text{... (i)}$$

ii) Consider X(s) acting alonge, R(s) = Y(s) = 0

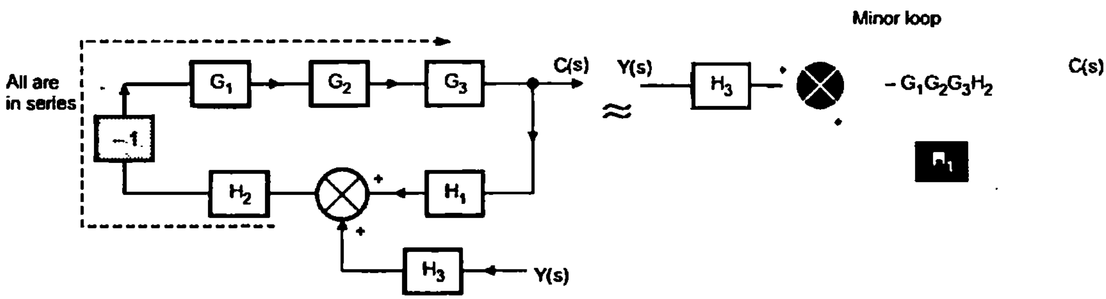
When R(s) = 0 , Summing point at R(s) will vanish, but sign of feedback signal at that summing point is negative. So it is necessary to carry on that sign by adding a block of '–1' in series with that signal.

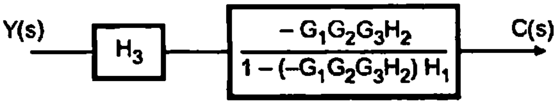




$$\therefore \frac{C(s)}{X(s)} = \frac{G_3 G_4}{1 + G_1 G_2 G_3 H_1 H_2} \quad \text{.... (ii)}$$

iii) Consider Y(s) acting alone, R(s) = X(s) = 0





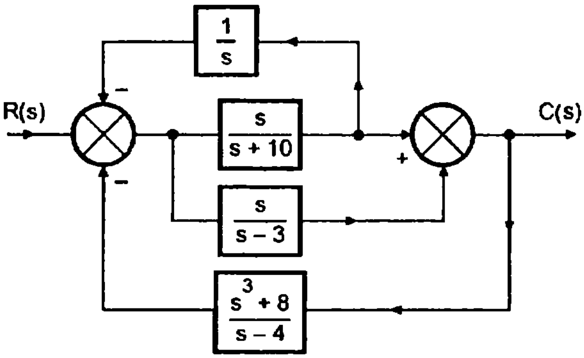
$$\therefore \frac{C(s)}{Y(s)} = \frac{-G_1 G_2 G_3 H_2 H_3}{1 + G_1 G_2 G_3 H_1 H_2} \quad \text{.... (iii)}$$

From (i), (ii) and (iii) we can write the total output as

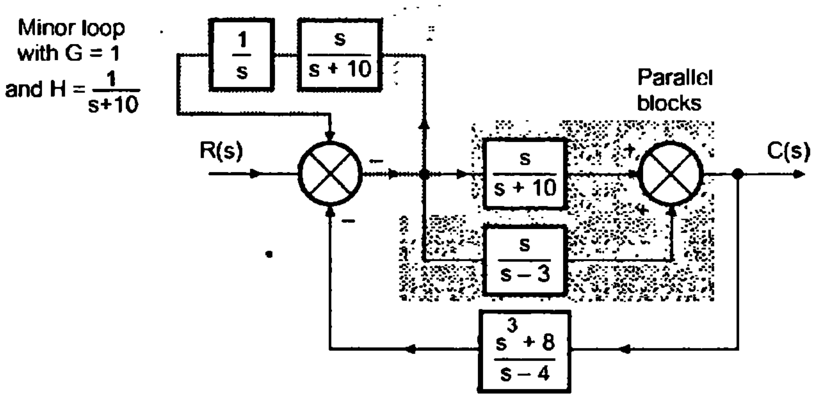
$$C(s) = \frac{G_1 G_2 G_3 R(s) + G_3 G_4 X(s) - G_1 G_2 G_3 H_2 H_3 Y(s)}{1 + G_1 G_2 G_3 H_1 H_2}$$

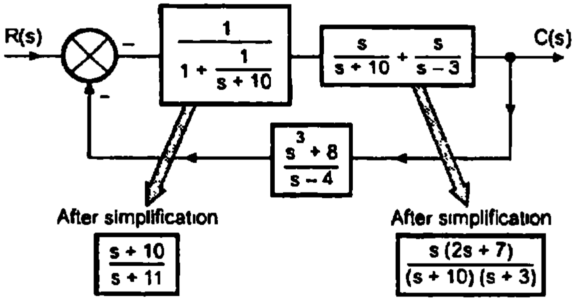
Example 5.18 : *Reduce the block diagram and obtain the transfer function C(s)/R(s).*

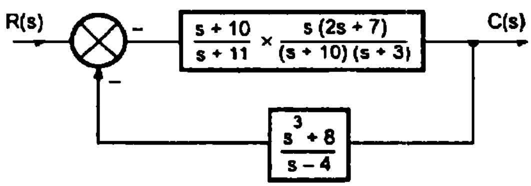
(M.U. : May-96)



Solution : No series, parallel combination and no minor feedback loop exists. So shifting takeoff point before the block of $\left(\frac{s}{s+10}\right)$.



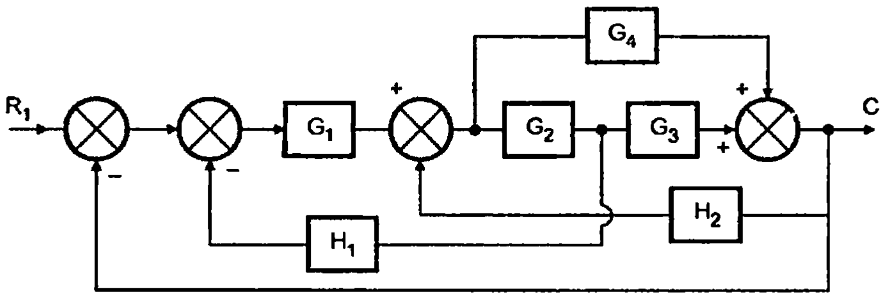




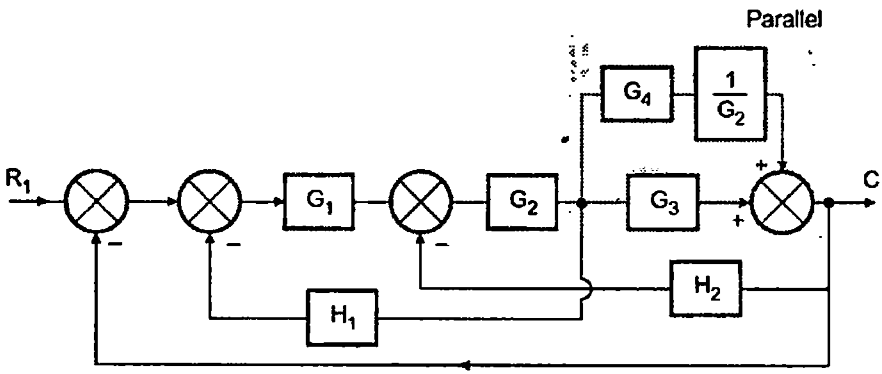
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{s(2s+7)}{(s+11)(s-3)}}{1+\dfrac{s(2s+7)}{(s+11)(s-3)}\cdot\dfrac{(s^3+8)}{(s-4)}} = \frac{s(2s+7)(s-4)}{(s+11)(s-3)(s-4)+s(2s+7)(s^3+8)}$$

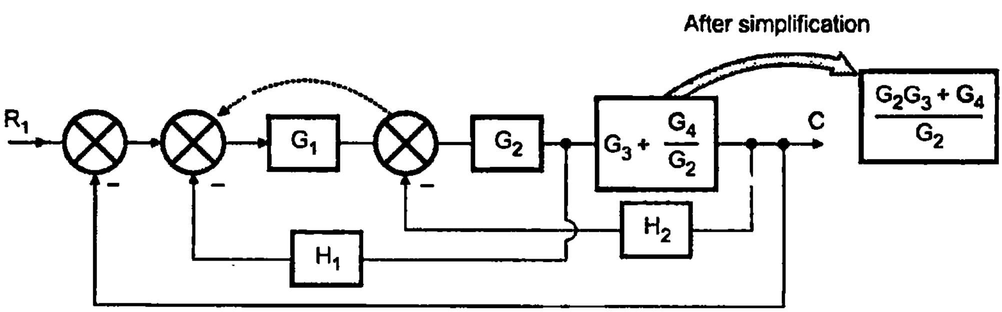
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{s(2s^2-s-28)}{2s^5+7s^4+s^3+20s^2-9s+132}}$$

Example 5.19 : *Determine C/R ratio for the system shown below.* **(M.U. : May-95)**

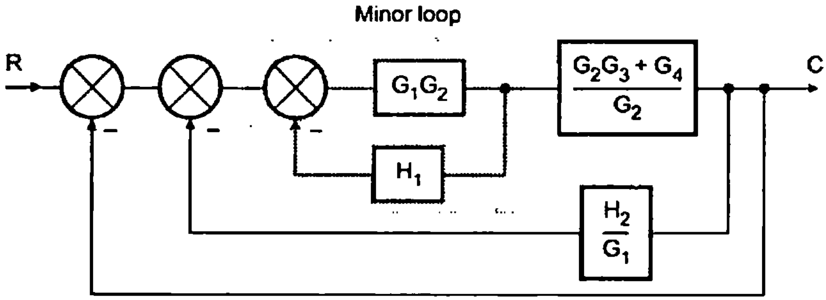


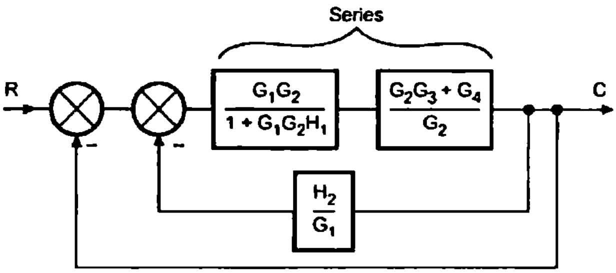
Solution : Shifting takeoff point which is before G_2 to, after G_2.

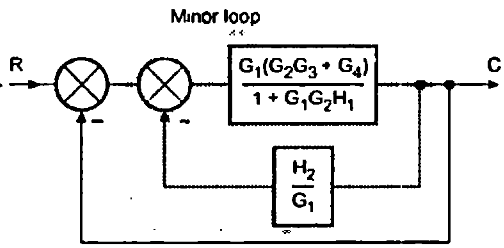


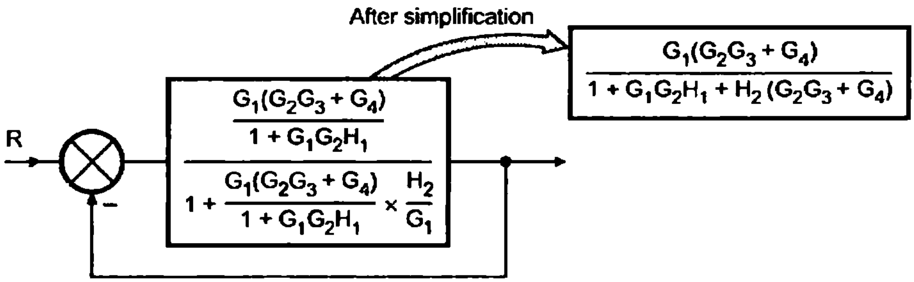


Shifting summing point before the block 'G_1' and interchanging using associative law,





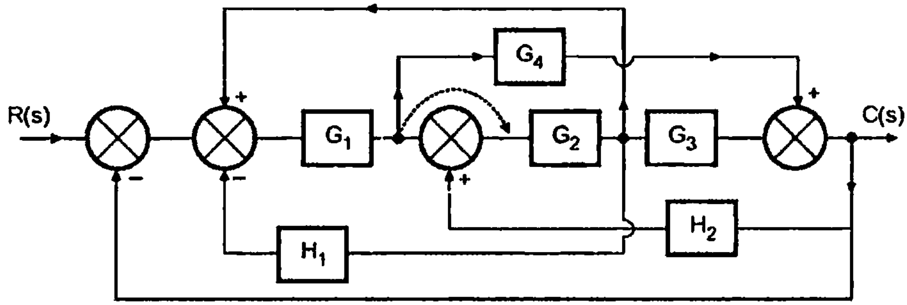




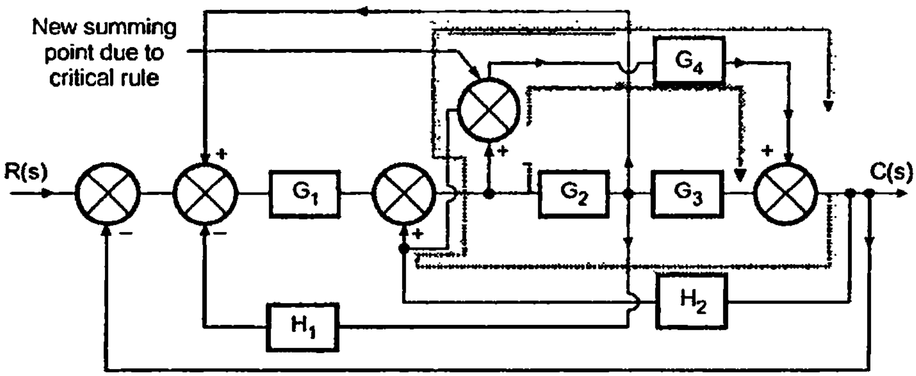
$$\therefore \quad \frac{C}{R} = \frac{\dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + H_2(G_2G_3 + G_4)}}{1 + \dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + H_2(G_2G_3 + G_4)}}$$

$$\therefore \quad \boxed{\frac{C}{R} = \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}}$$

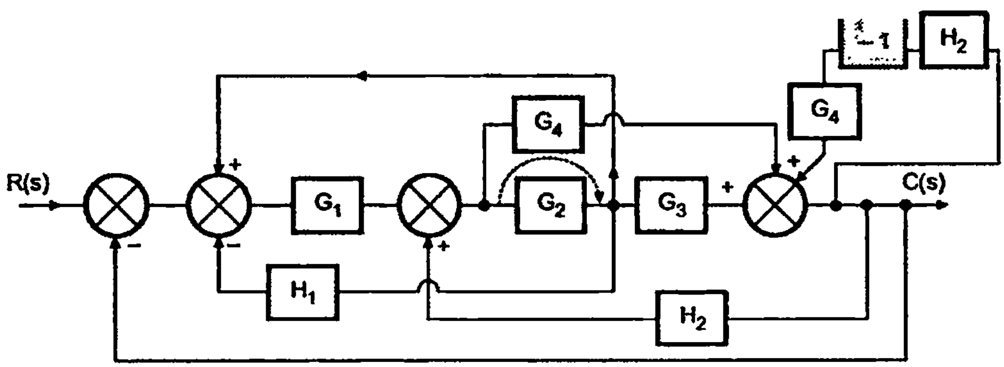
Example 5.20 : *For the block diagram shown, obtain C(s)/R(s) by using reduction rules.* (M.U. : Jan.-92)



Solution : Shift the takeoff point, to the right of summing point. This is the "Critical rule" discussed earlier as rule 10 and 11. In this problem it is necessary to use this rule, which is generally not used to solve simple problems.

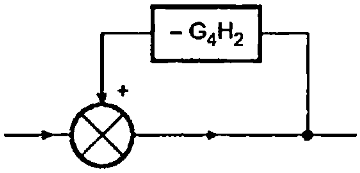


New summing point gets added due to use of critical rule . This summing point can be eliminated by separating the two paths which are linked by that summing point. The paths are shown dotted. So block diagram reduces as,



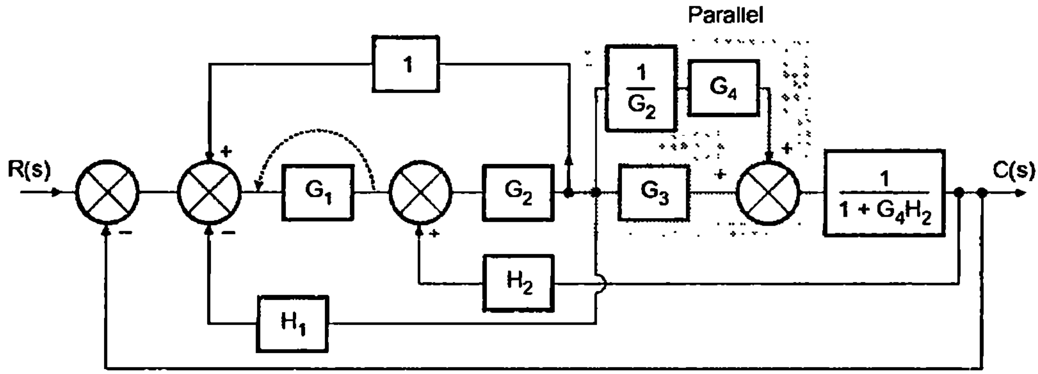
While removing summing point, as sign of one of the signal is negative, the block of transfer function '–1' is connected series with that signal.

So now there exists a minor feedback loop

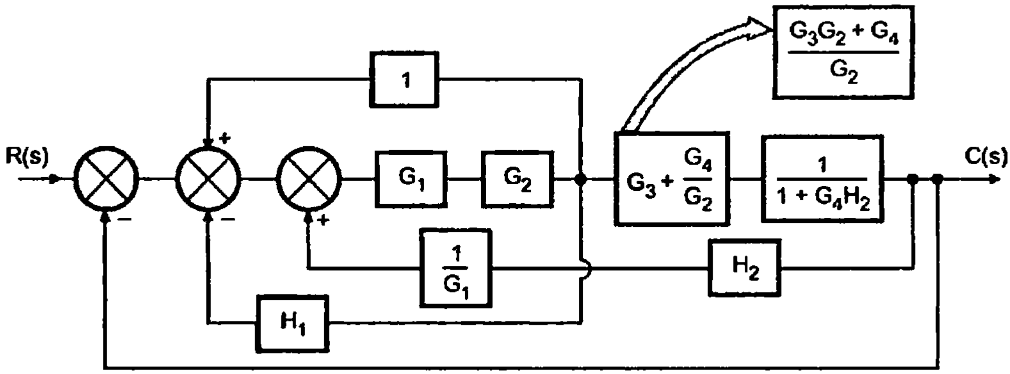


$$\approx \frac{1}{1-(-G_4H_2)} = \frac{1}{1+G_4H_2}$$

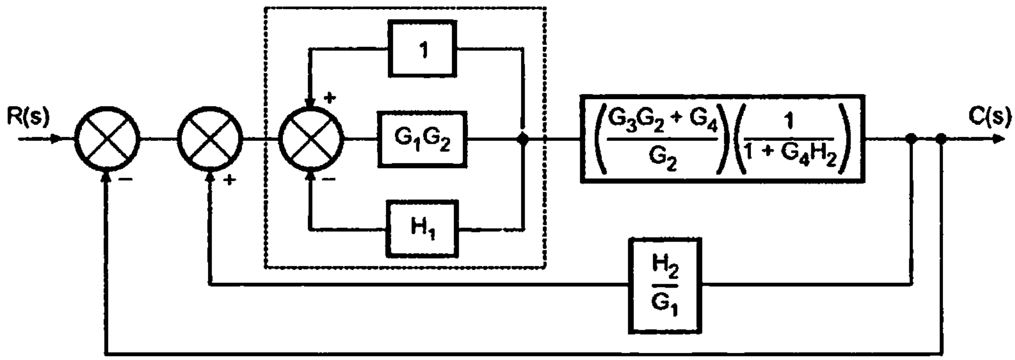
Shifting take off to the right of G_2



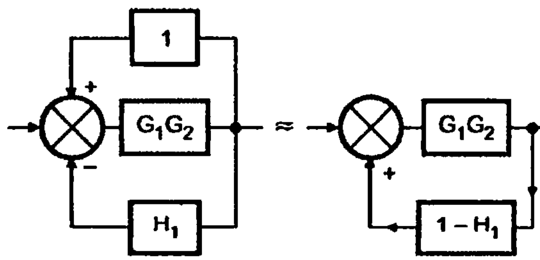
Combining blocks in parallel and shifting summing point to the left of 'G_1'.



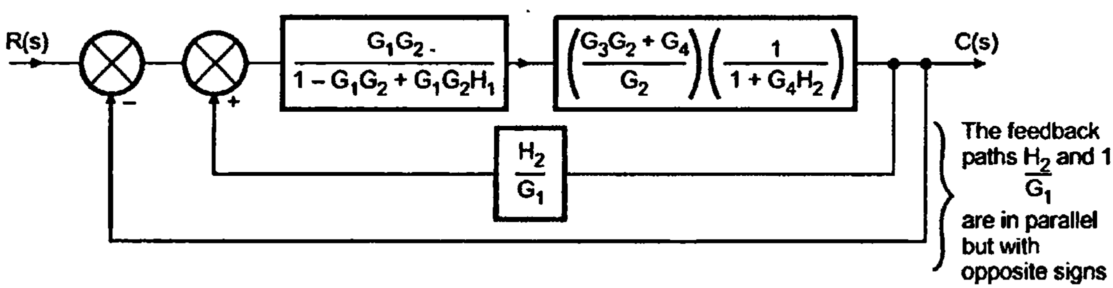
Interchanging the summing points using associative law.



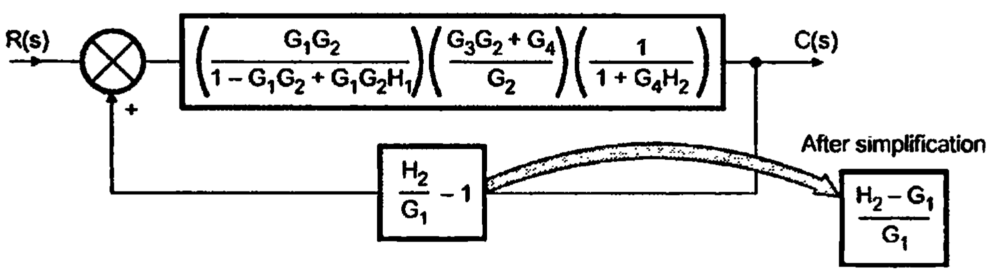
Solving minor feedback loop. The block of '1' and 'H_1' are in parallel so loop becomes,



$$\approx \frac{G_1G_2}{1-(1-H_1)\,G_1G_2} = \frac{G_1G_2}{1-G_1G_2+G_1G_2H_1}$$



Combining the feedback blocks which are in parallel.

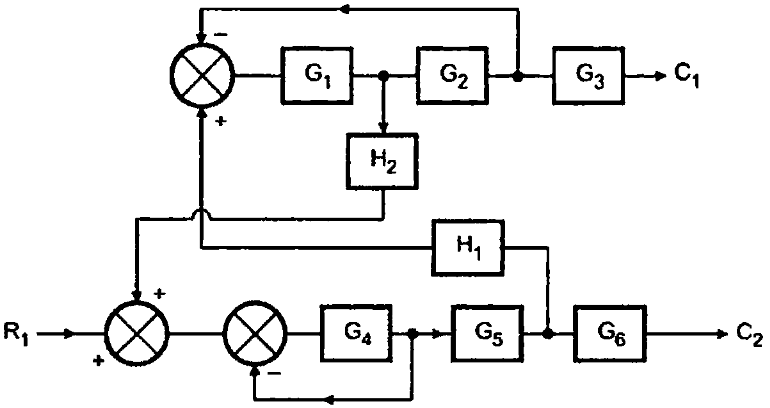


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\left(\frac{G_1 G_2}{1 - G_1 G_2 + G_1 G_2 H_1}\right)\left(\frac{G_3 G_2 + G_4}{G_2}\right)\left(\frac{1}{1 + G_4 H_2}\right)}{1 - \left(\frac{G_1 G_2}{1 - G_1 G_2 + G_1 G_2 H_1}\right)\left(\frac{G_3 G_2 + G_4}{G_2}\right)\left(\frac{1}{1 + G_4 H_2}\right)\left(\frac{H_2 - G_1}{G_1}\right)}$$

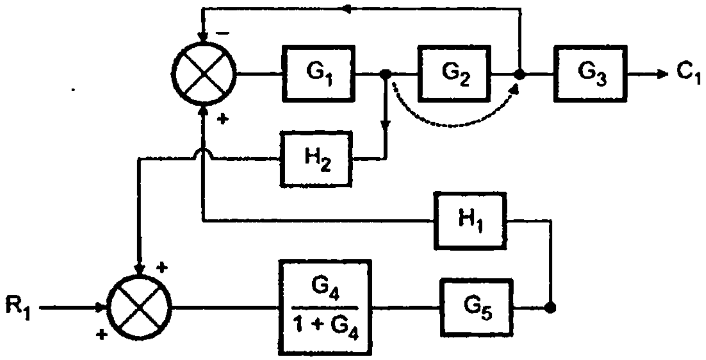
$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 - G_1 G_2 + G_1 G_2 H_1 - G_1 G_2 G_4 H_2 + G_1 G_2 G_4 H_1 H_2 + G_1 G_2 G_3 + G_1 G_4 - G_2 G_3 H_2}$$

Example 5.21 : *In the given block diagram, obtain the transfer function of the system* C_1/R_1.

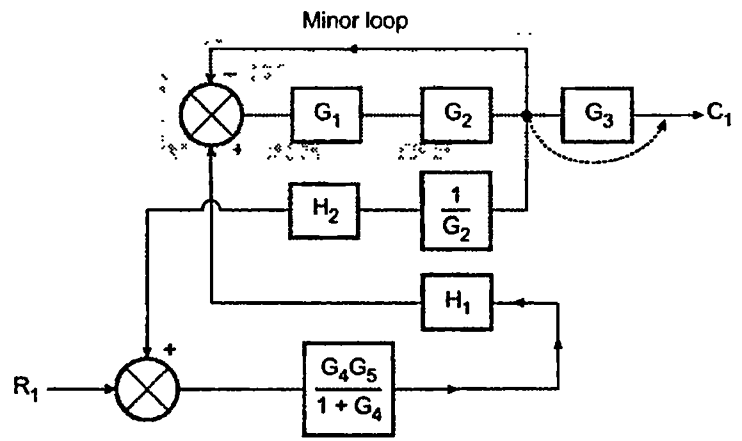
(M.U. : Nov.-96)



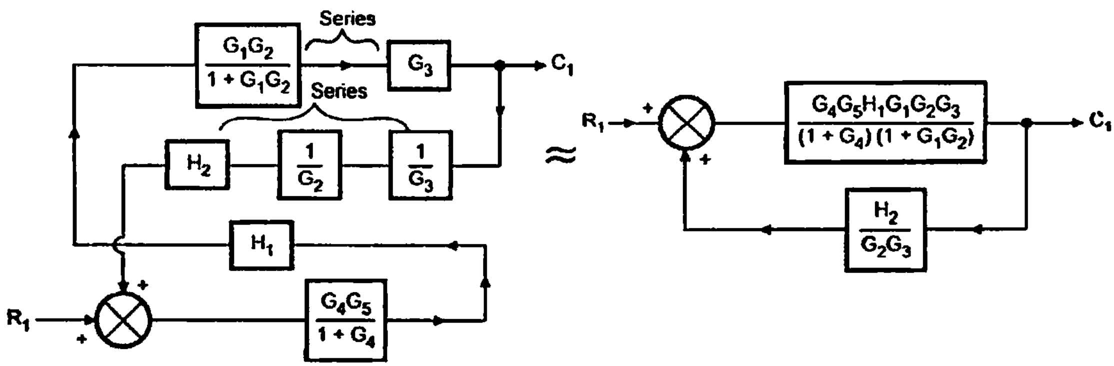
Solution : Solving the minor feedback loop of 'G_4' and '1'.



As C_2 is not the focus of interest hence G_6 becomes meaningless in the block diagram, thus it is removed.



Solving minor feedback loop and shifting takeoff point.

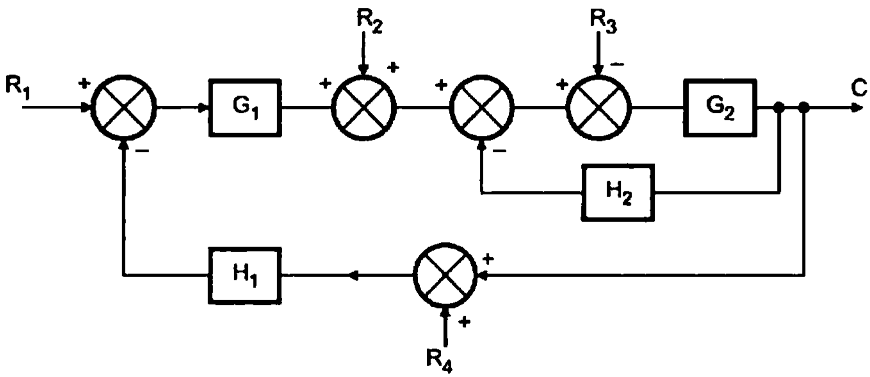


$$\therefore \quad \frac{C_1}{R_1} = \frac{\dfrac{G_1 G_2 G_3 G_4 G_5 H_1}{(1+G_1 G_2)\ (1+G_4)}}{1-\dfrac{G_1 G_2 G_3 G_4 G_5 H_1}{(1+G_1 G_2)\ (1+G_4)} \cdot \dfrac{H_2}{G_2 G_3}}$$

$$\therefore \quad \boxed{\frac{C_1}{R_1} = \frac{G_1 G_2 G_3 G_4 G_5 H_1}{1+G_1 G_2+G_4+G_1 G_2 G_4-G_1 G_4 G_5 H_1 H_2}}$$

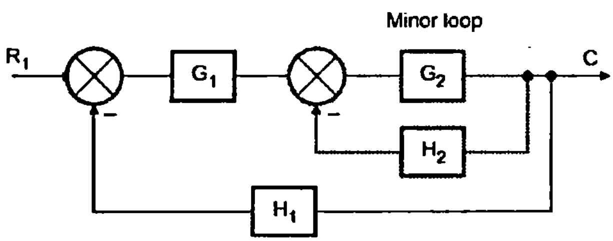
Example 5.22 : *Find C using block diagram reduction techniques*

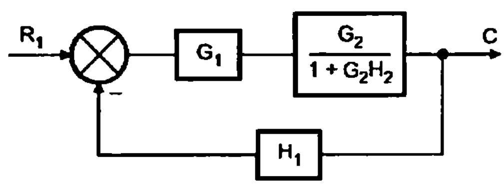
(M.U. - May-98, Dec.-06)



Solution : Consider R_1 alone R_2, R_3, R_4 are zero.

> **Note :** Whenever R_1 is zero, as the sign of feedback at R_1 is negative, while removing summing point at R_1, do not forget to insert a block of '–1' to consider effect of negative sign.

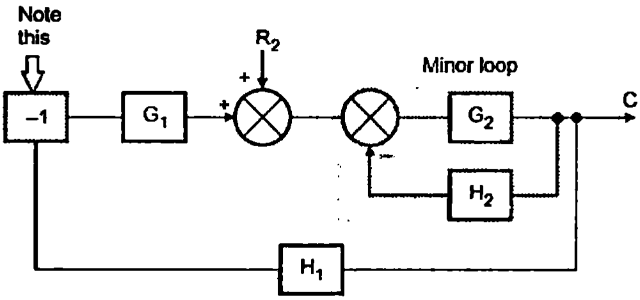


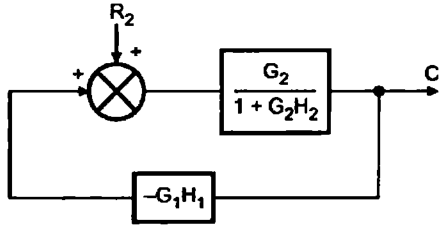


$$\therefore \quad \frac{C}{R_1} = \frac{\dfrac{G_1G_2}{1+G_2H_2}}{1+\dfrac{G_1G_2H_1}{1+G_2H_2}} = \frac{G_1G_2}{1+G_2H_2+G_1G_2H_1}$$

$$\therefore \quad \boxed{C = \frac{G_1G_2R_1}{1+G_2H_2+G_1G_2H_1}} \quad \text{... due to } R_1$$

Consider R_2 alone, with $R_3 = R_1 = R_4 = 0$

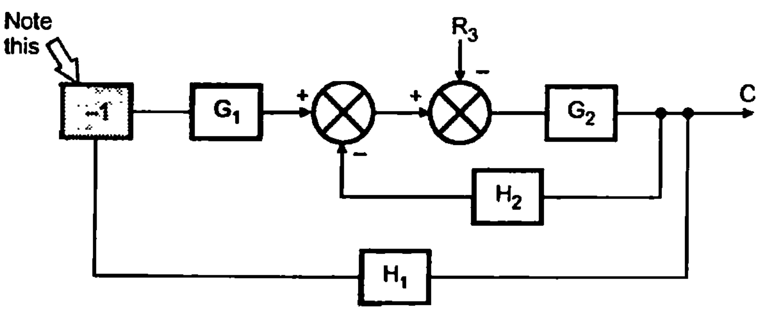




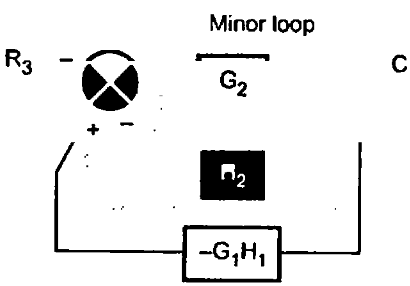
$$\therefore \qquad \frac{C}{R_2} = \frac{\dfrac{G_2}{1+G_2H_2}}{1-\left(\dfrac{G_2}{1+G_2H_2}\right)(-G_1H_1)}$$

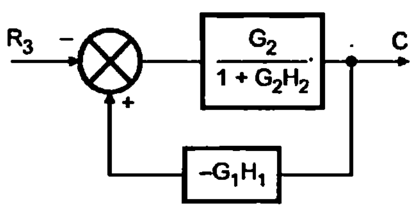
$$\therefore \qquad \boxed{C = \frac{G_2R_2}{1+G_2H_2+G_1G_2H_1}}$$

Consider R_3 alone, $R_1 = R_2 = R_4 = 0$



Combining two summing points we get,

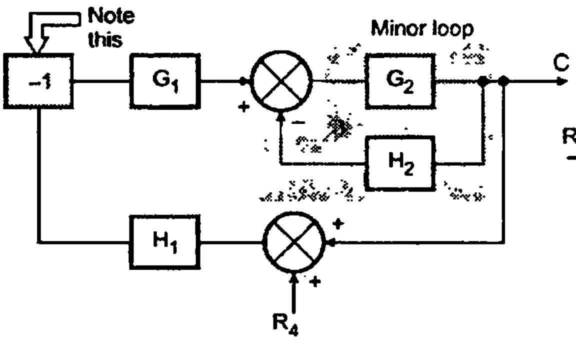


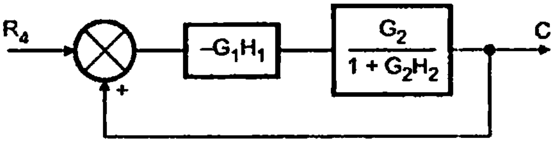


$$\therefore \quad \frac{C}{-R_3} = \frac{\dfrac{G_2}{1+G_2H_2}}{1-\left(\dfrac{G_2}{1+G_2H_2}\right)(-G_1H_1)} = \frac{G_2}{1+G_2H_2+G_1G_2H_1}$$

$$\therefore \quad \boxed{C = \frac{-R_3G_2}{1+G_2H_2+G_1G_2H_1}}$$

Consider R_1 alone, with $R_1 = R_2 = R_3 = 0$.





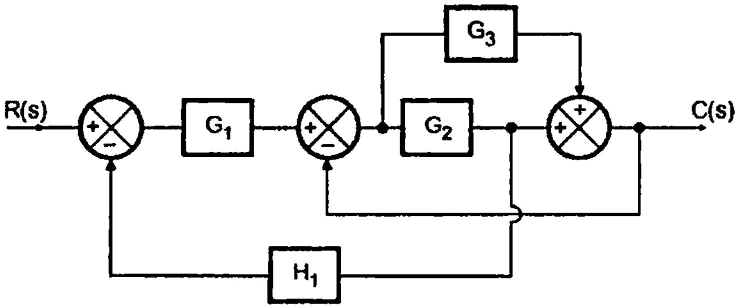
$$\therefore \quad \frac{C}{R_4} = \frac{\dfrac{-G_1G_2H_1}{1+G_2H_2}}{1-(-G_1H_1)\left(\dfrac{G_2}{1+G_2H_2}\right)} = \frac{-G_1G_2H_1}{1+G_2H_2+G_1G_2H_1}$$

$$\therefore \quad \boxed{C = \frac{-G_1G_2H_1R_4}{1+G_2H_2+G_1G_2H_1}}$$

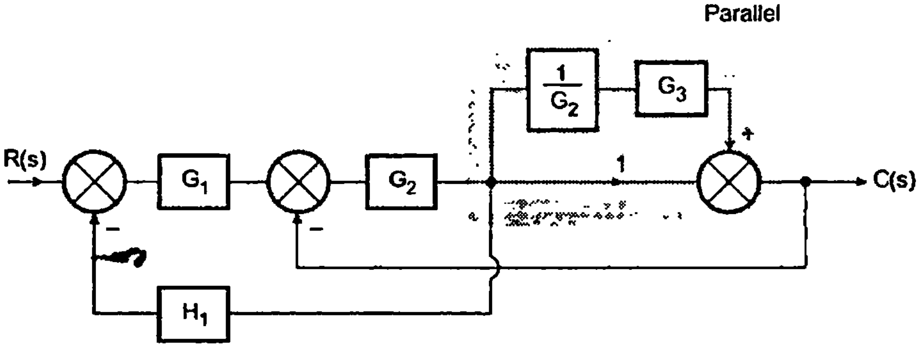
Combining all the values of C, we get

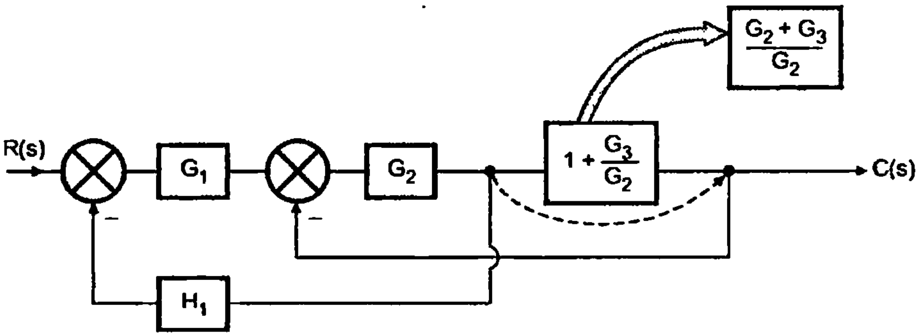
$$\boxed{C = \frac{G_1G_2R_1+G_2(R_2-R_3)-G_1G_2H_1R_4}{1+G_2H_2+G_1G_2H_1}}$$

Exampel 5.23 : *Determine the transfer function* $\frac{C(s)}{R(s)}$ *using block diagram reduction technique for the block diagram shown -* **(M.U.: May-2003)**

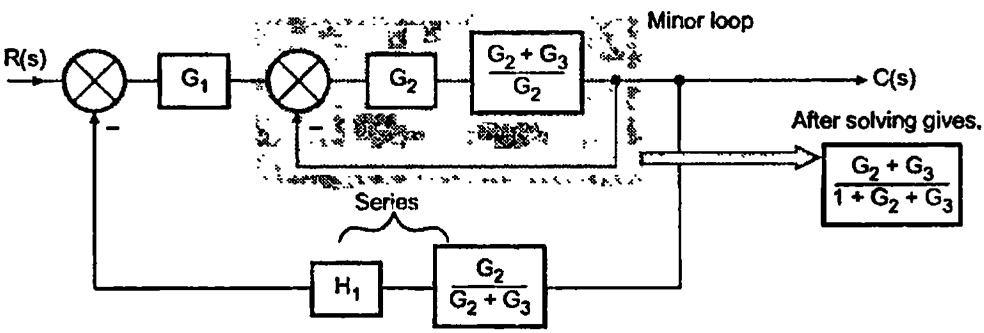


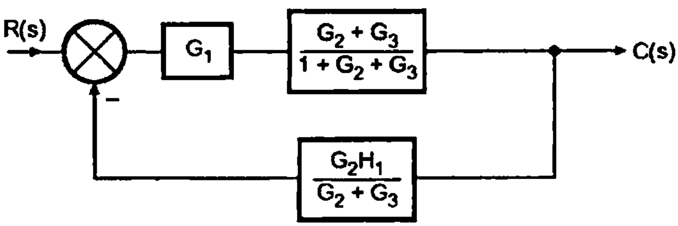
Solution : Shifting takeoff point of G_3 to the right of G_2 we get,





Shifting takeoff point towards C(s) and interchanging takeoff points,





$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{G_1(G_2+G_3)}{1+G_2+G_3}}{1+\dfrac{G_1(G_2+G_3)}{(1+G_1+G_3)}\times\dfrac{G_2H_1}{(G_2+G_3)}} = \boxed{\frac{G_1(G_2+G_3)}{1+G_2+G_3+G_1G_2H_1}}$$

Example 5.24 : *Determine the closed loop transfer function for the system shown.*
(M.U. : Dec.-2003)

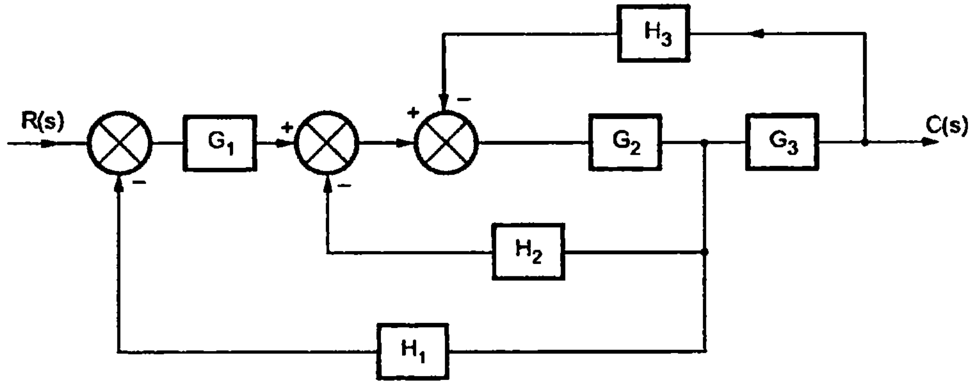
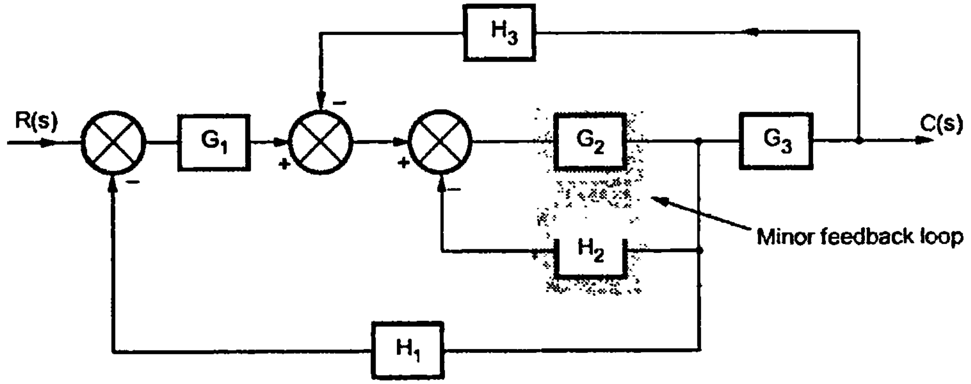
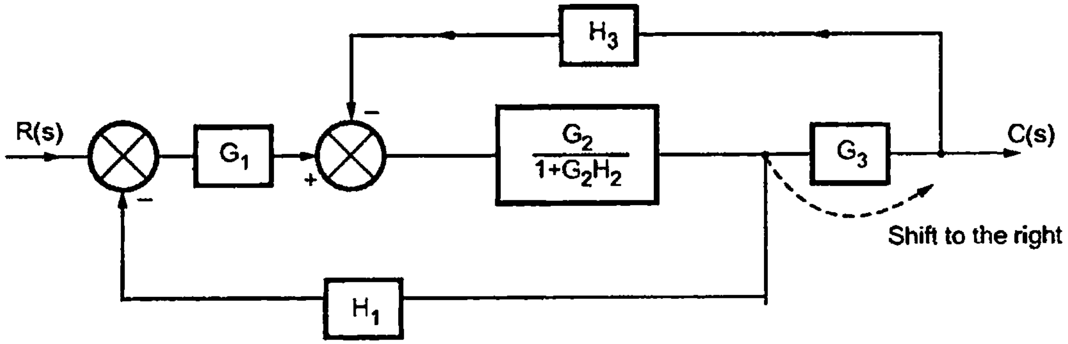
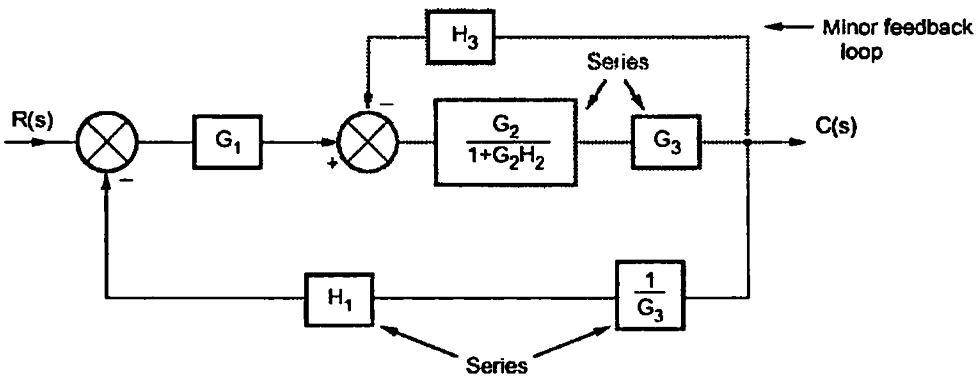


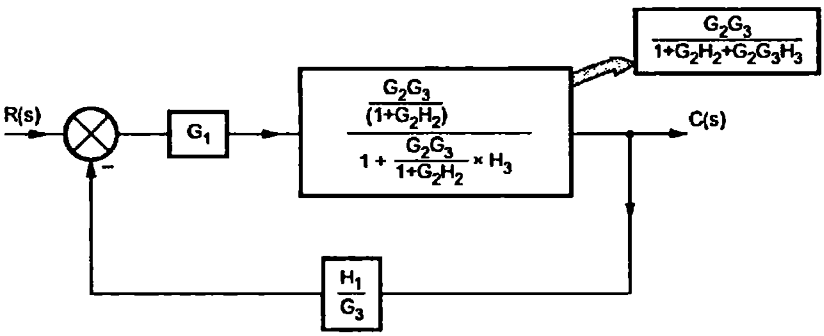
Fig. 5.47

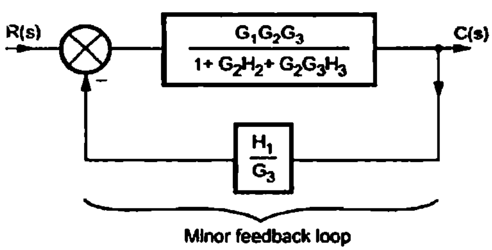
Solution : Interchange the positions of two internal summing points, using associative law.











$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3}{1+G_2H_2+G_2G_3H_3}}{1+\dfrac{G_1G_2G_3}{1+G_2H_2+G_2G_3H_3}\times\dfrac{H_1}{G_3}}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1\,G_2G_3}{1+G_2H_2+G_2G_3H_3+G_1G_2H_1}}$$

Example 5.25 : *Obtain the transfer function* $\frac{C(s)}{R(s)}$. **(M.U. : May-2004)**

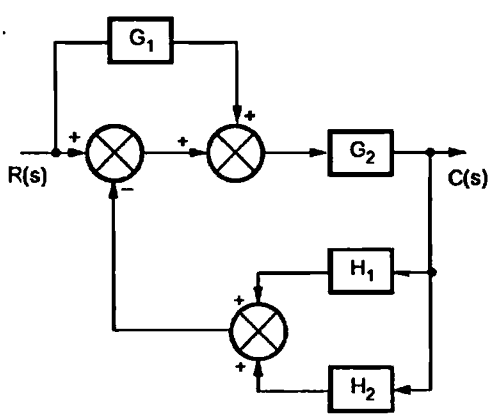
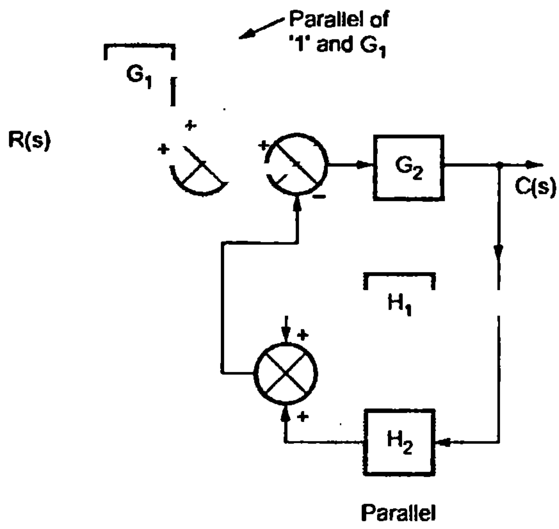
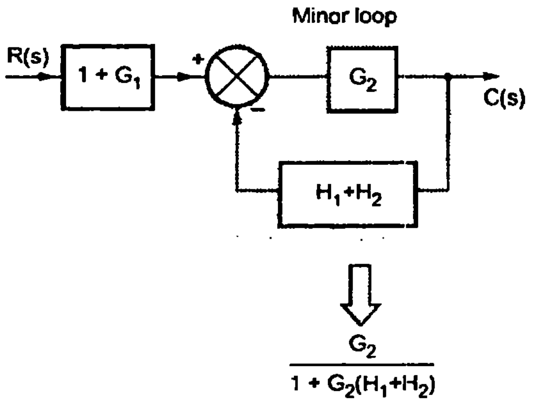


Fig. 5.48

Solution : Interchange the positions of two summing points using **associative law.**





$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{(1+G_1)\,G_2}{1+G_2\,(H_1+H_2)}}$$

Example 5.26 : *Find the overall transfer function.* **(M.U. : May-2005, May-2007)**

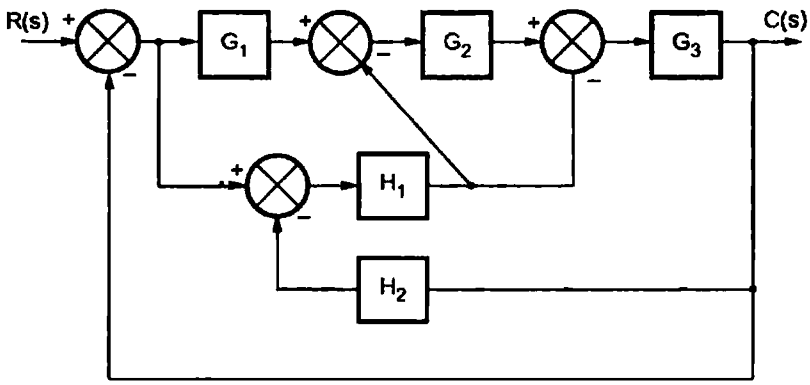
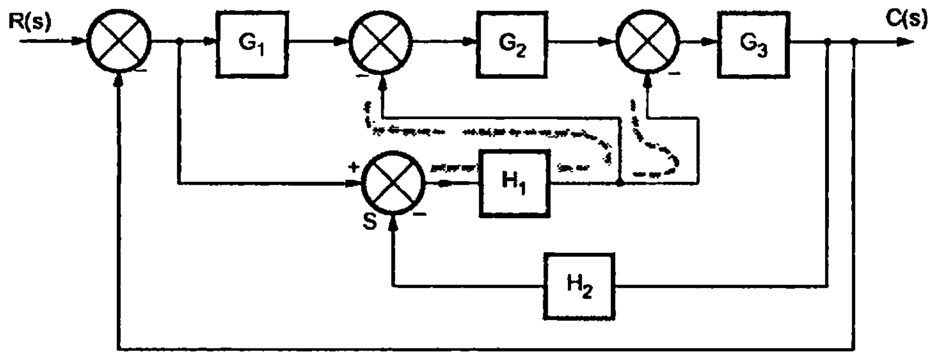
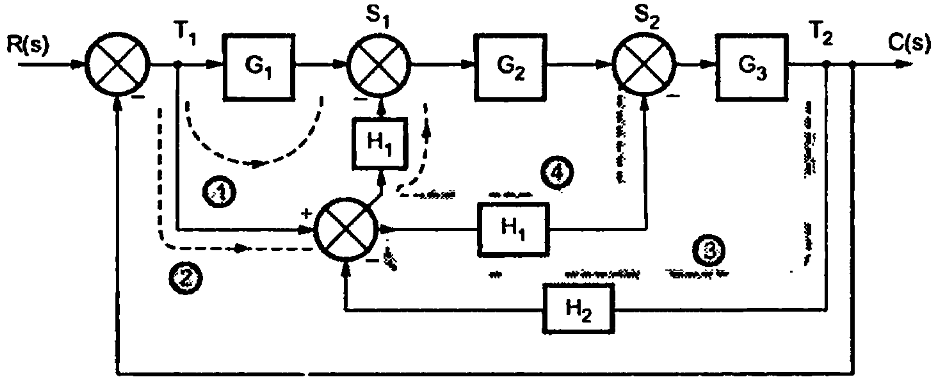


Fig. 5.49

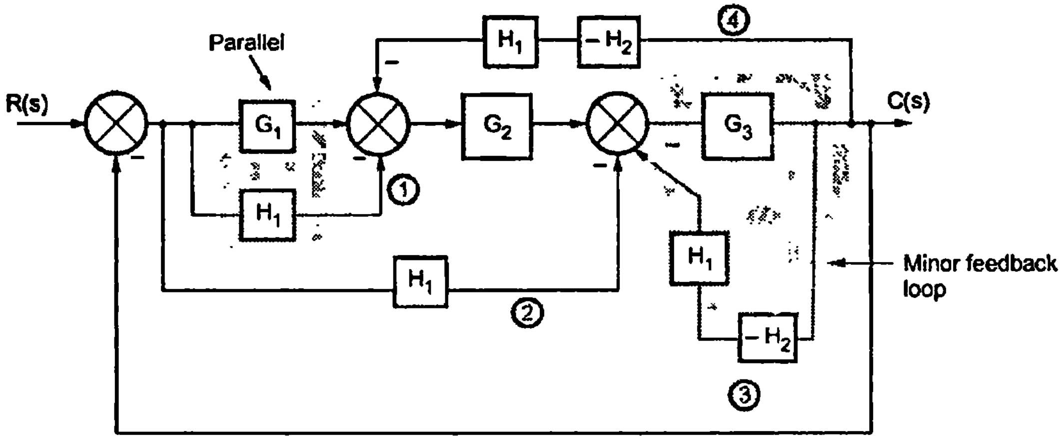
Solution : Separating the feedback paths,

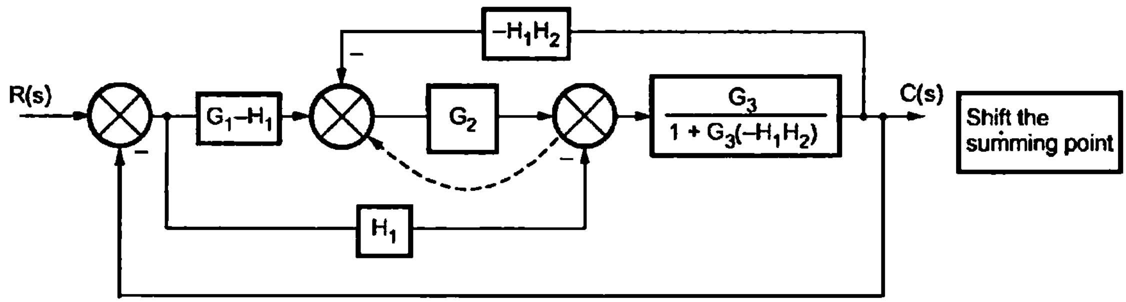


Separating the dotted paths from the summing point 'S'.

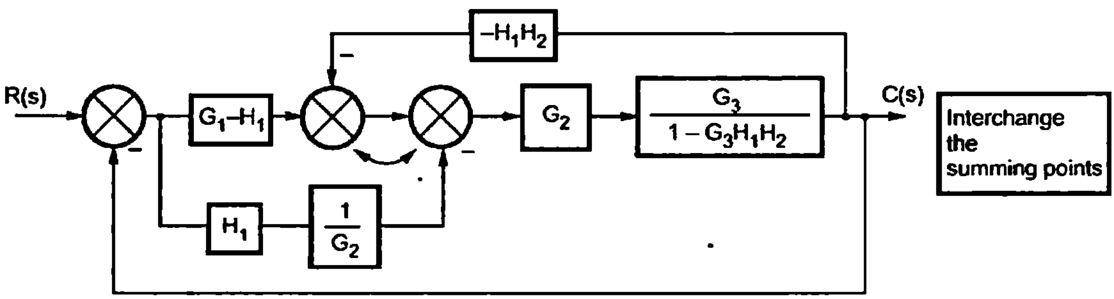


Separating the paths from T_1 to S_1, T_1 to S_2, T_2 to S_1 and T_2 to S_2.

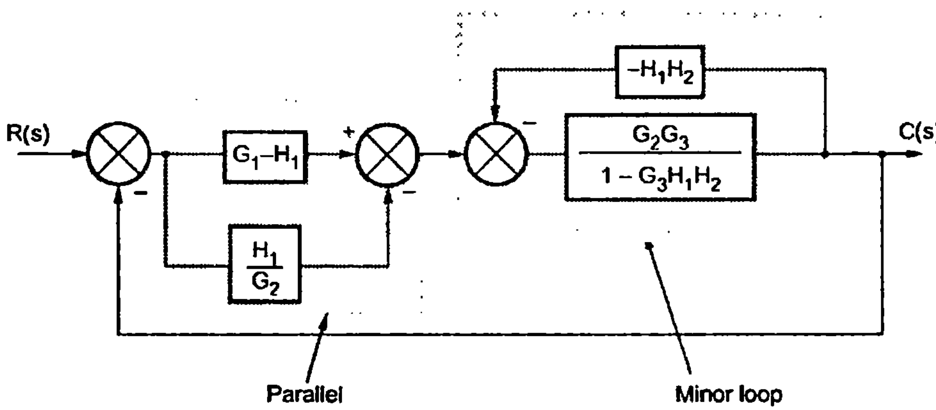




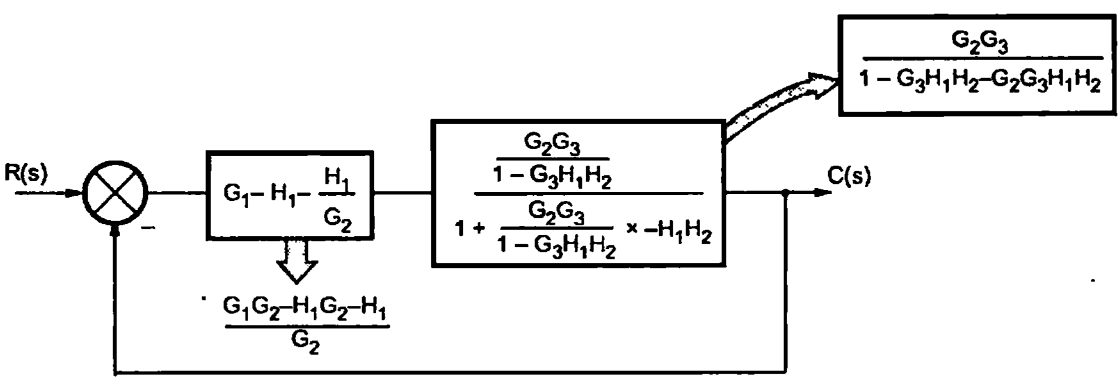
R(s)
G_1-H_1
G_2
$-H_1H_2$
$\frac{G_3}{1+G_3(-H_1H_2)}$
H_1
C(s)
Shift the summing point



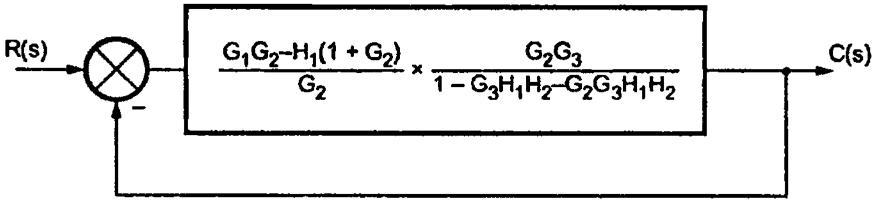
R(s)
G_1-H_1
$-H_1H_2$
G_2
$\frac{G_3}{1-G_3H_1H_2}$
H_1
$\frac{1}{G_2}$
C(s)
Interchange the summing points



R(s)
G_1-H_1
$\frac{H_1}{G_2}$
$-H_1H_2$
$\frac{G_2G_3}{1-G_3H_1H_2}$
C(s)
Parallel
Minor loop



$\frac{G_2G_3}{1-G_3H_1H_2-G_2G_3H_1H_2}$
R(s)
$G_1-H_1-\frac{H_1}{G_2}$
$\frac{G_1G_2-H_1G_2-H_1}{G_2}$
$\frac{\frac{G_2G_3}{1-G_3H_1H_2}}{1+\frac{G_2G_3}{1-G_3H_1H_2}\times -H_1H_2}$
C(s)



$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{[G_1G_2 - H_1(1+G_2)]G_3}{1-G_3H_1H_2 - G_2G_3H_1H_2}}{1+\dfrac{[G_1G_2 - H_1(1+G_2)]G_3}{1-G_3H_1H_2-G_2G_3H_1H_2}}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 - H_1G_3 - H_1G_2G_3}{1-G_3H_1H_2 - G_2G_3H_1H_2 + G_2G_2G_3 - H_1G_3 - H_1G_2G_3}}$$

Example 5.27 : *For the system represented by the block diagram shown, evaluate the closed loop transfer functions when the input R is i) at station I and in ii) at station II.*

(M.U. : May-2006)

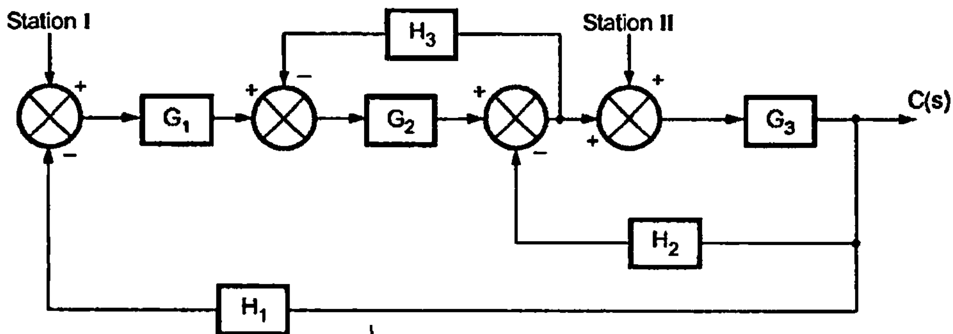
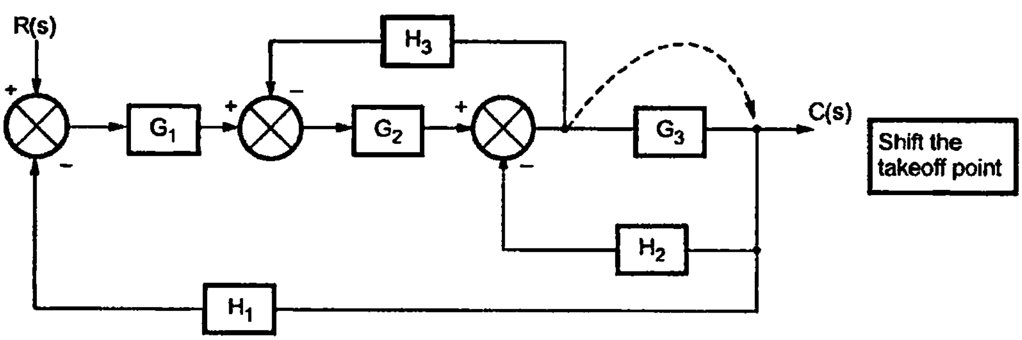
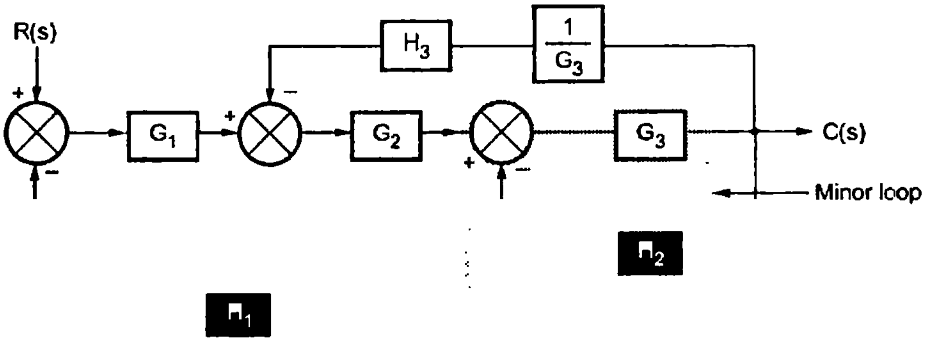
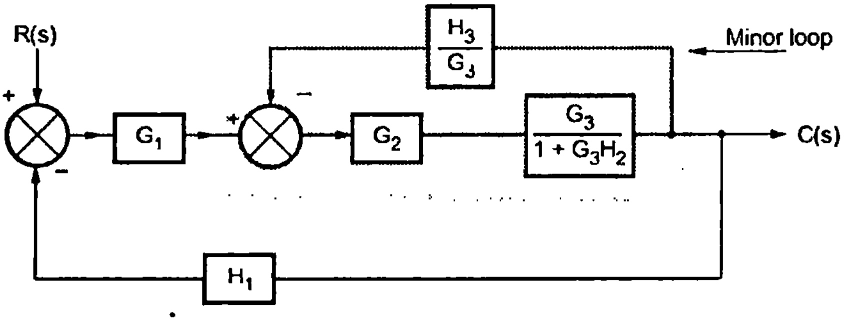


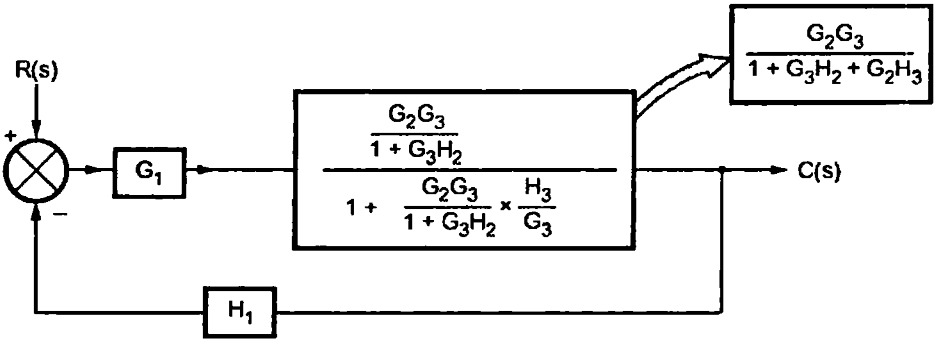
Fig. 5.50

Solution : i) Input R at station I





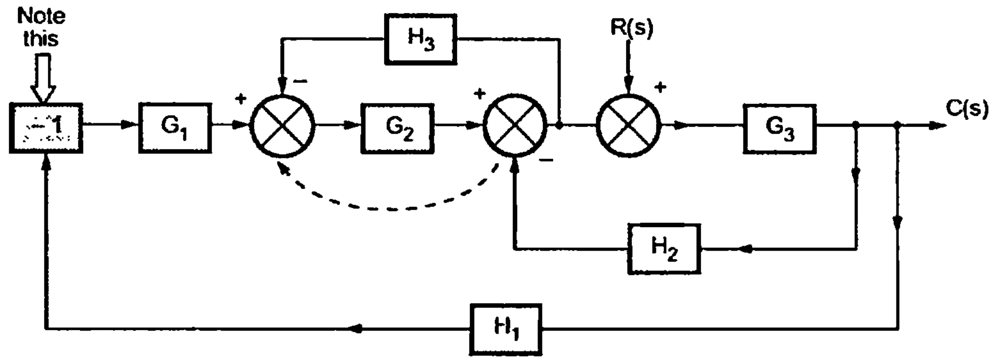




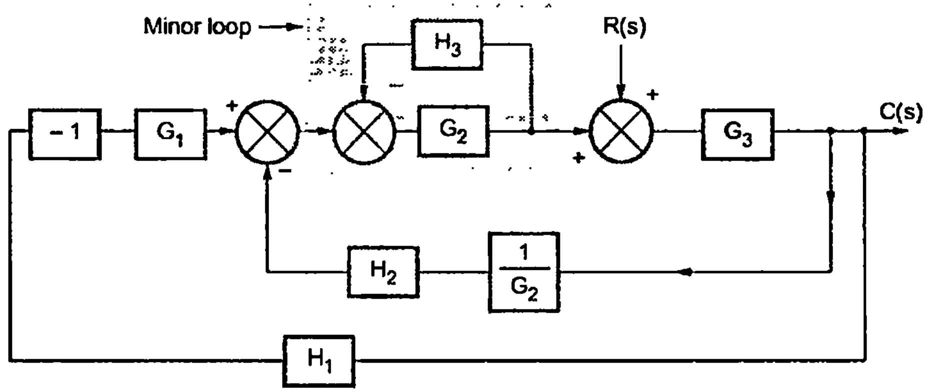
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}\times H_1} = \boxed{\frac{G_1\,G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}}$$

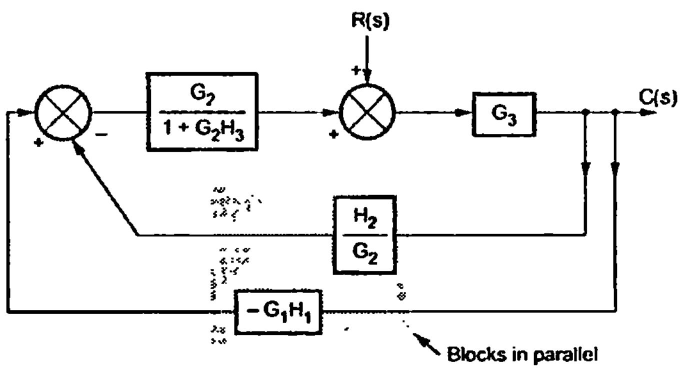
ii) Input R at station II

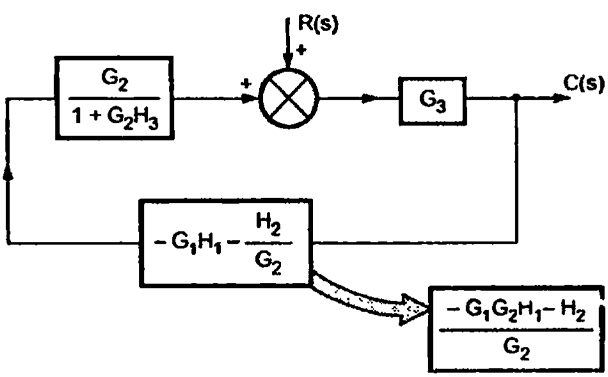
Note that at station I, though input is zero while removing the summing point, keep the negative sign of H_1 as it is. Thus add a block of '$-$ 1' as shown in the system.

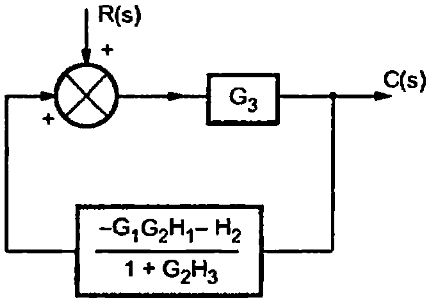


Shift the summing point to the left of G_2 and interchange the two summing points.







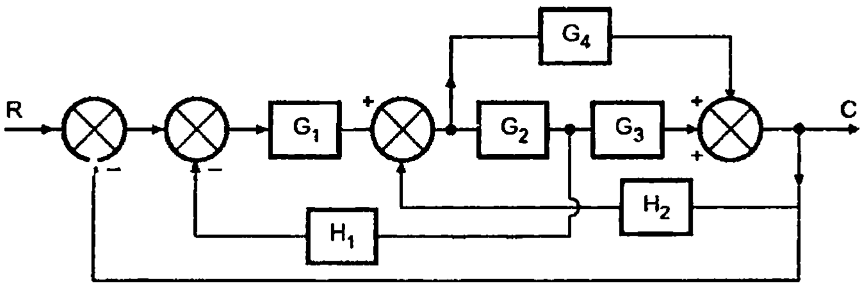


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_3}{1 - \dfrac{G_3(-G_1G_2H_1 - H_2)}{1 + G_2H_3}}$$

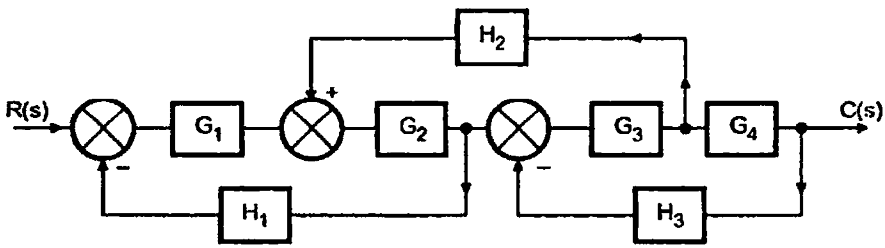
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_3(1 + G_2H_3)}{1 + G_2H_3 + G_3H_2 + G_1G_2G_3H_1}}$$

Review Questions

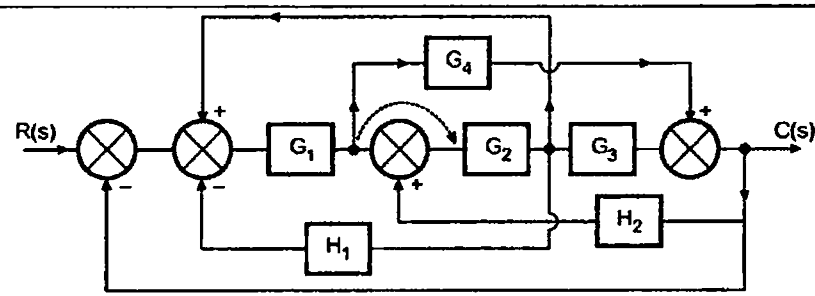
1. *What is block diagram representation ? Explain with suitable example.*
2. *State advantages and disadvantages of the block diagram reduction technique.*
3. *Explain the block diagram reduction rules.*
4. *Determine C/R ratio for the system shown below.*



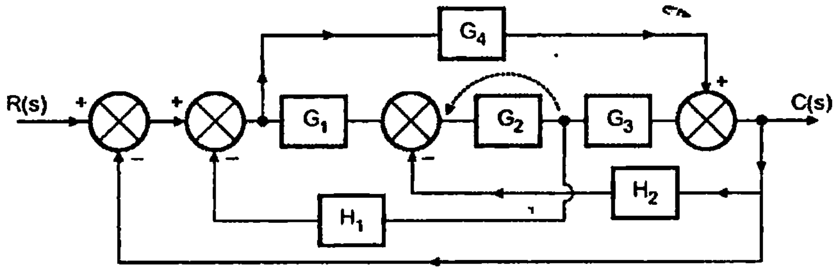
5. *Determine the transfer function of the system using block diagram reduction of the system shown.*



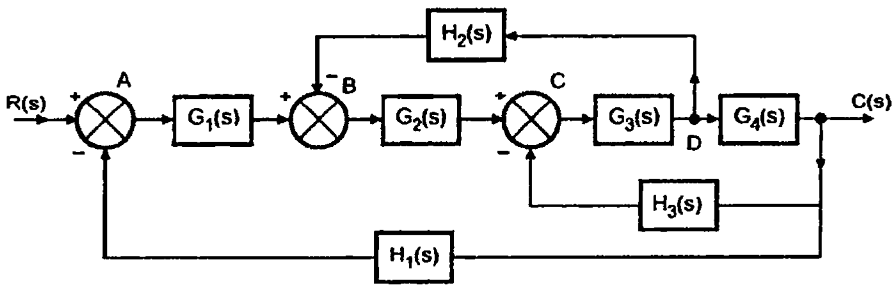
6. *For the block diagram shown, obtain C(s)/R(s) by using reduction rules.*



7. *Use block diagram reduction rules to obtain the transfer function of the block diagram shown below.*

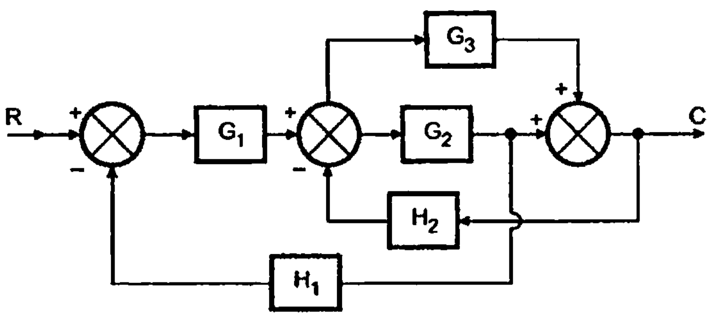


8. *Reduce the block diagram of the multiloop system shown in figure below.*



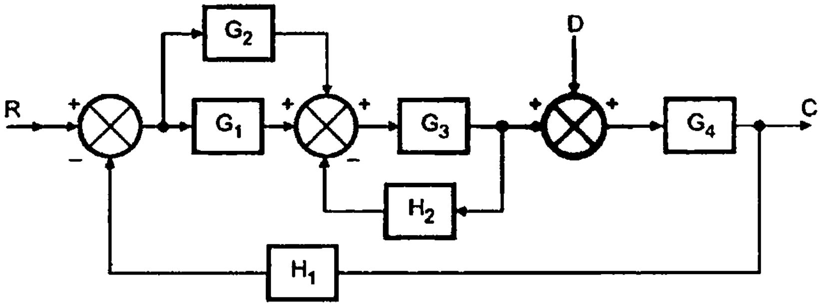
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_3 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4 H_1}\right)$$

9. *Determine the overall transfer function relating C and R for the system whose block diagram is shown in figure.*



$$\left(\text{Ans. : } \frac{C}{R} = \frac{G_1 G_2 + G_1 G_3}{1 + G_2 H_2 + G_1 G_2 H_1 - G_1 G_2 G_3 H_1 H_2}\right)$$

10. *Determine the ratio* $\frac{C}{R}, \frac{C}{D}$ *and the total output for the system whose block diagram is shown in following figure.*



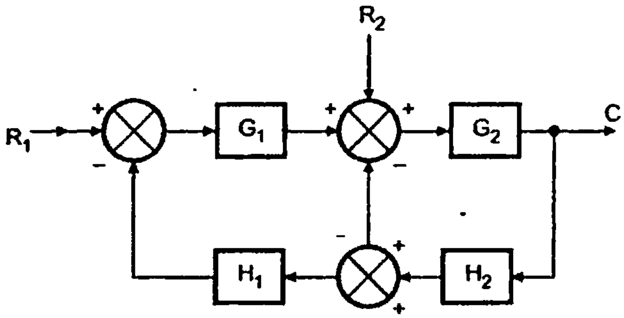
$$\left(\text{Ans.}: \frac{C}{R} = \frac{G_1 G_3 G_4 + G_2 G_3 G_4}{1 + G_3 H_2 + G_1 G_3 G_4 H_1 + G_2 G_3 G_4 H_1}\right.$$

$$\frac{C}{D} = \frac{G_4 (1 + G_3 H_2)}{1 + G_3 H_2 + G_1 G_3 G_4 H_1 + G_2 G_3 G_4 H_1}$$

$$\text{Total output} = \frac{G_1 G_3 G_4 + G_2 G_3 G_4}{1 + G_3 H_2 + G_1 G_3 G_4 H_1 + G_2 G_3 G_4 H_1} R$$

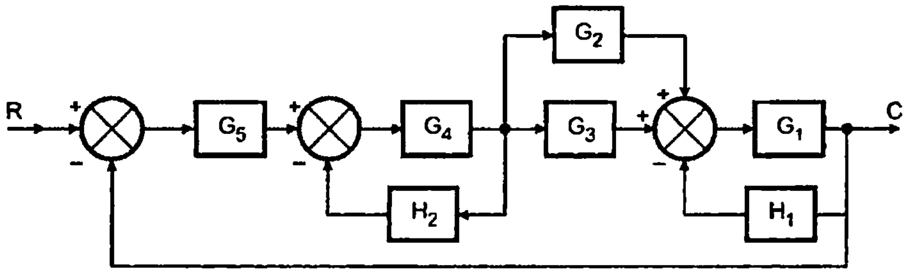
$$\left. + \frac{G_4 (1 + G_3 H_2)}{1 + G_3 H_2 + G_1 G_3 G_4 H_1 + G_2 G_3 G_4 H_1} D\right)$$

11. *Derive an expression for the total output for the system represented by the block diagram in following figure.*



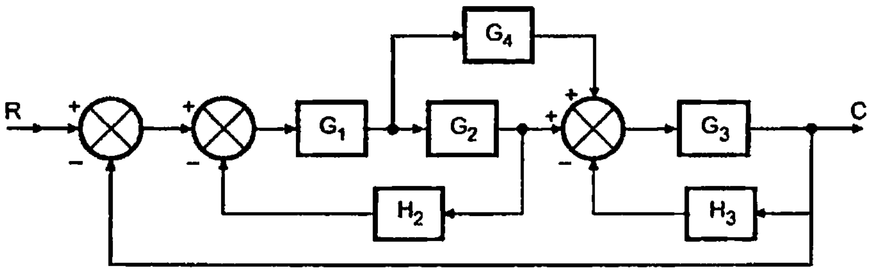
$$\left(\text{Ans.}: C = \frac{G_1 G_2 R_1 + G_2 R_2}{1 + G_1 G_2 H_1 H_2 + G_2 H_2}\right)$$

12. *Use block diagram reduction methods to obtain the equivalent transfer function from R to C.*



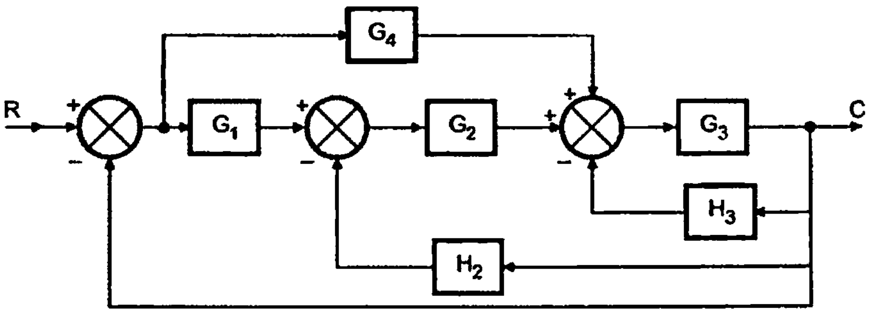
$$\left(\text{Ans. : } \frac{C}{R} = \frac{G_5 G_4 (G_2 + G_3)(G_1)}{(1 + G_4 H_2)(1 + G_1 H_1) + G_5 G_4 (G_2 + G_3) G_1}\right)$$

13. *Use block diagram reduction methods to obtain the equivalent transfer function from R to C .*



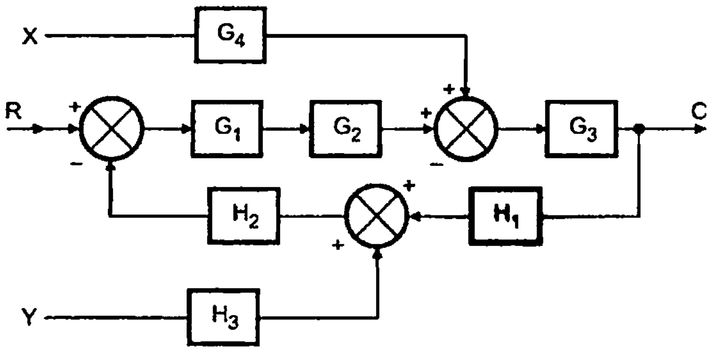
$$\left(\text{Ans. : } \frac{C}{R} = \frac{G_1 G_3 (G_2 + G_4)}{(1 + G_1 G_2 H_2)(1 + G_3 H_3) + G_1 G_3 (G_2 + G_4)}\right)$$

14. *Find the equivalent transfer function for the figure shown below.*



$$\left(\text{Ans. : } \frac{C}{R} = \frac{G_3 G_4 + G_1 G_2 G_3}{1 + G_3 H_3 + G_2 H_2 G_3 + G_3 G_4 + G_1 G_2 G_3}\right)$$

15. *Using block diagram reduction, find the transfer function from each input to the output C.*



Ans. : (Ans. : $\frac{C}{R} = \frac{G_1\,G_2\,G_3}{1+G_1\,G_2\,G_3\,H_1\,H_2}$, $\frac{C}{X} = \frac{G_4\,G_3}{1+G_3\,G_1\,G_2\,H_1\,H_2}$,

$\frac{C}{Y} = \frac{-G_1\,G_2\,G_3\,H_2\,H_3}{1+G_1\,G_2\,G_3\,H_1\,H_2}$)

6

Signal Flow Graph Representation

6.1 Background

There is one more way of representating systems particularly when set of equations describing the system is available. This representation which is obtained from the equations, which shows how signal flows in the system is called **signal flow graph representation.** As it uses the equations of the system which consist of various variables of the system, the variables of the system plays a base role in signal flow graph. Thus we can define signal flow graph as -

The graphical representation of the variables of a set of linear algebraic equations representing the system is called signal flow graph representation.

Let us see which are the important elements constituting the signal flow graph.

As variables are important elements of the set of equations for the system, these are represented first in signal flow graph by small circles called **nodes** of signal flow graph. **Each node represents a separate variable of the system.**

All the dependent and independent variables are represented by the nodes. The relationships between various nodes are represented by joining the nodes as per the equations. The lines joining the nodes are called **branches.** The branch is associated with the transfer function and an arrow. The transfer function represents mathematical operation on one variable to produce the other variable. The arrow indicates the flow of signal and **signal can travel only along an arrow.**

e.g. Consider a simple equation,

$$V = IR$$

Where V = Voltage

I = Current

R = Resistance which is parameter of the system.

Fig. 6.1

This is nothing but simple Ohm's law. Now while representing this equation by signal flow graph , first the variables voltage V and current I, are represented by nodes and they are connected by the branch as shown in the Fig. 6.1.

This represents that **voltage V depends on value of current I.** And the relationship between the two is through resistance R. So signal I gets multiplied by R to generate variable V. So R becomes **branch transfer function** or **branch gain** joining I and V. The direction of arrow is from I to V. This is shown in the Fig. 6.2.

Fig. 6.2

So all the branches represent the cause and effect relationship existing between the various variables. The branch transfer function is also called **branch gain** or **branch transmittance** in signal flow graph terminology.

6.2 Properties of Signal Flow Graph

1) The signal flow graph is applicable only to **linear time invariant systems.**
2) The signal in the system flows along the branches and along the arrowheads associated with the branches.
3) The signal gets multiplied by the branch gain or branch transmittance when it travels along it.
   e.g. Consider signal flow graph shown in the Fig. 6.3.
   The signal from x_1 gets multiplied by 4 when it travels along the branch joining x_1 to x_2. So we can say value of x_2 is 4 times the value of x_1. It shows dependence of x_2 on x_1.

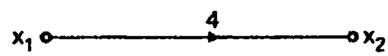


Fig. 6.3

4) **The value of variable represented by any node is an algebraic sum of all the signals entering at the node**

   e.g. Consider the variable x_2. At that node, 3 signals are entering from x_1, x_3 and x_4. So value of x_2 depends on the variables x_1, x_3 and x_4. The branch gains indicate the exact contribution of each variable in generating x_2. So value of x_2 is algebraic sum of all such signals entering. So we can write,

$$x_2 = 4x_1 - 2x_3 + 3x_4$$

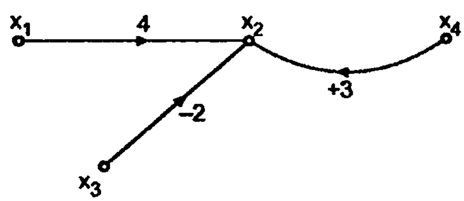


Fig. 6.4

5) The value of the variable represented by any node is available to all the branches leaving that node. **The number of branches leaving a node does not affect the value of variable represented by that node.**

   e.g. Consider signal flow graph represented in Fig. 6.5. The value of x_2 can be obtained from signals entering at x_2.

   i.e. $$x_2 = 5x_1 - 2x_3$$

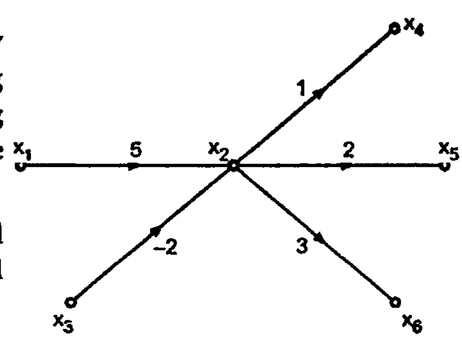


Fig. 6.5

Now there are three branches leaving x_2, joining to x_4, x_5 and x_6 that means x_4, x_5 and x_6 variables depend on x_2.

So we can write $x_4 = x_2$, $x_5 = 2x_2$, $x_6 = 3x_2$

i.e. for all branches leaving from x_2, the value of x_2 available is same and number of such branches do not affect the value of x_2.

Key Point : *The value of a variable represented by node depends only on the signals entering and this value is as it is available to all the branches leaving from that node.*

6) For a given system signal flow graph is not unique. Many other graphs can be drawn by writing system equations in different manner.

Key Point: *The signal flow graph is not the unique property of the system.*

6.3 Terminology used in Signal Flow Graph

Consider a signal flow graph shown in the Fig. 6.6

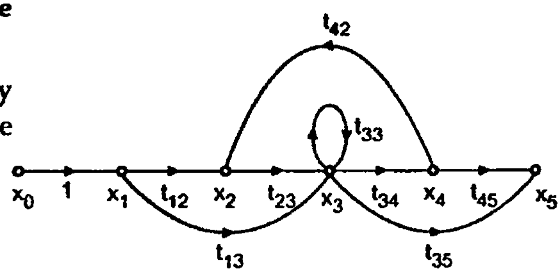


Fig. 6.6

i) **Source Node :** The node having only outgoing branches is known as source or input node. e.g. x_0 is source node.

ii) **Sink Node :** The node having only incoming branches is known as sink or output node. e.g. x_5 is sink node.

iii) **Chain Node :** A node having incoming and outgoing branches is known as chain node. e.g. x_1, x_2, x_3 and x_4.

iv) **Forward Path :** A path from the input to output node is defined as forward path.

e.g. $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$ First forward path

$x_0 - x_1 - x_3 - x_4 - x_5$ Second forward path.

$x_0 - x_1 - x_3 - x_5$ Third forward path.

$x_0 - x_1 - x_2 - x_3 - x_5$ Fourth forward path.

v) **Feedback Loop :** A path which originates from a particular node and terminating at the same node, travelling through atleast one other node, without tracing any node twice is called feedback loop. For example, $x_2 - x_3 - x_4 - x_2$.

vi) **Self Loop :** A feedback loop consisting of only one node is called self loop. i.e. t_{33} at x_3 is self loop. A self loop can not appear while defining a forward path or feedback loop as node containing it gets traced twice which is not allowed.

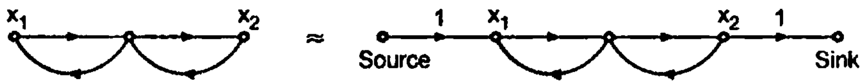
vii) **Path Gain :** The product of branch gains while going through a forward path is known as path gain. i.e. path gain for path $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$ is ,

$1 \times t_{12} \times t_{23} \times t_{34} \times t_{45}$. This can be also called **forward path gain.**

Key Point: *While tracing a forward path or a feedback loop, no node is to be traced twice.*

viii) **Dummy Node :** If there exists incoming and outgoing branches both at first and last node representing input and output variables, then as per definition these can not be called source and sink nodes. In such a case a separate input and output nodes can be created by adding branches with gain 1. Such nodes are called dummy nodes.

e.g.



(a) (b)

Fig. 6.7

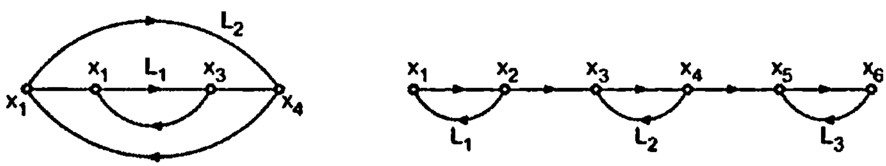
In the signal flow graph x_1 and x_2 are input and output variables but as per definition not input and output nodes. Such independent nodes can be generated by adding branches of gain 1 as shown in the Fig. 6.7 (b).

Note : Such creation of dummy nodes is not necessary. Without this also, signal flow graph can be analysed to get the overall transfer function.

Key Point: *Addition of branches of gain 1 is possible only before starting node and after the last node. In between the chain nodes such branches of gain 1 cannot be added.*

ix) **Non touching loops :** If there is no node common in between the two or more loops, such loops are said to be non touching loops.

The Fig. 6.8 (a)&(b) show a combination of non touching loops of two and three loops.

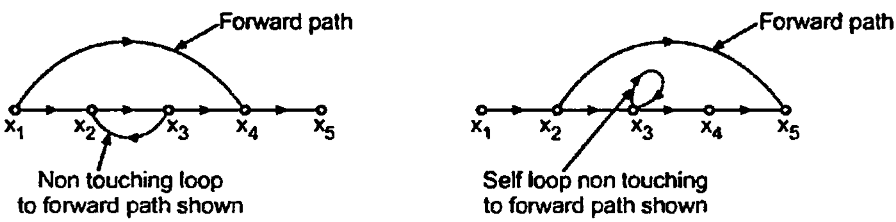


(a) Two non touching loops **(b) Three non touching loops**

Fig. 6.8

Similarly if there is no node common in between a forward path and a feedback loop, a loop is said to be non touching to that forward path.

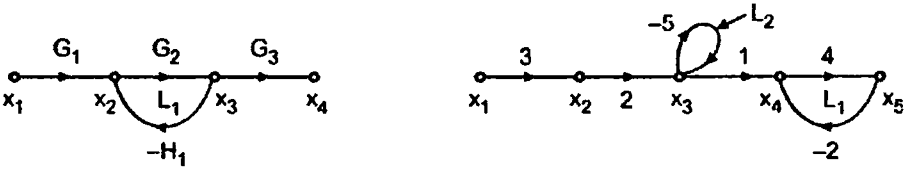
The Fig. 6.9 (a) and (b) shows such a loop which is non touching to a forward path.



(a) (b)

Fig. 6.9

x) **Loop Gain :** The product of all the gains of the branches forming a loop is called loop gain. For a self loop, gain indicated along it is its gain. Generally such loop gains are denoted by 'L' e.g. L_1, L_2 etc.



(a) (b)

Fig. 6.10

In the Fig. 6.10 (a), there is one loop with gain $L_1 = G_2 \times -H_1 = -G_2H_1$

In the Fig. 6.10 (b), there are two loops with gains.

$$L_1 = 4 \times -2 = -8 \text{ and other self loop with } L_2 = -5$$

6.4 Methods to Obtain Signal Flow Graph

6.4.1 From the System Equations

Steps :

1) *Represent each variable by a separate node.*
2) *Use the property that value of the variable represented by a node is an algebraic sum of all the signals entering at that node, to simulate the equations.*
3) *Coefficients of the variables in the equations are to be represented as the branch gains, joining the nodes in signal flow graph.*
4) *Show the input and output variables separately to complete signal flow graph.*

Example : Consider the system equations as say,

$V_1 = 2V_i + 3V_2$ ...(1)

$V_3 = 5V_2 + V_o$ ...(3)

$V_2 = 4V_1 + 5V_3 + 2V_2$ ...(2)

$V_o = 6V_3$ ...(4)

Let output be V_o and input be V_i where V_1 , V_2 , V_3 are the system variables.

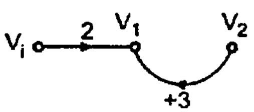


Fig. 6.11(a)

Equation (1) shows that V_1 depends on V_i and V_2. So there are two branches entering at node V_1 i.e. from V_i and V_2. But branch from V_2 to V_1 has direction from output to input hence to be shown as a feedback path as in the Fig. 6.11(a). Similarly all equations are to be simulated and joined to get complete signal flow graph.

Signal from output side towards input i.e. from, V_2 to V_1, V_3 to V_2 and so on are to be indicated as feedback paths.

In equation (2) there is component of V_2 itself, contributing to generate the variable V_2. This results in a self loop in a signal flow graph. The complete signal flow graph is shown in the Fig. 6.11(b).

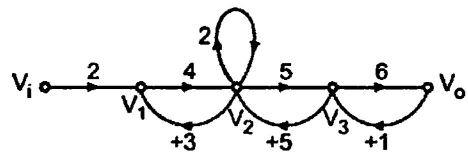


Fig. 6.11 (b)

6.4.2 From the Given Block Diagram

Steps :

1) *Name all the summing points and take off points in the block diagram.*
2) *Represent each summing and take off point by a separate node in signal flow graph*
3) *Connect them by the branches instead of blocks, indicating block transfer functions as the gains of the corresponding branches.*
4) *Show the input and output nodes separately if required, to complete signal flow graph.*

Example :

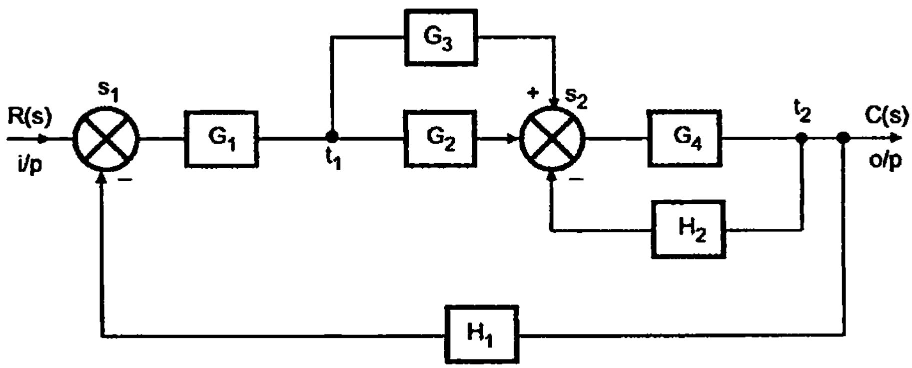


Fig. 6.12

Naming summing and take off points as shown in the Fig. 6.13.

Key Point: *Make sure that if summing and take off points are near each other in a given block diagram, they are to be represented by separate nodes in the corresponding signal flow graph.*

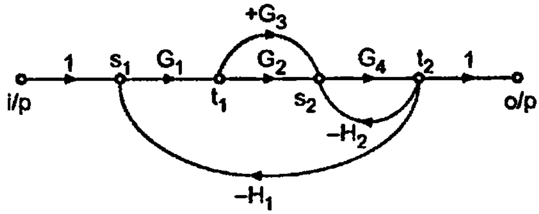


Fig. 6.13

6.5 Mason's Gain Formula

It is seen earlier that in block diagram representation, we have to apply reduction rules, one after the other to obtain simple form of the system and hence overall transfer function. We have to draw the reduced block diagram after every step. This is time consuming. In signal flow graph approach, once S.F.G is obtained, direct use of one formula leads to the overall system transfer function $\frac{C(s)}{R(s)}$. This formula is stated by Mason and hence referred as Mason's gain formula. The formula can be stated as :

$$\text{Overall T.F.} = \frac{\Sigma T_K \Delta_K}{\Delta}$$

where K = Number of forward paths

T_K = Gain of K^{th} forward path

Δ = System determinant to be calculated as :

Δ = 1–[Σ all individual feedback loop gains [including self loops]] + [Σ gain × gain product of all possible combinations of two non touching loops] – [Σ gain × gain × gain product of combinations of three non touching loops] +....

Δ_K = Value of above Δ by eliminating all loop gains and associated products which are touching to the K^{th} forward path.

Explanation : If we have identified in signal flow graph following information that,

Number of forward paths K = 3.

The gains i.e. product of branch gains involved in defining various forward paths are denoted as T_1, T_2 and T_3.

Now number of loops including self loops are say 3 and their gains are say L_1, L_2, and L_3.

Out of these three, $L_1 L_2$ and $L_1 L_3$ are the combinations of two non touching loops.

There is no combination of three non touching loops.

∴ Δ = 1 – [Σ all individual and self loop gains] + [Σ gain × gain product of all combinations of two non touching loops]

∴ $$\Delta = 1 - [L_1 + L_2 + L_3] + [L_1 L_2 + L_1 L_3]$$

Now for Δ_K i.e. Δ_1, Δ_2 and Δ_3 consider each forward path separately.

For T_1, say all loops L_1, L_2 and L_3 are touching to T_1 hence all loop gains are to be eliminated from Δ to get Δ_1.

$$\therefore \quad \Delta_1 = 1$$

For T_2, say L_2 is non touching and L_1, L_3 are touching. So L_1, L_3 and associative products i.e. $L_1\ L_2$ and $L_1\ L_3$ are to be eliminated to get Δ_2, L_2 will exist.

$$\therefore \quad \Delta_2 = 1 - L_2$$

For T_3, say L_1 and L_2 both are non touching and L_3 is touching. So L_3 and associated product $L_1\ L_3$ must be eliminated L_1, L_2 and product $L_1\ L_2$ as both are non touching to T_3 will exist in Δ_3

$$\therefore \quad \Delta_3 = 1 - L_1 - L_2 + L_1 L_2$$

Hence T.F. can be obtained by substituting these values in

$$\text{T.F.} = \frac{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3}{\Delta}$$

➠ **Example 6.1 :** *Find the overall T.F. by using Mason's gain formula for the signal flow graph given in the Fig. 6.14*

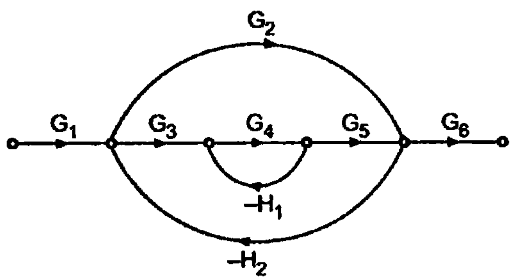


Fig. 6.14

Solution : Two forward paths, K = 2,

$$T_1 = G_1 G_3 G_4 G_5 G_6$$

$$T_2 = G_1 G_2 G_6$$

Loops are,

$$L_1 = -G_4 H_1$$

$$L_2 = -G_3 G_4 G_5 H_2$$

$$L_3 = -G_2 H_2$$

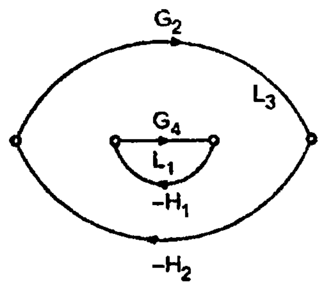


Fig. 6.14(a) Non touching loops

Out of these, L_1 and L_3 is combination of 2 non touching loops

$$\Delta = 1 - [L_1 + L_2 + L_3] + [L_1 L_3]$$

Δ_1 = Eliminate L_1, L_2, L_3 as all are touching to T_1 from Δ

$\therefore \quad \Delta_1 = 1$

Δ_2 = Eliminate L_2 and L_3, as they are touching to T_2, from Δ. But L_1 is non touching hence keep it as it is in Δ.

$\therefore \quad \Delta_2 = 1 - [L_1]$

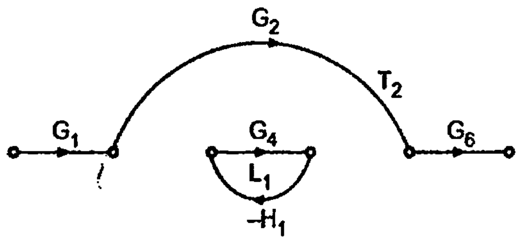


Fig. 6.14(b) L_1 Non touching to T_2

Substitute in Mason's gain formula,

$$T.F. = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

$$T.F. = \frac{G_1 G_3 G_4 G_5 G_6 [1] + G_1 G_2 G_6 [1 + G_4 H_1]}{1 + G_4 H_1 + G_3 G_4 G_5 H_2 + G_2 H_2 + G_2 G_4 H_1 H_2}$$

6.6 Comparison of Block Diagram and Signal Flow Graph Methods

The comparison of block diagram representation and signal flow graph is given in a tabular form as :

Sr. No.	Block Diagram	Signal Flow Graph
1.	Basic importance given is to the elements and their transfer functions.	Basic importance given is to the variables of the systems.
2.	Each element is represented by a block.	Each variable is represented by a separate node.
3.	Transfer function of the element is shown inside the corresponding block.	The transfer function is shown along the branches connecting the nodes.
4.	Summing points and takeoff points are separate.	Summing and takeoff points are absent. Any node can have any number of incoming and outgoing branches.
5.	Feedback path is present from output to input.	Instead of feedback path, various feedback loops are considered for the analysis.

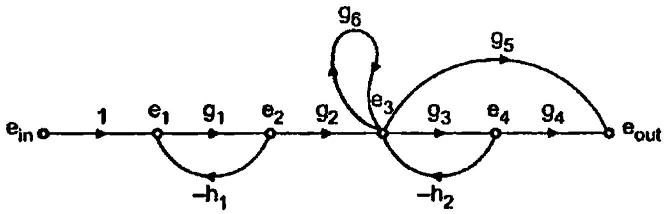
6.	For a minor feedback loop present, the formula $\frac{G}{1\pm GH}$ can be used.	Gains of various forward paths and feedback loops are just the product of associative branch gains. No such formula $\frac{G}{1\pm GH}$ is necessary.
7.	Block diagram reduction rules can be used to obtain the resultant transfer function.	The Mason's Gain Formula is available which can be used directly to get resultant transfer function without reduction of signal flow graph.
8.	Method is slightly complicated and time consuming as block diagram is required to be drawn time to time after each step of reduction.	No need to draw the signal flow graph again and again. Once drawn, use of Mason's Gain Formula gives the resultant transfer function.
9.	Concept of self loop is not existing in block diagram approach.	Self loops can exist in signal flow graph approach.
10.	Applicable only to linear time invariant systems.	Applicable to linear time invariant systems.

6.7 Application of the General Gain Formula between Output Nodes and Non Input Nodes

It was derived earlier that Mason's gain formula is used to get a relation between output node and input node called transfer function.

But often, it is required to calculate the relation between output node variable and a non input node variable.

Example :



Now it is required to calculate $\frac{e_{out}}{e_2}$ i.e. dependence of e_{out} on e_2 and e_2 is not input variable.

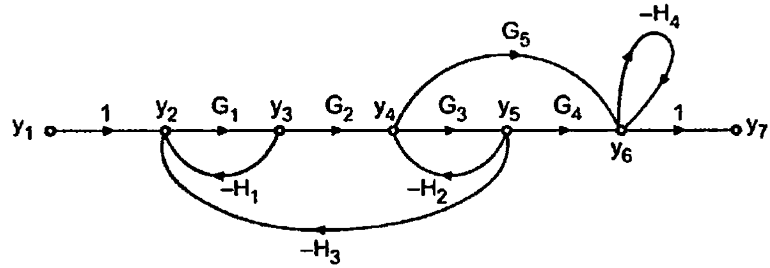
So $\frac{e_{out}}{e_2}$ can be expressed as

$$\frac{e_{out}}{e_2} = \frac{\dfrac{e_{out}}{e_{in}}}{\dfrac{e_2}{e_{in}}} = \frac{\left.\dfrac{\Sigma T_K \Delta_K}{\Delta}\right|_{\text{from } e_{in} \text{ to } e_2}}{\left.\dfrac{\Sigma T_K \Delta_K}{\Delta}\right|_{\text{from } e_{in} \text{ to } e_{out}}}$$

Since Δ is independent of inputs and outputs,

$$\frac{e_{out}}{e_2} = \frac{\Sigma T_K \Delta_K \big|_{\text{from } e_{in} \text{ to } e_{out}}}{\Sigma T_K \Delta_K \big|_{\text{from } e_{in} \text{ to } e_2}}$$

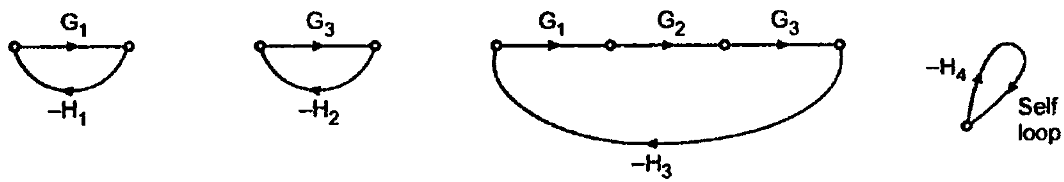
Example 6.2 : *Calculate* $\frac{y_7}{y_2}$ *of the system, whose signal flow graph is given below.*



Solution : Forward paths for y_1 to y_7 are two

$$T_1 = G_1 G_2 G_3 G_4, \quad T_2 = G_1 G_2 G_5$$

Individual feedback loops are



$L_1 = -G_1 H_1$ $\quad L_2 = -G_3 H_2$ $\quad L_3 = -G_1 G_2 G_3 H_3$ $\quad L_4 = -H_4$

Combinations of two non touching loops

$$L_1 L_2 = +G_1 G_3 H_1 H_2, \qquad L_1 L_4 = +G_1 H_1 H_4$$

$$L_2 L_4 = +G_3 H_2 H_4, \qquad L_3 L_4 = +G_1 G_2 G_3 H_3 H_4$$

One combination of three non touching

$$L_1 L_2 L_4 = -G_1 G_3 H_1 H_2 H_4$$

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1 L_2 + L_1 L_4 + L_2 L_4 + L_3 L_4] - [L_1 L_2 L_4]$$

$\Delta_1 = 1$ all loops are touching

$\Delta_2 = 1$ all loops are touching

$$\frac{y_7}{y_1} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_5}{\Delta}$$

Now to find the ratio $\frac{y_2}{y_1}$

Forward paths for y_1 to y_2 is one. $T_1 = 1$ Now Δ is same as above.

and $\Delta_1 = 1 - L_2 - L_4 + L_2 L_4$ ... L_2 and L_4 nontouching to $T_1 = 1$

$$= 1 + G_3 H_2 + H_4 + G_3 H_2 H_4$$

$$\therefore \quad \frac{y_2}{y_1} = \frac{T_1 \Delta_1}{\Delta} = \frac{1 + G_3 H_2 + H_1 + G_3 H_2 H_4}{\Delta}$$

$$\therefore \quad \boxed{\frac{y_7}{y_2} = \frac{\frac{y_7}{y_1}}{\frac{y_2}{y_1}} = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_5 (1 + G_3 H_2)}{1 + G_3 H_2 + H_4 + G_3 H_2 H_4}}$$

6.8 Application of Mason's Gain Formula to Electrical Network

To apply Mason's gain formula, it is necessary to draw the signal flow graph of the network. This section explains simple method of writing network equations from which signal flow graph can be easily obtained. Applying Mason's gain formula to the signal flow graph, transfer function of the network is obtained.

General procedure to solve such network is as follows,

i) Find out the Laplace transform of the given network and redraw the network in s-domain.

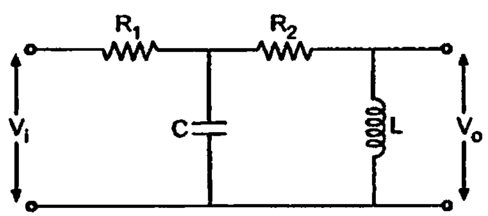
ii) Write down the equations for the different branch currents and node voltages.

iii) Simulate each equation by drawing corresponding signal flow graph.

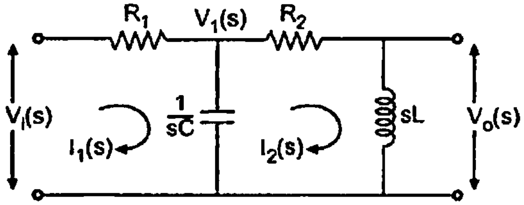
iv) Combine all signal flow graphs to get total signal flow graph for the given network.

v) Use Mason's gain Formula to derive the transfer function of the given network.

Example 6.3 : *Find the T.F. for the given network.*



Solution : Laplace Transform of the given network is as shown in following figure.



Key Point: *Assume the network variables alternately as the loop current and node voltage and then write the equations by analysing the horizontal and vertical branches alternately.*

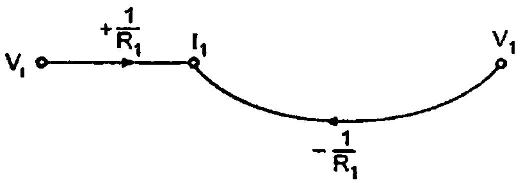
$$I_1(s) = \frac{(V_i - V_1)}{R_1} \quad \text{... (I)}$$

$$V_1 = (I_1 - I_2)\frac{1}{sC} \quad \text{... (II)}$$

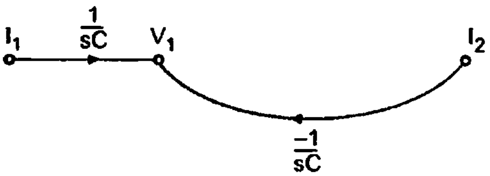
$$I_2 = \frac{(V_1 - V_o)}{R_2} \quad \text{...(III)}$$

$$V_o = I_2\, sL \quad \text{...(IV)}$$

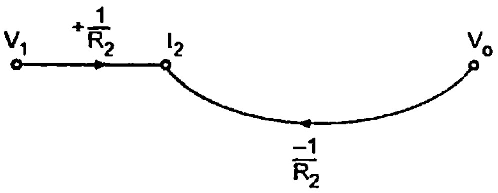
S.F.G. for equation (I) :



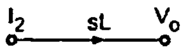
S.F.G. for equation (II) :



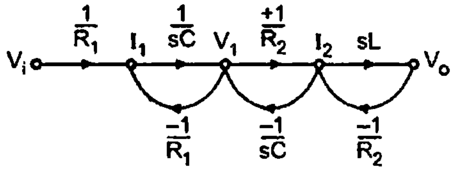
S.F.G. for equation (III) :



S.F.G. for equation (IV) :



Total Signal Flow Graph for the network is as follows.



Use Mason's gain formula to find $\frac{V_o}{V_i}$.

$$\frac{V_o}{V_i} = \frac{\Sigma\, T_K\ \Delta_K}{\Delta}\ ;\ \text{Number of forward paths} = 1$$

$$\therefore \quad \frac{V_o}{V_i} = \frac{T_1\ \Delta_1}{\Delta}$$

$$T_1 = \frac{L}{R_1\, R_2 C}$$

Individual feedback loops are,

$$L_1 = -\frac{1}{sR_1\, C},\qquad L_2 = -\frac{1}{sR_2\, C},\qquad L_3 = -\frac{sL}{R_2}$$

L_1 and L_3 are non touching.

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3] + [L_1\, L_3]$$

$$\Delta = 1 + \frac{1}{sR_1\, C} + \frac{1}{sR_2\, C} + \frac{sL}{R_2} + \frac{L}{R_1\, R_2\, C}$$

As all loops are touching to T_1 , $\Delta_1 = 1$,

$$\therefore \quad \frac{V_o}{V_i} = \frac{\frac{L}{R_1\, R_2\, C}}{1 + \frac{1}{sR_1\, C} + \frac{1}{sR_2\, C} + \frac{sL}{R_2} + \frac{L}{R_1\, R_2\, C}}$$

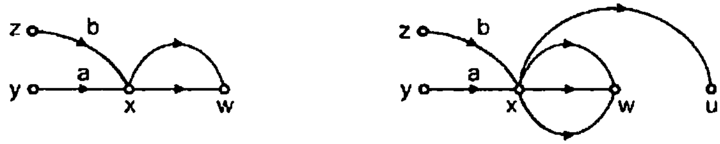
$$\therefore \quad \frac{V_o}{V_i} = \frac{sL}{sR_1\, R_2\, C + R_2 + R_1 + s^2\, L R_1\, C + sL}$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{sL}{s^2 L R_1 C + s[L + R_1 R_2 C] + (R_1 + R_2)}}$$

6.9 Obtaining Block Diagram from Signal Flow Graph

To obtain the block diagram from given signal flow graph, it is necessary to write the set of system equations representing the given signal flow graph. Assume suitable node variables, write the equation for every node. While writing the equation remember that the

value of the variable represented by a node is an algebraic sum of all the signals entering at that node. The number of outgoing branches have no effect on the value of the node variable. For example consider the part of signal flow graph shown in the Fig. 6.15.



(a) (b)

Fig. 6.15

In both the case

$$x = ay + bz$$

And the number of outgoing branches have no effect on the value of node variable x.

This set of equations can be represented by block diagram, simulating each equation separately.

Key Point: *For any + or – sign in equation, there exists a summing point while for each branch gain of signal flow graph, there exists a block of same transfer function as branch gain, in the block diagram.*

For example, consider the equation.

$$V_1 = 2V_2 - 4V_3$$

Here 2 and 4 are the gains; correspondingly there exists the blocks of transfer functions 2 and 4 and a summing point for a minus sign. The block diagram simulation of above equation is shown in the Fig. 6.16.

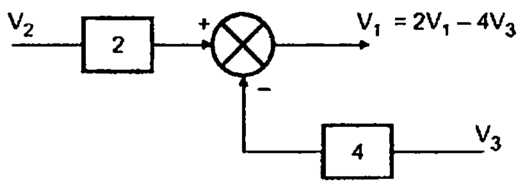
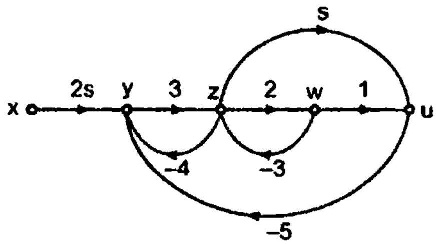


Fig. 6.16

Simulating all the equations in the same manner and joining all of them, the required block diagram can be obtained.

Example 6.4 : *Obtain the block diagram for the signal flow graph, shown in the figure below.*



Solution : Write the equations for various node variables y, z, w and u. The input node is x as there are only outgoing branches.

$$y = (2s)x - (4)z - (5)u \quad \text{... (1)}$$

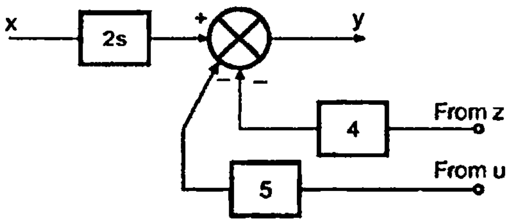
$$z = (3)y - (3)w \quad \text{... (2)}$$

$$w = (2)z \quad \text{... (3)}$$

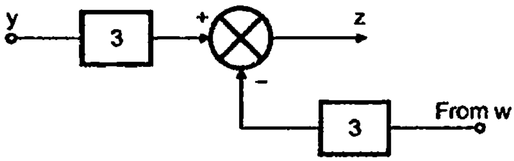
$$u = (1)w + (s)z \quad \text{... (4)}$$

The block diagram simulations of various equations are,

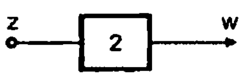
Equation (1)



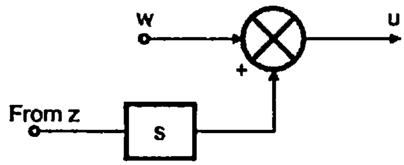
Equation (2),



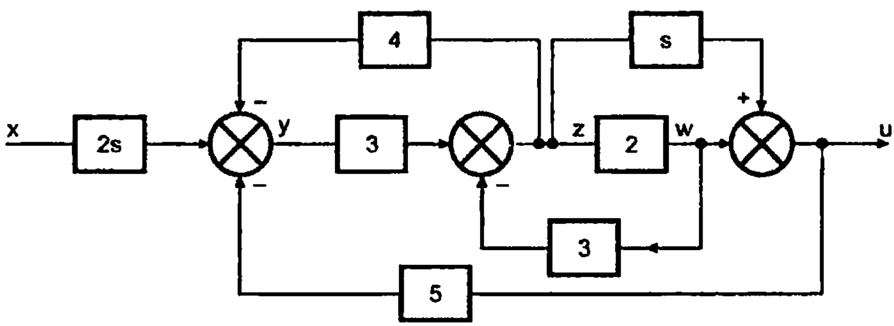
Equation (3),



Equation (4),

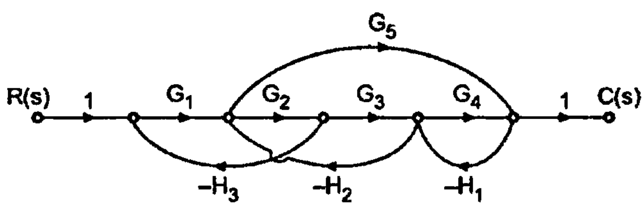


Combining all the simulations and repositioning branches for convenience we get the complete block diagram as,



Examples with Solutions

Example 6.5 : *Find* $\frac{C(s)}{R(s)}$ *for S.F.G. shown in following figure.*



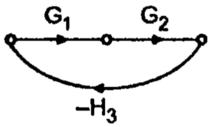
Solution : Number of forward paths = K = 2

$$\therefore \qquad \text{T.F.} = \frac{\sum_{K=1}^{2} T_K \Delta_K}{\Delta} \qquad \text{using Mason's gain Formula}$$

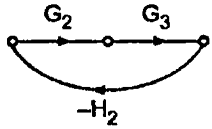
$$T_1 = G_1 G_2 G_3 G_4$$

$$T_2 = G_1 G_5$$

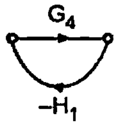
Individual feedback loops,



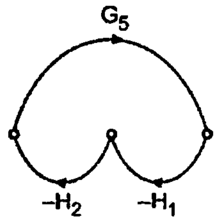
$L_1 = -G_1G_2H_3$



$L_2 = -G_2G_3H_2$



$L_3 = -G_4H_1$



$L_4 = +G_5H_1H_2$

Loops L_1 and L_3 are non touching loops.

$$\therefore \quad \Delta = [L_1 + L_2 + L_3 + L_4] + [L_1 L_3]$$

$$= 1 - [-G_1G_2H_3 - G_2G_3H_2 - G_4H_1 + G_5H_1H_2] + [G_1G_2G_4H_1H_3]$$

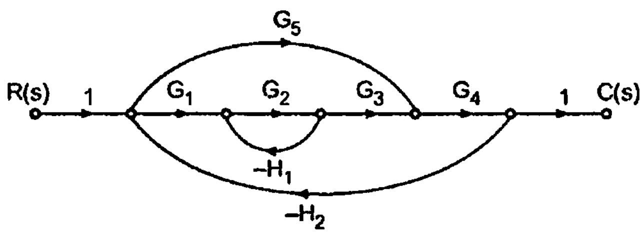
Consider T_1 , all loops are touching $\therefore \ \Delta_1 = 1$

Consider T_2, all loops are touching $\therefore \ \Delta_2 = 1$

$$\therefore \text{ T.F.} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 G_4 \cdot 1 + G_1 G_5 \cdot 1}{1 + G_1 G_2 H_3 + G_2 G_3 H_2 + G_4 H_1 - G_5 H_1 H_2 + G_1 G_2 G_4 H_1 H_3}$$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 + G_1 G_5}{1 + G_1 G_2 H_3 + G_2 G_3 H_2 + G_4 H_1 - G_5 H_1 H_2 + G_1 G_2 G_4 H_1 H_3}$$

Example 6.6 : *Find* $\frac{C(s)}{R(s)}$ *by using Mason's gain formula.*



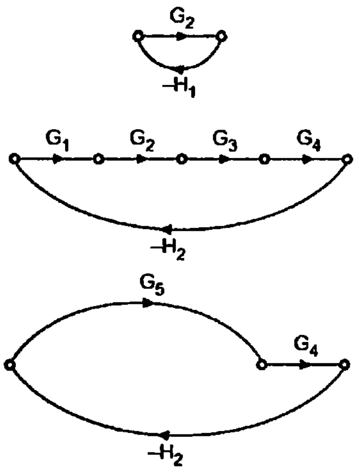
Solution : Number of forward paths K = 2

Mason's gain formula,

$$\text{T.F.} = \frac{\sum_{K=1}^{2} T_K \Delta_K}{\Delta} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

$$T_1 = G_1 G_2 G_3 G_4, \quad T_2 = G_5 G_4$$

Individual feedback loops are,



$L_1 = -G_2H_1$

$L_2 = -G_1G_2G_3G_4H_2$

$L_3 = -G_5G_4H_2$

L_1 and L_3 are two non touching loops.

$$\therefore \quad \Delta = 1-[L_1 + L_2 + L_3]+[L_1 L_3]$$

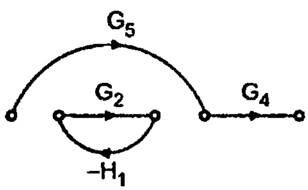
$$= 1-[-G_2 H_1 - G_1 G_2 G_3 G_4 H_2 - G_5 G_4 H_2]+[G_2 H_1 G_5 G_4 H_2]$$

$$= 1+G_2 H_1 + G_1 G_2 G_3 G_4 H_2 + G_5 G_4 H_2 + G_2 G_5 G_4 H_1 H_2$$

For T_1 all loops are touching

$\therefore \ \Delta_1 = 1$ eliminating all loop gains and products from Δ

Consider T_2,



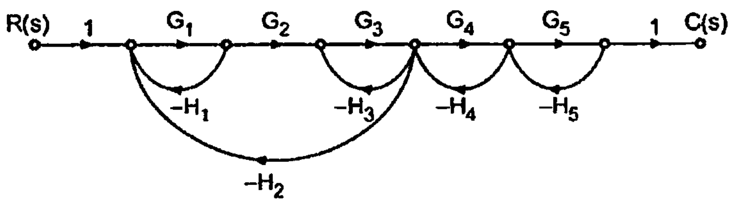
L_1 is non touching to T_2.

$$\therefore \quad \Delta_2 = 1-[L_1]=1-(-G_2H_1)=1+G_2H_1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 G_4 \cdot 1 + G_5 G_4 (1+G_2 H_1)}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 + G_4 G_5 (1+G_2 H_1)}{1+G_2 H_1 + G_1 G_2 G_3 G_4 H_2 + G_5 G_4 H_2 + G_2 G_5 G_4 H_1 H_2}}$$

Example 6.7 : *Find* $\frac{C(s)}{R(s)}$

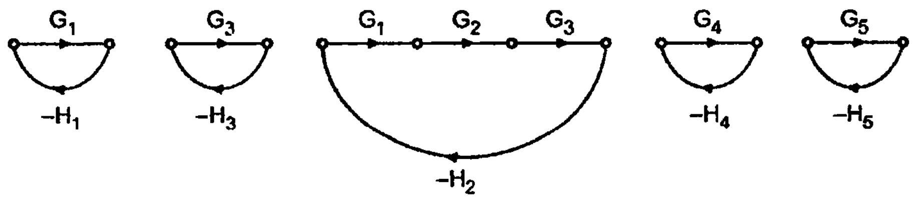


Solution : Number of forward paths = K = 1

$\therefore$ $$\text{T.F.} = \frac{T_1 \Delta_1}{\Delta}$$ .... Mason's gain formula

$$T_1 = G_1 G_2 G_3 G_4 G_5$$

Individual feedback loops,



$L_1 = -G_1H_1$ $L_2 = -G_3H_3$ $L_3 = -G_1G_2G_3H_2$ $L_4 = -G_4H_4$ $L_5 = -G_5H_5$

Combinations of two non touching loops,

i) L_1 and L_2 ii) L_1 and L_5 iii) L_1 and L_4

iv) L_2 and L_5 v) L_3 and L_5

Combination of three non touching loops, is L_1, L_2 and L_5.

$$\therefore \quad \Delta = 1-[L_1+L_2+L_3+L_4+L_5]+[L_1L_2+L_1L_5+L_1L_4+L_2L_5+L_3L_5]-[L_1L_2L_5]$$

$$\therefore \quad \Delta = 1+G_1H_1+G_3H_3+G_1G_2G_3H_2+G_4H_4+G_5H_5$$
$$+G_1G_3H_1H_3+G_1G_4H_1H_4+G_1G_5H_1H_5+G_3G_5H_3H_5$$
$$+G_1G_2G_3G_5H_2H_5+G_1G_3G_5H_1H_3H_5$$

Now considering $T_1 = G_1 G_2 G_3 G_4 G_5$

All loops are touching to this forward path hence,

$$\Delta_1 = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3 G_4 G_5 \cdot 1}{\Delta}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 G_5}{\begin{array}{l} 1+G_1 H_1+G_3 H_3+G_1 G_2 G_3 H_2+G_4 H_4+G_5 H_5 \\ +G_1 G_3 H_1 H_3+G_1 G_4 H_1 H_4+G_1 G_5 H_1 H_5 \\ +G_3 G_5 H_3 H_5+G_1 G_2 G_3 G_5 H_2 H_5 \\ +G_1 G_3 G_5 H_1 H_3 H_5 \end{array}}$$

Example 6.8 : *Construct the signal flow graph for the following set of system equations.*

$$Y_2 = G_1 Y_1 + G_3 Y_3 \quad \text{... (1)}$$

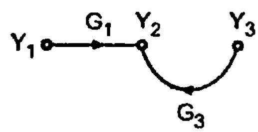
$$Y_3 = G_4 Y_1 + G_2 Y_2 + G_5 Y_3 \quad \text{... (2)}$$

$$Y_4 = G_6 Y_2 + G_7 Y_3 \quad \text{... (3)}$$

where Y_4 is output. Find transfer function $\frac{Y_4}{Y_1}$.

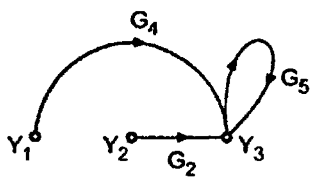
Solution : System node variables are Y_1, Y_2, Y_3, Y_4

Consider equation 1 : This indicates Y_2 depends on Y_1 and Y_3



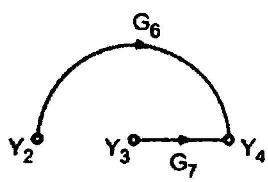
S.F.G. for equation (1)

Consider equation 2 : This indicates Y_3 depends on Y_1, Y_2 and Y_3



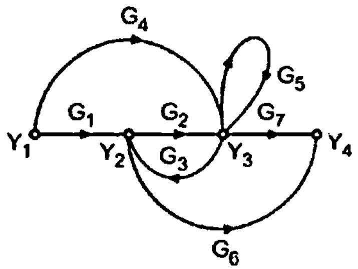
S.F.G. for equation (2)

Consider equation 3 : This indicates Y_4 depends on, Y_3 and Y_2



S.F.G. for equation (3)

Combining all three we get, complete S.F.G. as shown,



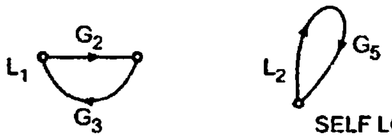
No. of forward paths = K = 4

$$\therefore \quad \text{T.F.} = \sum_{K=1}^{4} \frac{T_K \Delta_K}{\Delta} = \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3 + T_4 \Delta_4}{\Delta}$$

... Mason's gain formula

$T_1 = G_1 G_2 G_7$, $T_2 = G_4 G_7$, $T_3 = G_1 G_6$, $T_4 = G_4 G_3 G_6$

Individual loops are,



$L_1 = G_2 G_3$ $\qquad$ $L_2 = G_5$

$$\therefore \quad \Delta = 1 - [L_1 + L_2] = 1 - G_2 G_3 - G_5$$

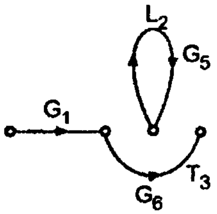
No non touching loop combinations.

Consider T_1, both loops are touching $\therefore \Delta_1 = 1$

T_2, both loops are touching $\therefore \Delta_2 = 1$

T_3, for this `G_5' self loop is non touching, $\therefore \Delta_3 = 1 - G_5$

T_1, both loops are touching $\therefore \Delta_4 = 1$



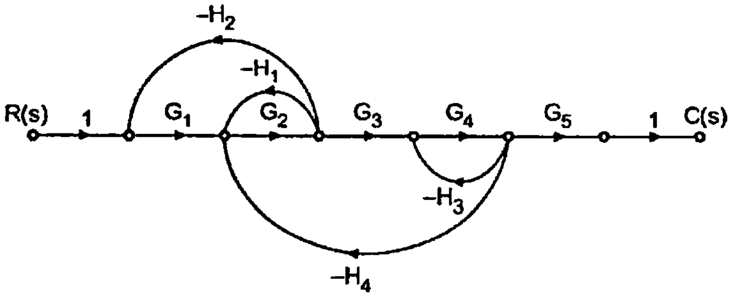
L_2 non touching to T_3

$$\frac{Y_4}{Y_1} = \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3 + T_4 \Delta_4}{\Delta}$$

$$= \frac{G_1 G_2 G_7 .1 + G_4 G_7 .1 + G_1 G_6 (1 - G_5) + G_4 G_3 G_6 .1}{\Delta}$$

$$\therefore \quad \boxed{\frac{Y_4}{Y_1} = \frac{G_1\, G_2\, G_7 + G_4\, G_7 + G_1\, G_6\, (1 - G_5) + G_4 G_3 G_6}{1 - G_2\, G_3 - G_5}}$$

Example 6.9 : *Find* $\frac{C(s)}{R(s)}$

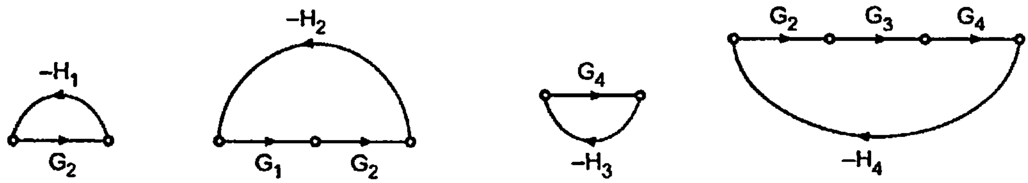


Solution : Number of forward paths = K = 1

$$\therefore \quad T.F. = \frac{\sum_{K=1} T_K\, \Delta_K}{\Delta} = \frac{T_1\, \Delta_1}{\Delta} \qquad \text{Mason's gain formula}$$

$$T_1 = G_1\, G_2\, G_3\, G_4\, G_5$$

Individual feedback loops :



$L_1 = -G_2\, H_1$ $\quad L_2 = -G_1\, G_2\, H_2$ $\quad L_3 = -G_4\, H_3$ $\quad L_4 = -G_2\, G_3\, G_4\, H_4$

Combinations of two non touching loops.

i) L_1 and L_3 ii) L_2 and L_3

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1\, L_3 + L_2\, L_3]$$

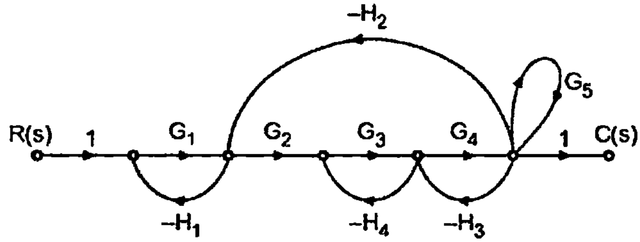
$$= 1 + G_2\, H_1 + G_1\, G_2\, H_2 + G_4\, H_3 + G_2\, G_3\, G_4\, H_4 + G_2\, G_4\, H_1\, H_3 + G_1\, G_2\, G_4\, H_2\, H_3$$

Consider $T_1 = G_1\, G_2\, G_3\, G_4\, G_5$ All loops are touching $\therefore \Delta_1 = 1$

$$\frac{C(s)}{R(s)} = \frac{T_1\, \Delta_1}{\Delta} = \frac{G_1\, G_2\, G_3\, G_4\, G_5 \cdot 1}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 G_5}{1 + G_2 H_1 + G_1 G_2 H_2 + G_4 H_3 + G_2 G_3 G_4 H_4 + G_2 G_4 H_1 H_3 + G_1 G_2 G_4 H_2 H_3}}$$

Example 6.10 : *Find* $\frac{C(s)}{R(s)}$

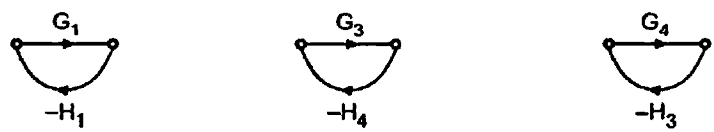


Solution : Number of forward paths = K = 1

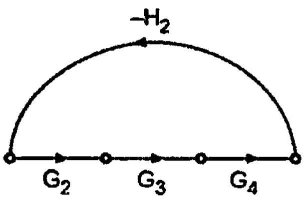
$$\text{T.F.} = \frac{T_1 \Delta_1}{\Delta} \qquad \text{... By Mason's gain formula}$$

$$\therefore \quad T_1 = G_1 G_2 G_3 G_4$$

Individual feedback loops



$L_1 = -G_1 H_1$ $\quad L_2 = -G_3 H_4$ $\quad L_3 = -G_4 H_3$



$L_4 = -G_2 G_3 G_4 H_2$ $\quad L_5 = G_5$

Combinations of two non touching loops

i) L_1 and L_2 ii) L_1 and L_3 iii) L_1 and L_5 iv) L_2 and L_5

Combination of three non touching loops.

i) L_1 , L_2 and L_5

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5] + [L_1 L_2 + L_1 L_3 + L_1 L_5 + L_2 L_5] - [L_1 L_2 L_5]$$

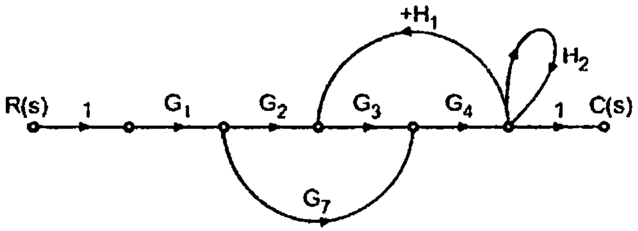
$$\Delta = 1 + G_1 H_1 + G_3 H_4 + G_4 H_3 + G_2 G_3 G_4 H_2 - G_5 + G_1 G_3 H_1 H_4 + G_1 G_4 H_1 H_3 - G_1 H_1 G_5 - G_3 H_4 G_5 - G_1 G_3 G_5 H_1 H_4$$

$$\Delta_1 = 1 \qquad \text{... All loops are touching to the forward path.}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3 G_4 \cdot 1}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1+G_1 H_1+G_3 H_4+G_4 H_3+G_2 G_3 G_4 H_2+G_5+G_1 G_3 H_1 H_4 + G_1 G_4 H_1 H_3-G_1 H_1 G_5-G_3 G_5 H_4-G_1 G_3 G_5 H_1 H_1}}$$

Example 6.11 : *Find* $\frac{C(s)}{R(s)}$

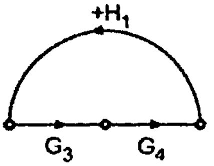


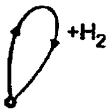
Solution : Number of forward paths = K = 2

$$\text{T.F.} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} \qquad \text{... By Mason's gain formula}$$

$T_1 = G_1 G_2 G_3 G_4$ and $T_2 = G_1 G_7 G_4$

Individual loops





SELF LOOP

$L_1 = +G_3 G_1 H_1$ $\qquad$ $L_2 = +H_2$

No combination of non touching loops

$$\therefore \quad \Delta = 1-[L_1 + L_2] = 1 - G_3 G_4 H_1 - H_2$$

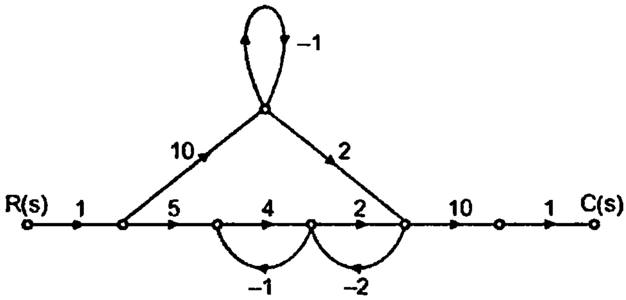
Consider $\quad T_1 \Rightarrow$ both loops are touching $\quad \therefore \Delta_1 = 1$

$\qquad T_2 \Rightarrow$ both loops are touching $\quad \therefore \Delta_2 = 1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 G_4 \cdot 1 + G_1 G_7 G_4 \cdot 1}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 + G_1 G_7 G_4}{1 - G_3 G_4 H_1 - H_2}}$$

Example 6.12 : *Find* $\frac{C(s)}{R(s)}$

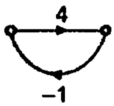


Solution : Number of forward paths K = 2

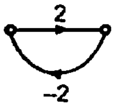
$$\therefore \quad \text{T.F.} = \frac{C(s)}{R(s)} = \frac{\sum_{K=1}^{2} T_K \Delta_K}{\Delta} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} \quad \text{... Mason's gain formula}$$

$$T_1 = 5 \cdot 4 \cdot 2 \cdot 10 = 400 \text{ and } T_2 = 1 \cdot 10 \cdot 2 \cdot 10 = 200$$

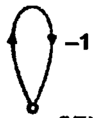
Individual loops are



$L_1 = -4$



$L_2 = -4$



$L_3 = -1$

Combinations of two non-touching loops are

i) L_1 and L_3 and ii) L_2 and L_3

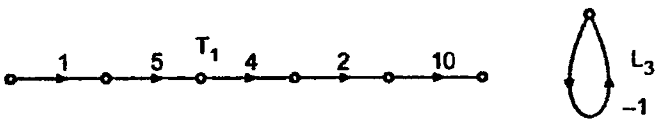
No combination of three non-touching loops :

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3] + [L_1 L_3 + L_2 L_3]$$

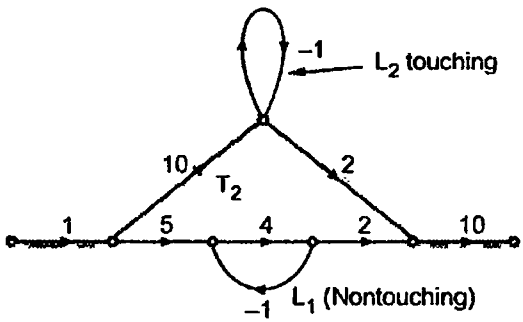
$$= 1 - [-4 - 4 - 1] + [4 + 4] = 1 + 9 + 8 = 18$$

Consider T_1 , L_3 loop is non touching

$$\therefore \quad \Delta_1 = 1 - L_3 = 1 - [-1] = 2$$



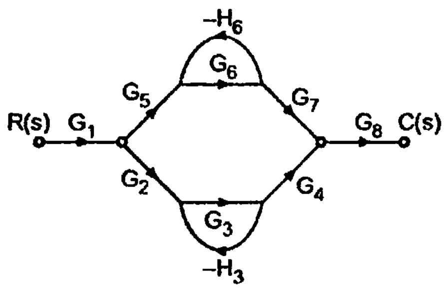
Consider T_2 , L_1 is nontouching



$$\Delta_2 = 1 - L_1 = 1 - [-4] = 5$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\,\Delta_1 + T_2\,\Delta_2}{\Delta} = \frac{400 \times 2 + 200 \times 5}{18} = \frac{800 + 1000}{18} = \mathbf{100}$$

Example 6.13 : *Find* $\frac{C(s)}{R(s)}$ *by Mason's gain formula.*



Solution : Number of forward paths = K = 2

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\sum_{K=1}^{2} T_K\ \Delta_K}{\Delta} = \frac{T_1\,\Delta_1 + T_2\,\Delta_2}{\Delta} \qquad \text{... Mason's gain formula}$$

$\therefore$ $T_1 = G_1\, G_5\, G_6\, G_7\, G_8$ $\quad T_2 = G_1\, G_2\, G_3\, G_4\, G_8$

Individual loops are,

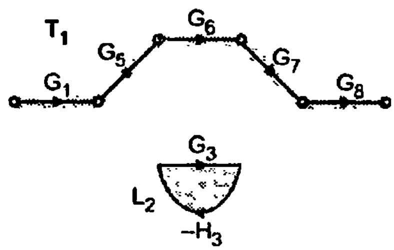
$L_1 = -\,G_6\, H_6$ $\qquad$ $L_2 = -\,G_3\, H_3$

Both L_1 and L_2 are non touching to each other.

$$\therefore \quad \Delta = 1 - [L_1 + L_2] + [L_1\, L_2]$$

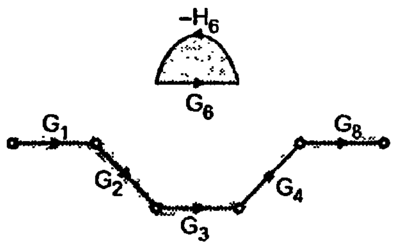
$$= 1 + G_6\, H_6 + G_3\, H_3 + G_3\, G_6\, H_3\, H_6$$

Consider T_1 , L_2 is non touching



$$\therefore \quad \Delta_1 = 1 - L_2 = 1 + G_3 H_3$$

Consider T_2 , L_1 is non touching



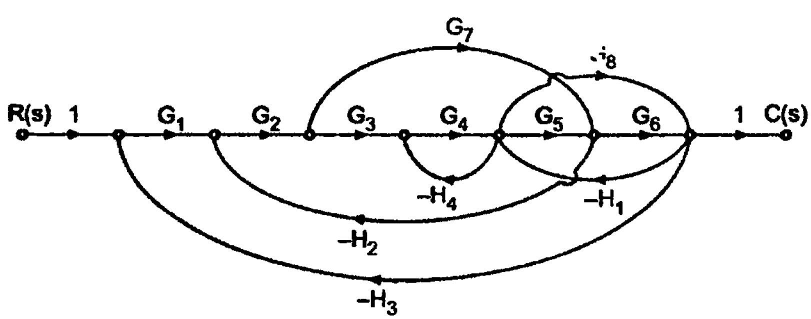
$$\therefore \quad \Delta_2 = 1 - L_1 = 1 + G_6 H_6$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_5 G_6 G_7 G_8 (1 + G_3 H_3) + G_1 G_2 G_3 G_4 G_8 (1 + G_6 H_n)}{\Delta}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_5 G_6 G_7 G_8 (1 + G_3 H_3) + G_1 G_2 G_3 G_4 G_8 (1 + G_6 H_6)}{1 + G_6 H_6 + G_3 H_3 + G_3 G_6 H_3 H_6}}$$

Example 6.14 : *Find* $\frac{C(s)}{R(s)}$



Solution : Number of forward paths = K = 3

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\sum_{K=1}^{3} T_K \Delta_K}{\Delta}$$

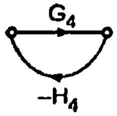
$$= \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3}{\Delta} \quad \text{... By Mason's gain formula}$$

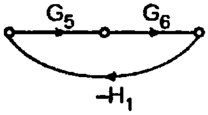
$$T_1 = G_1 G_2 G_3 G_4 G_5 G_6$$

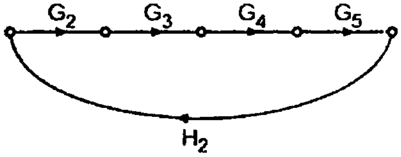
$$T_2 = G_1 G_2 G_7 G_6$$

$$T_3 = G_1 G_2 G_3 G_4 G_8$$

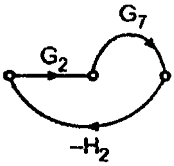
Individual feedback loops are :

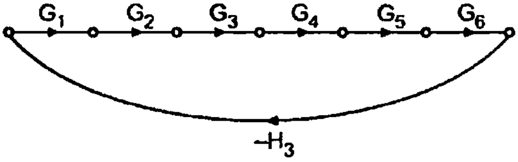


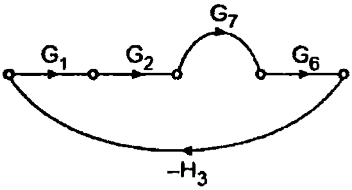




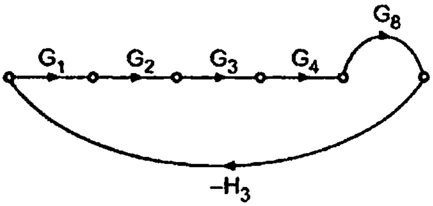
$L_1 = -G_4 H_4$ $\quad L_2 = -G_5 G_6 H_1$ $\quad L_3 = -G_2 G_3 G_4 G_5 H_2$

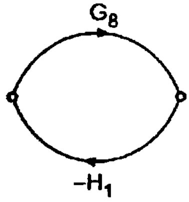






$L_4 = -G_2 G_7 H_2$ $\quad L_5 = -G_1 G_2 G_3 G_4 G_5 G_6 H_3$ $\quad L_6 = -G_1 G_2 G_7 G_6 H_3$





$L_7 = -G_1 G_2 G_3 G_4 G_8 H_3$ $\quad L_8 = -G_8 H_1$

Combinations of two non touching loops are :

i) L_1 and L_4 ii) L_4 and L_8 iii) L_1 and L_6

No combination of three non touching loops

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8] + [L_1 L_4 + L_4 L_8 + L_1 L_6]$$

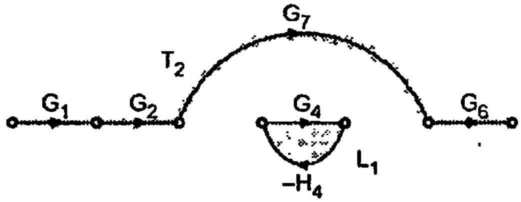
$$= 1 + G_4 H_4 + G_5 G_6 H_1 + G_2 G_3 G_4 G_5 H_2 + G_2 G_7 H_2$$

$$+ G_1 G_2 G_3 G_4 G_5 G_6 H_3 + G_1 G_2 G_3 G_4 G_8 H_3 + G_1 G_2 G_7 G_6 H_3 + G_8 H_1$$

$$+ G_4 H_4 G_2 G_7 H_2 + G_2 G_7 H_2 G_8 H_1 + G_4 H_4 G_1 G_2 G_7 G_6 H_3$$

Consider T_1, all loops are touching $\therefore \Delta_1 = 1$

For T_2 , only L_1 is non touching.



$$\therefore \quad \Delta_2 = 1 - L_1 = 1 + G_4 H_4$$

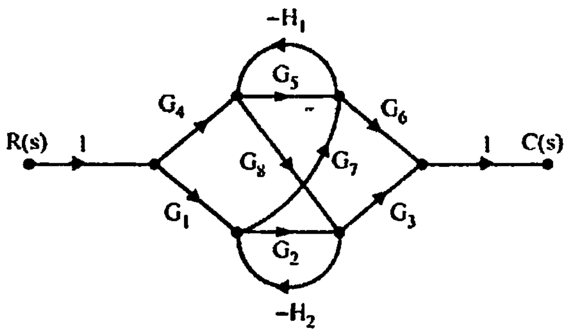
For T_3, all loops are touching $\therefore \Delta_3 = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3}{\Delta}$$

$$= \frac{G_1 G_2 G_3 G_4 G_5 G_6 \cdot 1 + G_1 G_2 G_7 G_6 \cdot (1 + G_4 H_4) + G_1 G_2 G_3 G_4 G_8 \cdot 1}{\Delta}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 G_5 G_6 + G_1 G_2 G_7 G_6 (1 + G_4 H_4) + G_1 G_2 G_3 G_4 G_8}{\left[\begin{array}{l} 1 + G_4 H_4 + G_5 G_6 H_1 + G_2 G_3 G_4 G_5 H_2 + G_2 G_7 H_2 \\ + G_1 G_2 G_3 G_4 G_5 G_6 H_3 + G_1 G_2 G_3 G_4 G_8 H_3 + G_1 G_2 G_7 G_6 H_3 \\ + G_8 H_1 + G_2 G_4 G_7 H_2 H_4 + G_2 G_7 G_8 H_1 H_2 + G_1 G_2 G_7 G_4 G_6 H_3 H_4 \end{array}\right]}$$

Example 6.15 : *Using Mason's gain formula, find the gain of the following system in figure below.*



Solution : The number of forward paths are K = 6

The forward path gains are,

$$T_1 = G_1G_2G_3, \qquad T_2 = G_4G_5G_6$$
$$T_3 = G_1G_7G_6, \qquad T_4 = G_4G_8G_3$$
$$T_5 = G_4G_8(-H_2)G_7G_6, \qquad T_6 = G_1G_7(-H_1)G_8G_3$$

The feedback loop gains are,

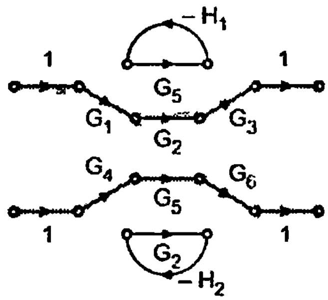
$$L_1 = -G_5 H_1 , L_2 = -G_2 H_2 , L_3 = + G_7 H_1 G_8 H_2$$

The two non touching loops are $L_1 L_2$

$$\therefore \quad L_1 L_2 = + G_2 G_5 H_1 H_2$$

$$\therefore \quad \Delta = 1-[L_1+L_2+L_3]+[L_1 L_2]$$

$$= 1+ G_5H_1 + G_2H_2 - G_7G_8H_1H_2 + G_2G_5H_1H_2$$



For T_1, L_1 is non touching.

$$\therefore \quad \Delta_1 = 1 - L_1 = 1+ G_5 H_1$$

For T_2 , L_2 is non touching.

$$\therefore \quad \Delta_2 = 1 - L_2 = 1+ G_2 H_2$$

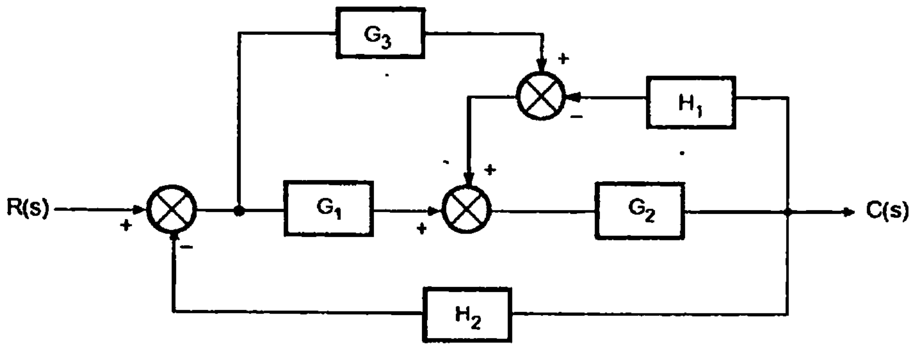
For T_3 to T_6 all loops are touching to all forward paths.

$$\therefore \quad \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$$

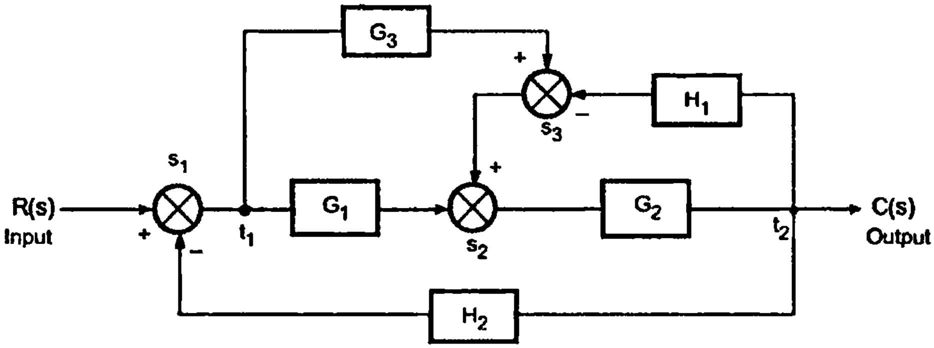
$$\therefore \quad \text{Gain} = \frac{\sum T_K \Delta_K}{\Delta} = \frac{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4 + T_5\Delta_5 + T_6\Delta_6}{\Delta}$$

$$\therefore \quad \boxed{\text{Gain} = \frac{G_1G_2G_3\ (1+G_5H_1) + G_1G_5G_6\ (1+G_2H_2) + G_1G_7G_6 + G_4G_8G_3 - G_4G_8G_7G_6H_2 - G_1G_3G_7G_8H_1}{1+G_5H_1 + G_2H_2 - G_7G_8H_1H_2 + G_2G_5H_1H_2}} \quad \text{... Ans}$$

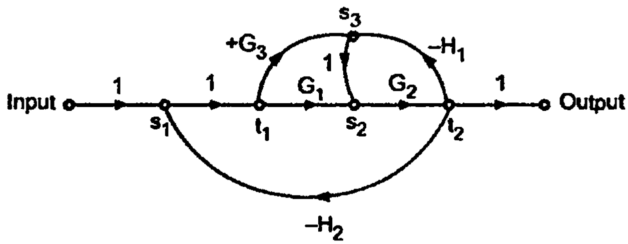
Example 6.16 : *Obtain the overall transfer function of the system shown by block diagram, using signal flow graph technique.*



Solution : Name the various summing and take off points to draw the signal flow graph as shown,



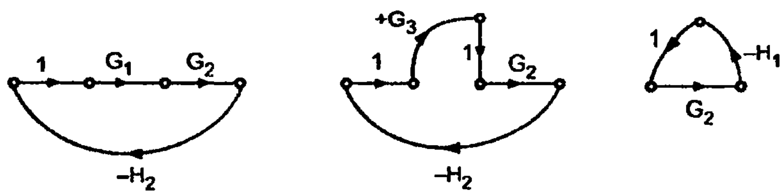
The corresponding signal flow graph is,



Forward path gains are,

$T_1 = G_1 G_2 \qquad T_2 = G_3 G_2$

The various loops are,



$L_1 = -G_1 G_2 H_2 \qquad L_2 = -G_2 G_3 H_2 \qquad L_3 = -G_2 H_1$

No combinations of non touching loops.

$\therefore \Delta = 1-[L_1+L_2+L_3] = 1+G_1 G_2 H_2 + G_2 G_3 H_2 + G_2 H_1$

For T_1, All loops are touching, $\therefore \Delta_1 = 1$

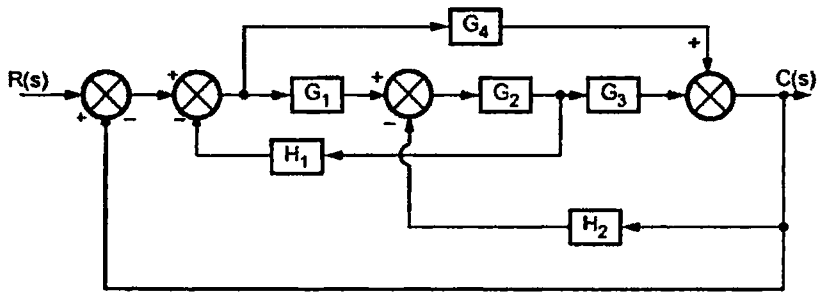
For T_2, All loops are touching, $\therefore \Delta_2 = 1$

According to Mason's gain formula,

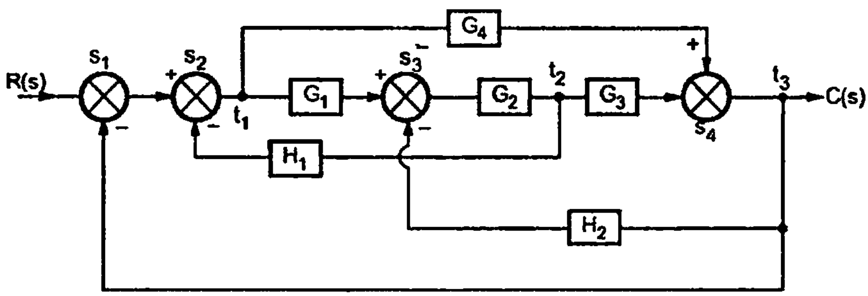
$$\frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

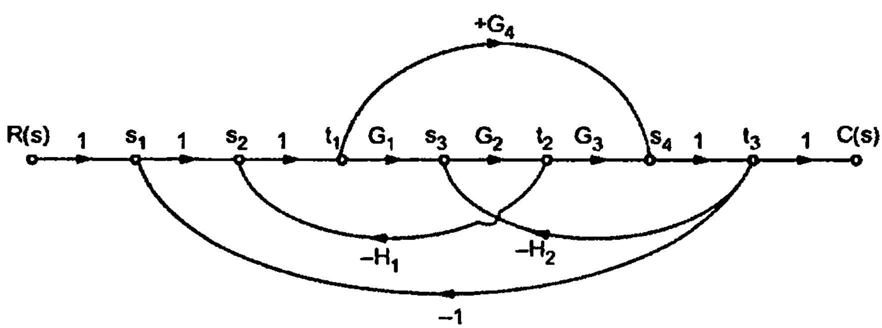
$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 + G_2 G_3}{1+G_1 G_2 H_2 + G_2 G_3 H_2 + G_2 H_1}}$$

Example 6.17 : *Determine the transfer function for the block diagram shown below by Mason's gain formula.*



Solution : Name the various summing and takeoff points to draw the signal flow graph.





The forward path gains are,

$$T_1 = G_1 G_2 G_3, \qquad T_2 = G_4$$

The various loop gains are,

$$L_1 = -G_1 G_2 H_1 \qquad L_2 = -G_2 G_3 H_2$$

$$L_3 = -G_1G_2G_3 \qquad L_4 = -G_4$$

and $$L_5 = 1 \times G_4 \times 1 \times (-H_2) \times G_2 \times (-H_1) = G_2 G_4 H_1 H_2$$

No combinations of non touching loops,

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5] = 1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 + G_4 - G_2 G_4 H_1 H_2$$

All the loops are touching to T_1 to T_2 hence,

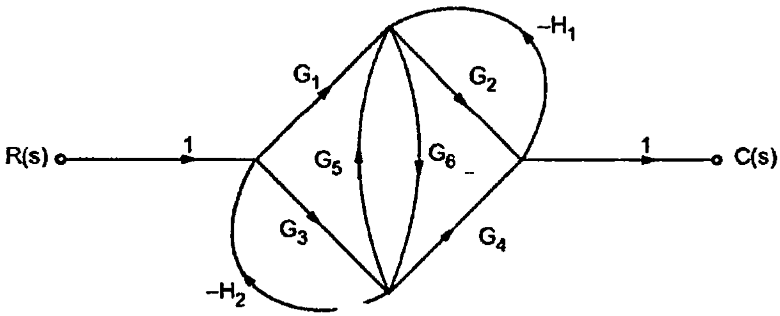
$$\Delta_1 = \Delta_2 = 1$$

Using Mason's gain formula,

$$\frac{C(s)}{R(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4 - G_2G_4H_1H_2}}$$

Example 6.18 : *For the signal flow graph of figure below, determine the transfer function $C(s) / R(s)$ using Mason's gain formula.*



Solution : Number of forward paths K = 4

$$T_1 = 1 \cdot G_1 \cdot G_2 \cdot 1 = G_1 G_2$$

$$T_2 = 1 \cdot G_3 \cdot G_4 \cdot 1 = G_3 G_4$$

$$T_3 = 1 \cdot G_1 \cdot G_6 \cdot G_4 \cdot 1 = G_1 G_6 G_4$$

$$T_4 = 1 \cdot G_3 \cdot G_5 \cdot G_2 \cdot 1 = G_2 G_3 G_5$$

Individual loops are

$$L_1 = -G_2 H_1, \quad L_2 = -G_3 H_2, \quad L_3 = G_5 G_6$$

$$L_4 = -G_4 H_1 G_6, \quad L_5 = -G_1 G_6 H_2$$

Combinations of two non touching loops are

i) L_1 and L_2

ii) No combination of three non touching loops

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_1 L_2)$$

$$= 1 - [(-G_2 H_1 - G_3 H_2 + G_5 G_6 - G_4 G_6 H_1 - G_1 G_6 H_2)] + [(-G_2 H_1)(-G_3 H_2)]$$

$\therefore \quad \Delta = 1+G_2 H_1 + G_3 H_2 - G_5 G_6 + G_4 G_6 H_1 + G_1 G_6 H_2 + G_2 G_3 H_1 H_2$

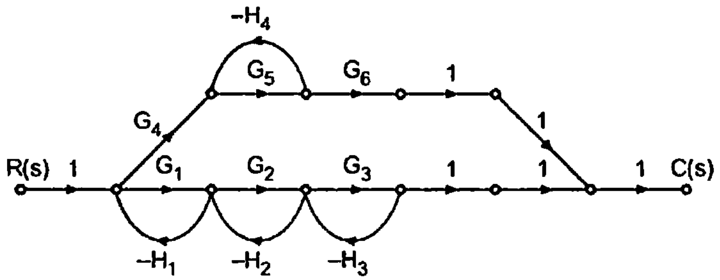
For all the forward paths all the loops are touching hence

$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$

Hence $\quad \dfrac{C(s)}{R(s)} = \dfrac{\Sigma T_K \Delta_K}{\Delta} = \dfrac{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4}{\Delta}$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 + G_3 G_4 + G_1 G_6 G_4 + G_2 G_3 G_5}{1+G_2 H_1 + G_3 H_2 - G_5 G_6 + G_5 G_6 H_1 + G_1 G_6 H_2 + G_2 G_3 H_1 H_2}} \quad \text{... (2)}$$

Example 6.19 : *Find* $\dfrac{C(s)}{R(s)}$ **(M.U. : May-97)**

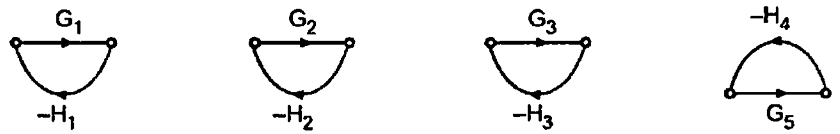


Solution : Number of forward paths = K = 2

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} \quad \text{... Mason's gain formula}$$

$T_1 = G_1 G_2 G_3 \quad$ and $\quad T_2 = G_4 G_5 G_6$

Individual feedback loops are,



$L_1 = -G_1 H_1 \qquad L_2 = -G_2 H_2 \qquad L_3 = -G_3 H_3 \qquad L_4 = -G_5 H_4$

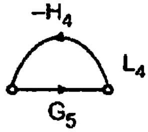
Combinations of two non-touching loops

i) L_1 and L_3 ii) L_1 and L_4 iii) L_2 and L_4 iv) L_3 and L_4

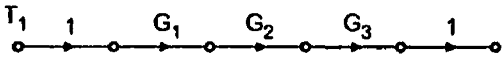
Combinations of three non-touching loops $\rightarrow L_1$, L_3 and L_4.

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1 L_3 + L_1 L_4 + L_2 L_4 + L_3 L_4] - [L_1 L_3 L_4]$$

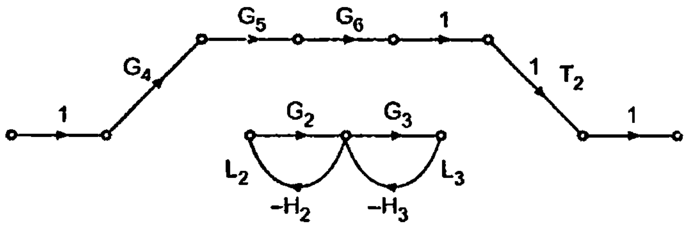
Consider T_1, For this `L_4' is non-touching



$$\therefore \Delta_1 = 1 - [L_4] = 1 + G_5 H_4$$



For T_2, L_2 and L_3 are non-touching



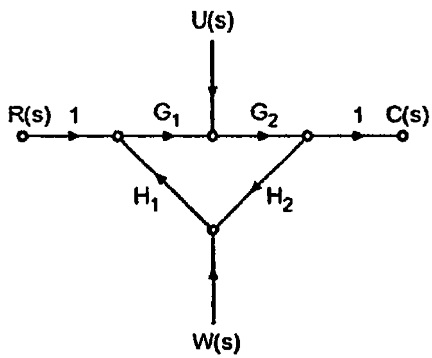
$$\therefore \Delta_2 = 1 - [L_2 + L_3]$$

$$= 1 + G_2 H_2 + G_3 H_3$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

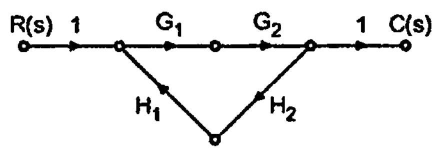
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 (1 + G_5 H_4) + G_4 G_5 G_6 (1 + G_2 H_2 + G_3 H_3)}{[1 + G_1 H_1 + G_2 H_2 + G_3 H_3 + G_5 H_4 + G_1 G_3 H_1 H_3 + G_1 G_5 H_1 H_4 + G_2 G_5 H_2 H_4 + G_3 G_5 H_3 H_4 + G_1 G_3 G_5 H_1 H_3 H_4]}$$

Example 6.20 : *Find the value of C(s).* **(M.U. : May-07, Dec.-06)**



Solution : As system has three inputs, considering each input separately,

Assuming U(s) = W(s) = 0, S.F.G. becomes,



$$T_1 = G_1 G_2$$

and $$L_1 = G_1 G_2 H_1 H_2$$

Using Mason's gain formula,

$$\Delta = 1-[L_1] = 1-G_1G_2H_1H_2$$

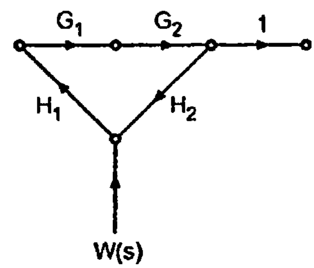
$$\Delta_1 = 1 \quad \text{as loop is touching to } T_1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\,\Delta_1}{\Delta} = \frac{G_1\,G_2}{1-G_1\,G_2\,H_1\,H_2}$$

$$\therefore \quad C(s) = R(s)\left[\frac{G_1\,G_2}{1-G_1\,G_2\,H_1\,H_2}\right] \quad \ldots (1)$$

Assume $U(s) = R(s) = 0$

∴ S.F.G. becomes as shown in following figure



$$T_1 = H_1\,G_1\,G_2$$

$$L_1 = H_2\,H_1\,G_1\,G_2$$

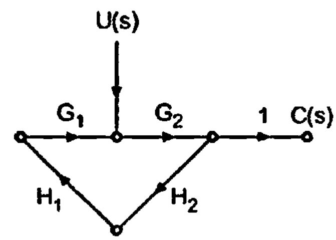
$$\therefore \quad \Delta = 1-L_1 = 1-G_1\,G_2\,H_1\,H_2$$

$$\Delta_1 = 1$$

$$\therefore \quad \frac{C(s)}{W(s)} = \frac{T_1\,\Delta_1}{\Delta} = \frac{H_1\,G_1\,G_2}{1-G_1\,G_2\,H_1\,H_2}$$

$$\therefore \quad C(s) = W(s)\left[\frac{H_1\,G_1\,G_2}{1-G_1\,G_2\,H_1\,H_2}\right] \quad \ldots (2)$$

Assume $R(s) = W(s) = 0$



$$T_1 = G_2$$

$$L_1 = G_2 \cdot H_1\,H_2\,G_1$$

$$\Delta = 1-L_1 = 1-H_1\,H_2\,G_1\,G_2$$

$$\Delta_1 = 1$$

$$\frac{C(s)}{U(s)} = \frac{T_1\,\Delta_1}{\Delta} = \frac{G_2 \cdot 1}{1-G_1\,G_2\,H_1\,H_2}$$

$$\therefore \quad C(s) = U(s)\left[\frac{G_2}{1-G_1\,G_2\,H_1\,H_2}\right] \quad \ldots (3)$$

∴ Total output is combination of all three individual output.

$$\therefore \quad \boxed{C(s) = \frac{G_1\,G_2\,R(s)+H_1\,G_1\,G_2\,W(s)+G_2\,U(s)}{1-G_1\,G_2\,H_1\,H_2}}$$

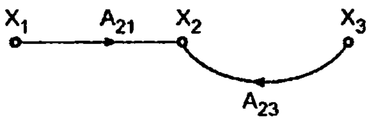
Example 6.21 : *Construct the signal flow graph for the following set of simultaneous equations.*

$$X_2 = A_{21}\ X_1 + A_{23}\ X_3$$
$$X_3 = A_{31}\ X_1 + A_{32}\ X_2 + A_{33}\ X_3$$
$$X_4 = A_{42}\ X_2 + A_{43}\ X_3$$

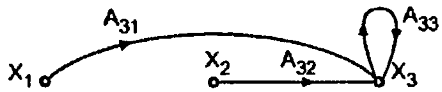
(M.U. : Dec.-96)

Solution : The value of the variable is the algebraic sum of all the signals entering at the node, representing that variable. The variables are X_1 , X_2 , X_3 , X_4 while the gains are A_{21}, A_{23}, A_{31}, A_{32}, A_{33}, A_{42} and A_{43}.

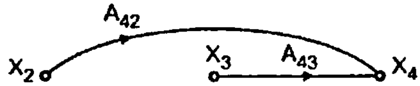
So selecting nodes representing variables and simulating differential equations.



... Equation (I)

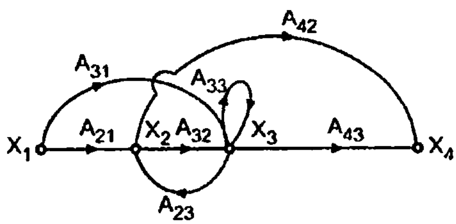


... Equation (II)



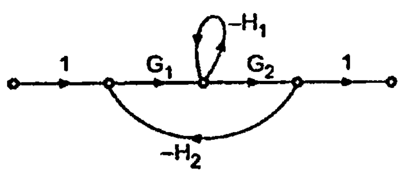
... Equation (III)

Therefore resultant signal flow graph is



Example 6.22 : *Find transfer functions for signal flow graphs given below*

i)

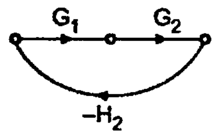


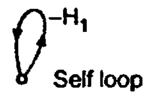
(M.U. : Nov.-95)

Solution : i) Forward paths K = 1

Forward path gain $T_1 = G_1 G_2$

Individual feedback loops are





$$L_1 = -G_1 G_2 H_2 \qquad L_2 = -H_1$$

There is no non touching loop combinations.

$$\therefore \quad \Delta = 1-[L_1+L_2] = 1-[-G_1 G_2 H_2 - H_1] = 1+G_1 G_2 H_2 + H_1$$

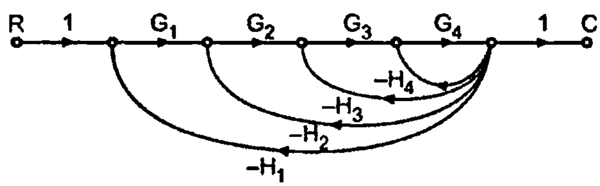
Both the loops are touching to the forward path

$$\therefore \quad \Delta_1 = 1$$

Using Mason's Gain Formula

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta} = \frac{G_1 G_2}{1+G_1 G_2 H_2 + H_1}}$$

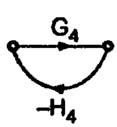
ii)

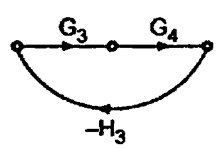


Solution : Forward paths K = 1

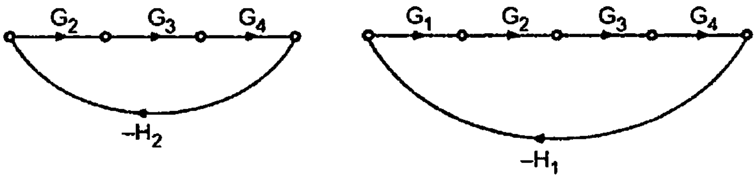
$$T_1 = G_1 G_2 G_3 G_4$$

Individual feedback loops





$$L_1 = -G_4 H_4 \qquad L_2 = -G_3 G_4 H_3$$



$L_3 = -G_2 G_3 G_4 H_2 \qquad L_4 = -G_1 G_2 G_3 G_4 H_1$

All loops are touching each other, no combination of non touching loops

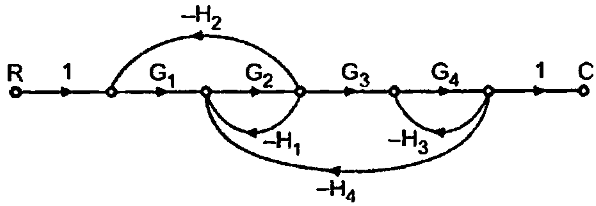
$$\therefore \qquad \Delta_1 = 1 - [L_1 + L_2 + L_3 + L_4]$$

$$= 1 + G_4 H_1 + G_3 G_4 H_3 + G_2 G_3 G_4 H_2 + G_1 G_2 G_3 G_4 H_1$$

All the loops are touching to the forward path hence $\Delta_1 = 1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3 G_4}{1 + G_4 H_4 + G_3 G_4 H_3 + G_2 G_3 G_4 H_2 + G_1 G_2 G_3 G_4 H_1}$$

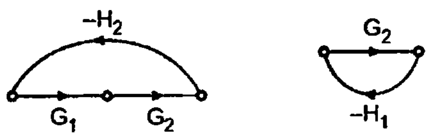
iii)



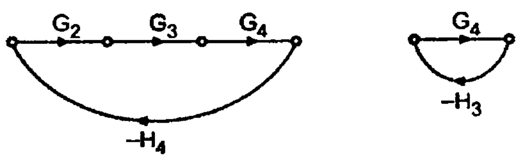
Solution : Forward path K = 1

$$T_1 = G_1 G_2 G_3 G_4$$

Individual feedback loops are



$L_1 = -G_1 G_2 H_2 \qquad L_2 = -G_2 H_1$



$L_3 = -G_2 G_3 G_4 H_4 \qquad L_4 = -G_4 H_3$

The loops L_2 and L_4 forms combination of two non touching loops. Similarly L_1 and L_4 also forms combination of two non touching loops.

$$\therefore \quad L_2 L_4 = G_2 G_4 H_1 H_3$$

$$L_1 L_4 = G_1 G_2 G_4 H_3 H_2$$

$$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_2 L_4 + L_1 L_4]$$

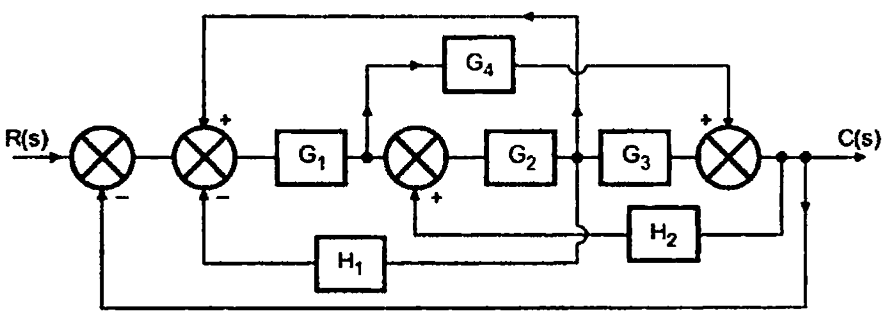
$$= 1 + G_1 G_2 H_2 + G_2 H_1 + G_2 G_3 G_4 + H_4 G_4 H_3 + G_2 G_4 H_1 H_3 + G_1 G_2 G_4 H_2 H_3$$

All loops are touching to forward path hence $\Delta_1 = 1$

$$\frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta}$$

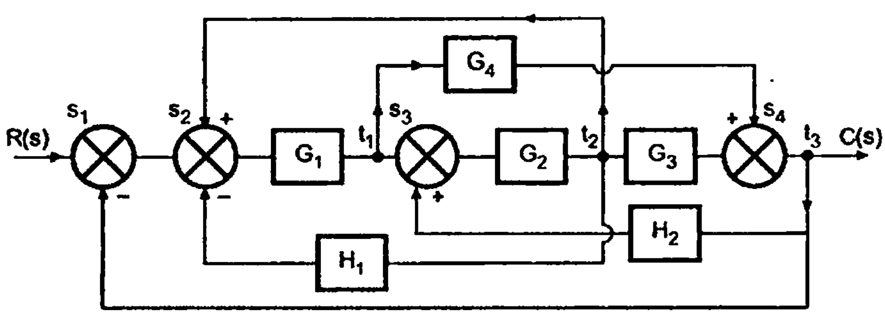
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_2 + G_2 H_1 + G_2 G_3 G_4 H_4 + G_4 H_3 + G_2 G_4 H_1 H_3 + G_1 G_2 G_4 H_2 H_3}}$$

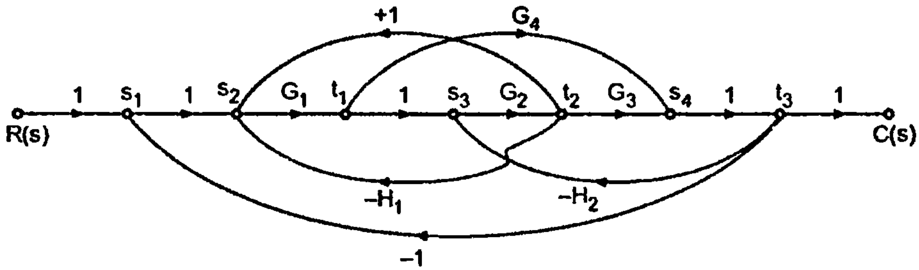
Example 6.23 : *Use Mason's Gain Formula to obtain $\frac{C(s)}{R(s)}$ of the system shown below.*



(M.U. : Jan-92)

Solution : Name all the summing and take off points and assuming each as a separate node draw the signal flow graph.

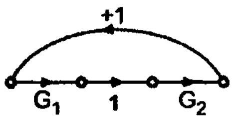


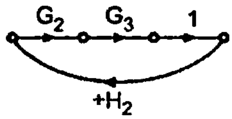


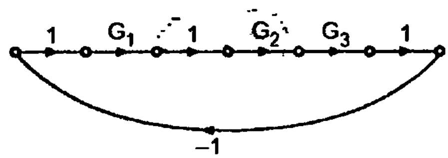
Number of forward paths K = 2

Forward path gains $T_1 = G_1 G_2 G_3 \quad T_2 = G_1 G_1$

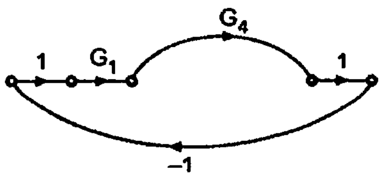
Individual feedback loops are

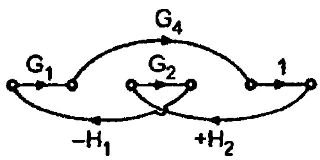


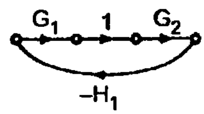




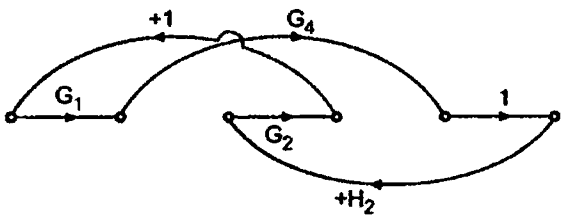
$L_1 = + G_1 G_2 \qquad L_2 = G_2 G_3 H_2 \qquad L_3 = - G_1 G_2 G_3$







$L_4 = - G_1 G_4 \qquad L_5 = - G_1 G_2 G_1 H_1 H_2 \qquad L_6 = - G_1 G_2 H_1$



$L_7 = G_1 G_2 G_4 H_2$

There is no combination of nontouching loops.

$$\therefore \qquad \Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7]$$

All loops are touching to both the forward paths

∴ Eliminating all loop gains from Δ we get,

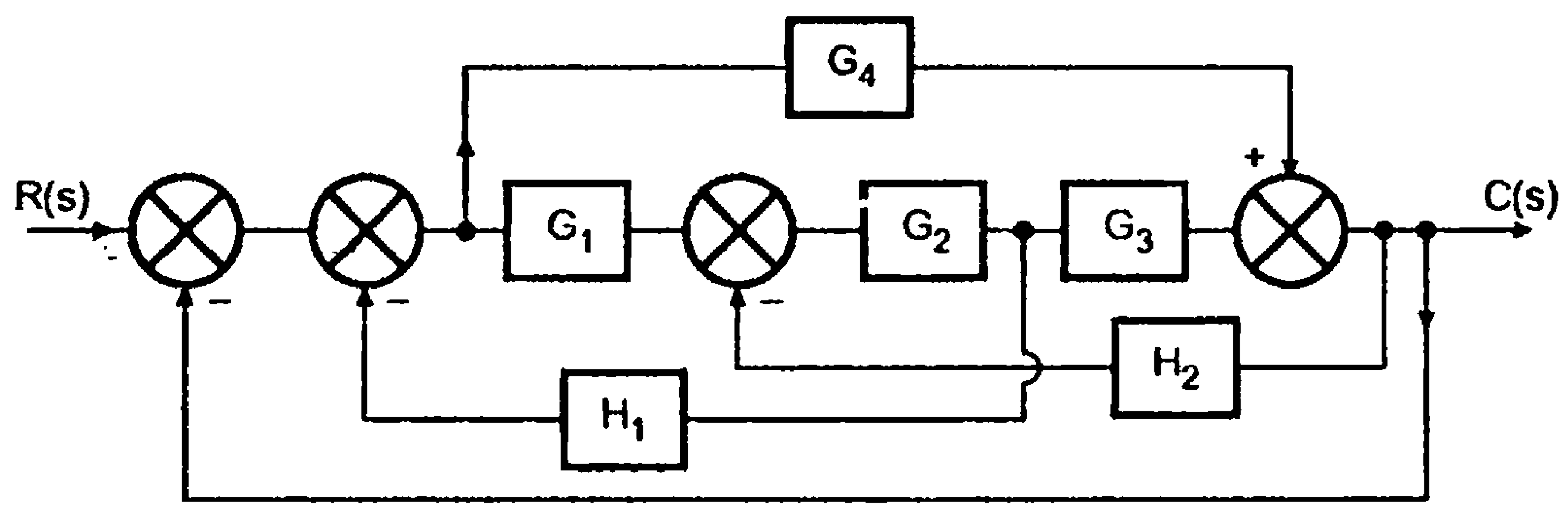
$$\Delta_1 = \Delta_2 = 1$$

∴ Using Mason's Gain formula,

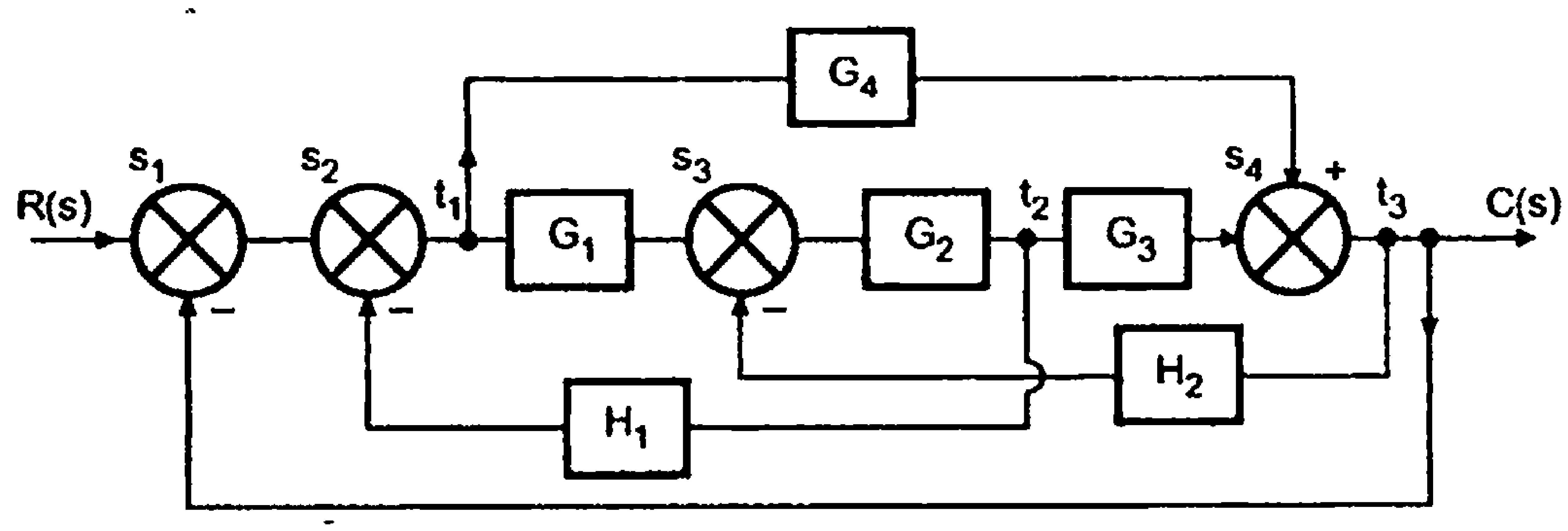
$$\frac{C(s)}{R(s)} = \frac{T_1\,\Delta_1 + T_2\,\Delta_2}{\Delta}$$

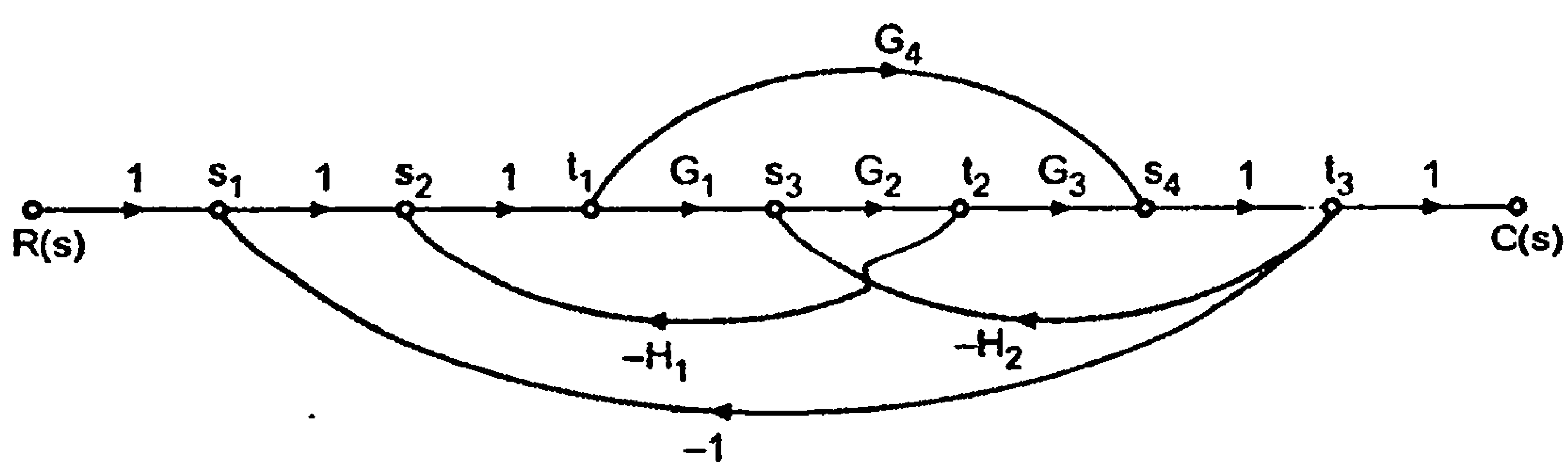
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 - G_1 G_2 - G_2 G_3 H_2 + G_1 G_2 G_3 + G_1 G_4 + G_1 G_2 G_4 H_1 H_2 + G_1 G_2 H_1 - G_1 G_2 G_4 H_2}}$$

Example 6.24 : *Draw the signal flow graph and hence obtain the transfer function of the system shown below.* **(M.U. : July-91)**



Solution : Name all the summing and take off points as shown below and representing each separately as a node draw the signal flow graph.





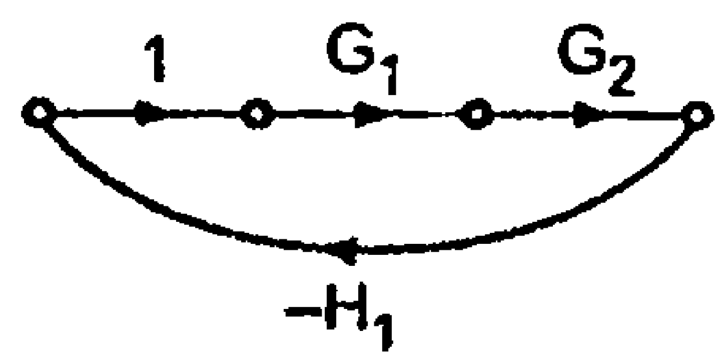
Number of forward paths are K = 2

Forward path gains are

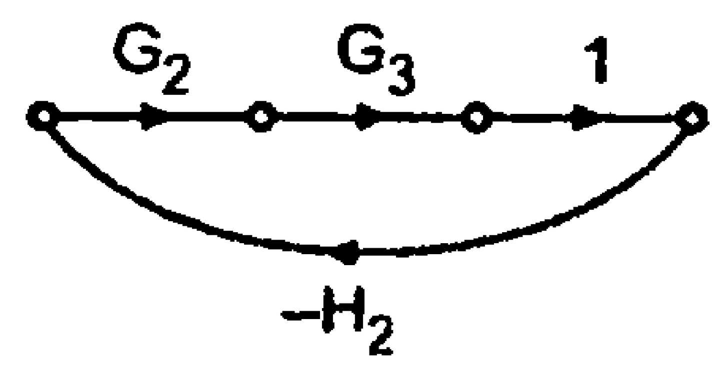
$$T_1 = G_1\,G_2\,G_3$$

$$T_2 = G_4$$

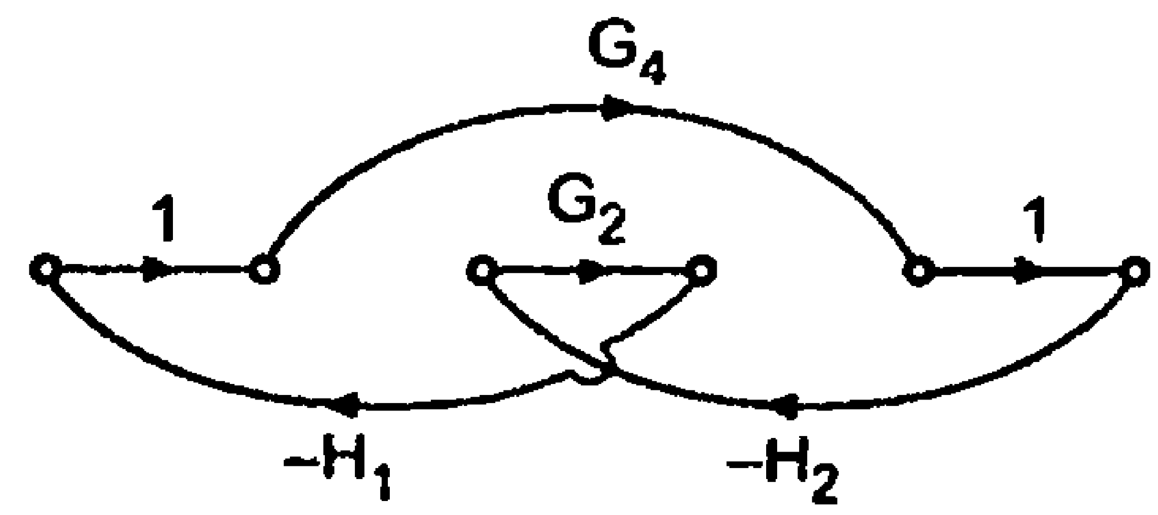
Individual feedback loops are



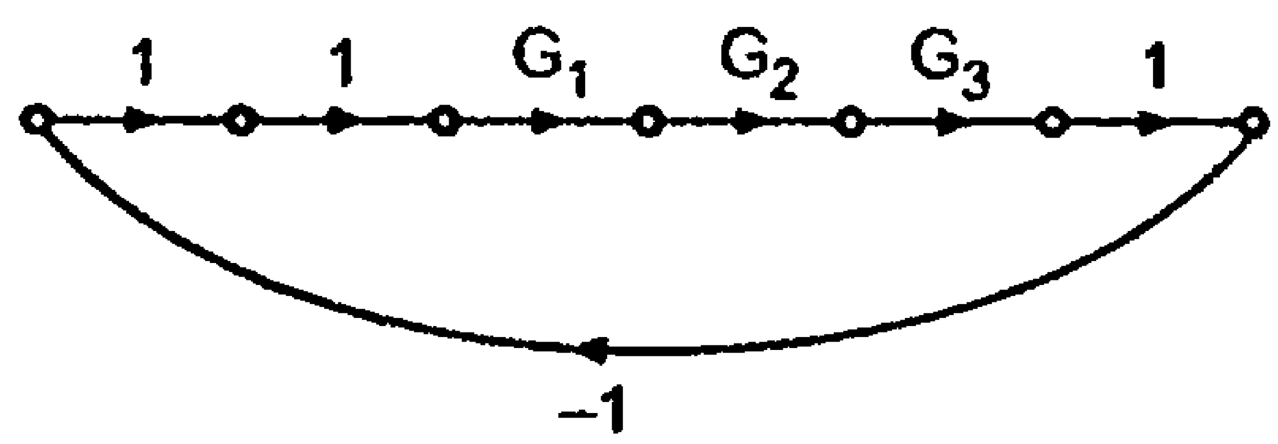
$L_1 = -G_1 G_2 H_1$



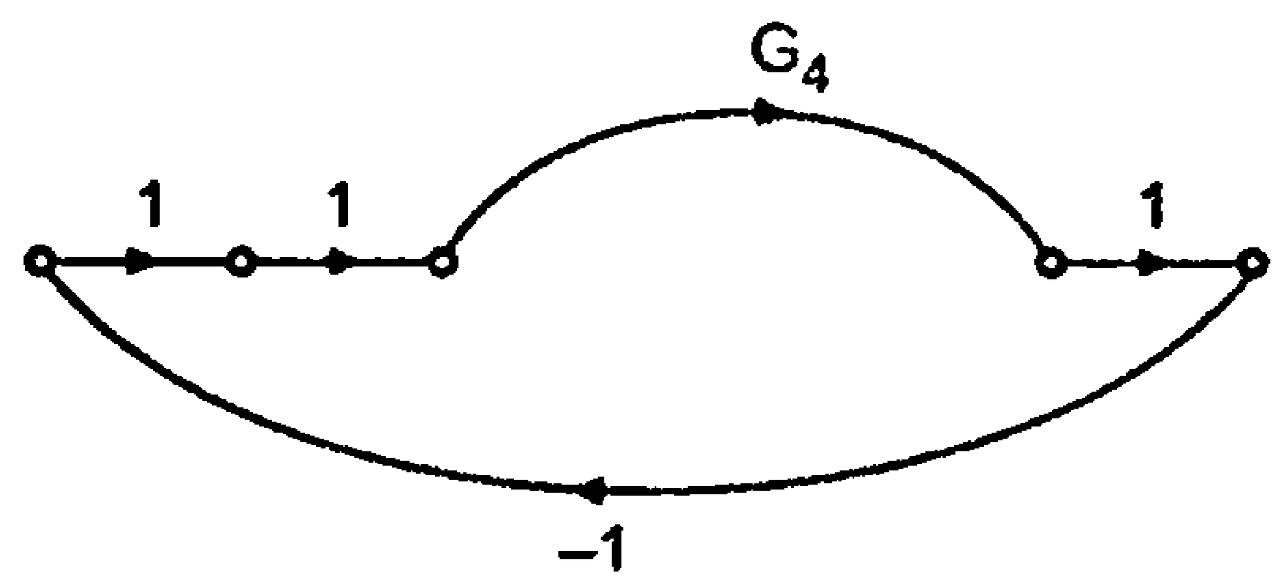
$L_2 = -G_2 G_3 H_2$



$L_3 = +G_2 G_4 H_1 H_2$



$L_4 = -G_1 G_2 G_3$



$L_5 = -G_4$

There are no combination of non touching loops

$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5]$

All the loops are touching to both the forward paths

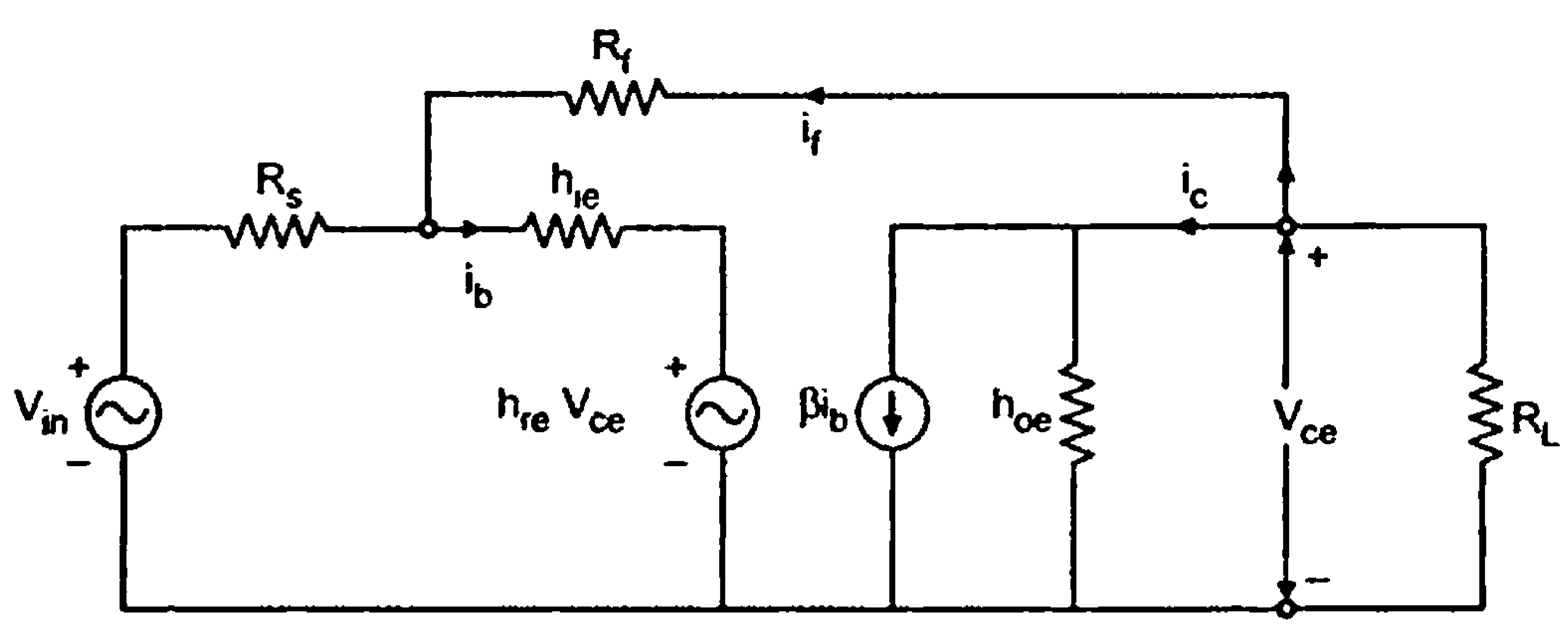
$\therefore \quad \Delta_1 = \Delta_2 = 1$

$\therefore$ Using Mason's gain formula

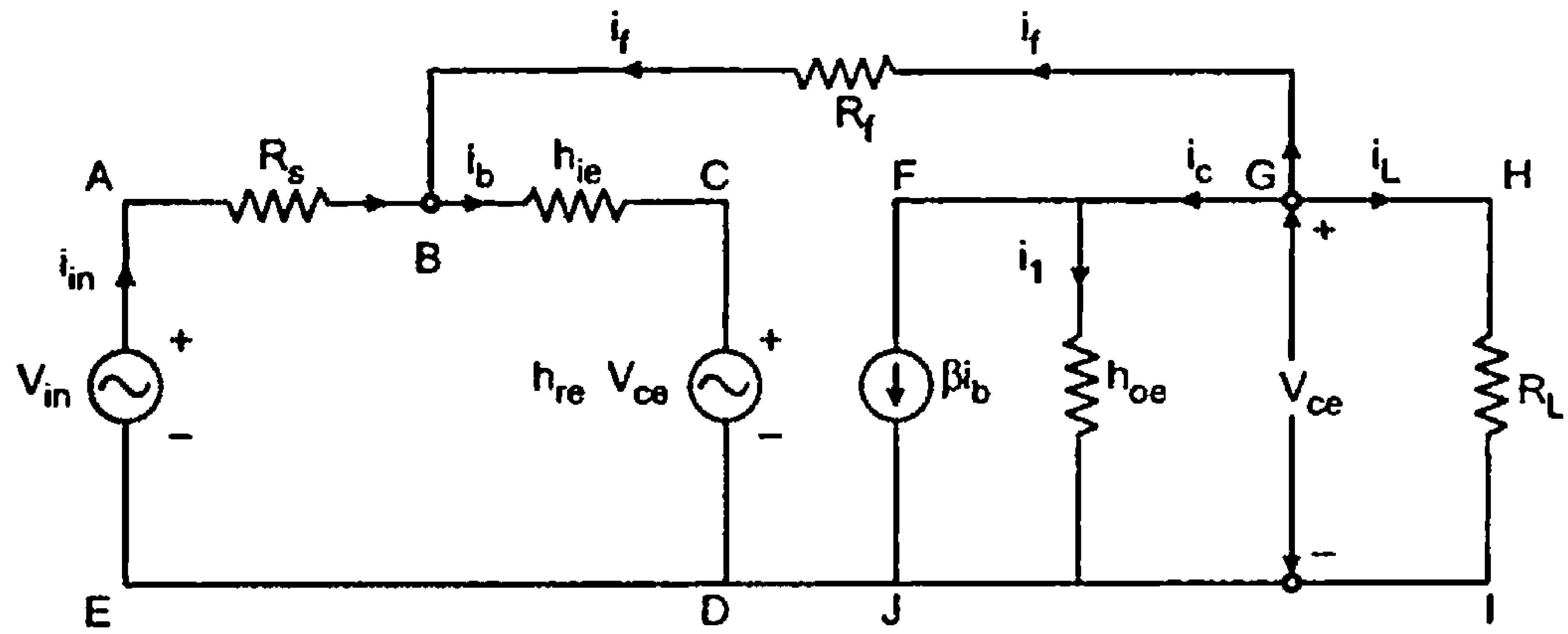
$$\frac{C(s)}{R(s)} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_4}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 + G_4 - G_2 G_4 H_1 H_2}}$$

Example 6.25 : *The small signal equivalent circuit of a common emitter transistor amplifier is shown below. The transistor amplifier includes a feedback resistor R_f. Obtain a signal flow graph model of the feedback amplifier and determine the input-output ratio (V_{ce} / V_{in}).* **(M.U. : Dec.-96)**



Solution : Let input current be i_{in} and load current be i_L.



At node B, $i_{in} + i_f = i_b$ ... (1)

Apply KVL to loop ABCDEA ,

$$V_{in} = i_{in} R_s + h_{ie} i_b + h_{re} V_{ce} \quad \text{... (2)}$$

Substituting $i_{in} = i_b - i_f$ from (1) in (2) we get

$$V_{in} = (i_b - i_f) R_s + h_{ie} i_b + h_{re} V_{ce}$$

$$V_{in} = i_b (R_s + h_{ie}) - i_f R_s + h_{re} V_{ce} \quad \text{... (3)}$$

The output equation is

$$V_{ce} = i_L R_L \quad \text{... (4)}$$

At node G,

$$i_c + i_f + i_L = 0 \quad \text{... (5)}$$

$\therefore$ $i_L = -(i_c + i_f)$ substitute in (4) we get

$$V_{ce} = -(i_c + i_f) R_L \quad \text{... (6)}$$

Now $i_c = i_1 + \beta i_b$

and $i_1 = h_{oe} V_{ce}$ as h_{oe} is admittance

$$\therefore \quad i_c = h_{oe} V_{ce} + \beta i_b \quad \text{... (7)}$$

Substituting (7) in (6) we get

$$V_{ce} = -[h_{ce} V_o + \beta i_b + i_f] R_L \quad \text{... (8)}$$

Applying KVL to the loop GBCDIG

$$i_f R_f + i_b h_{ie} + h_{re} V_{ce} - V_{ce} = 0$$

i.e. $$V_{ce} = i_f R_f + i_b h_{ie} + h_{re} V_{ce} \quad \text{... (9)}$$

Now the different variables are

$$V_{in} = \text{input} \qquad V_{ce} = \text{output}$$

and i_f , i_b are intermediate currents.

To sketch the signal flow graph let us obtain the equations for i_b , i_f and V_{ce}. V_{in} is input hence no equation for V_{in} is expected.

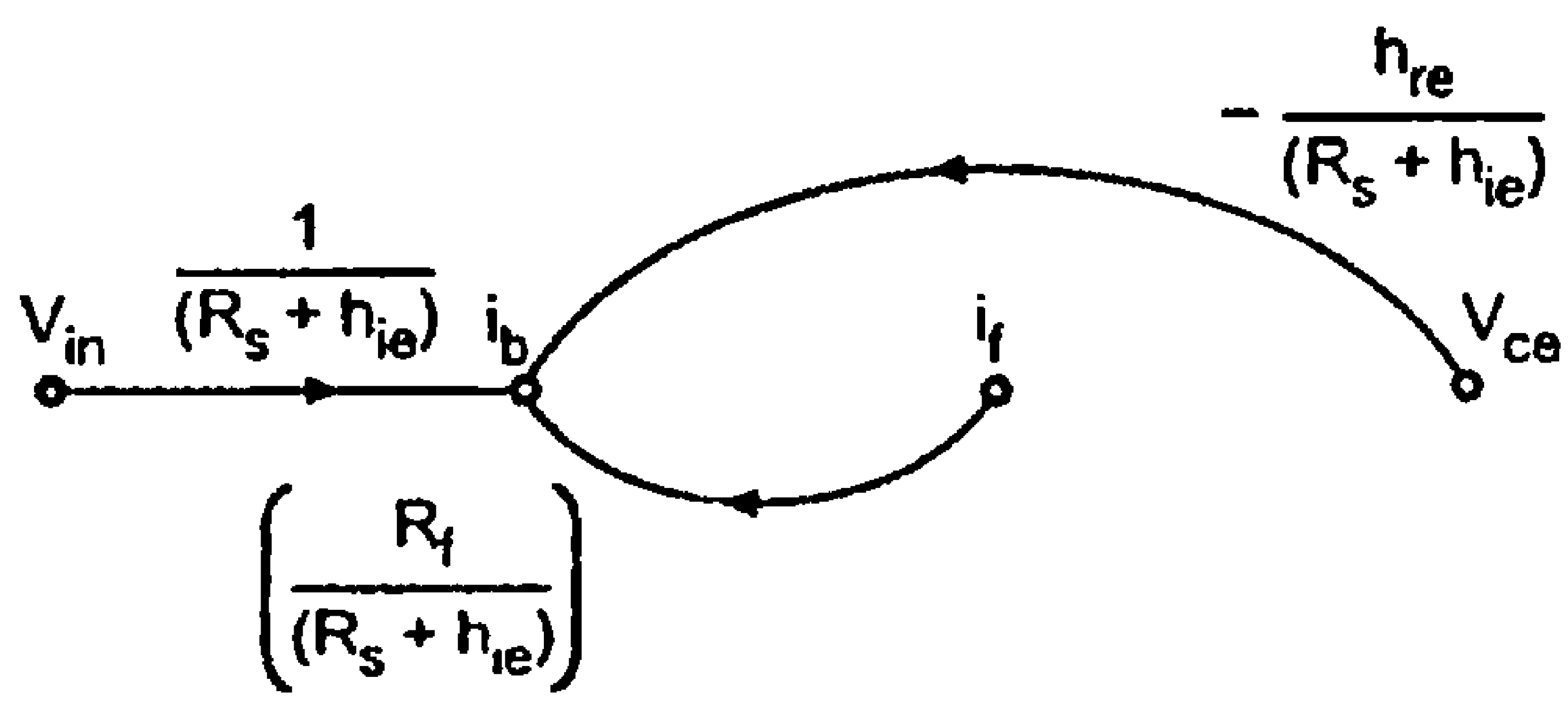
From (3) we can write equation for i_b as

$$i_b = \frac{V_{in} + i_f R_f - h_{re} V_{ce}}{(R_s + h_{ie})}$$

i.e.

$$i_b = \frac{1}{(R_s + h_{ie})} V_{in} + \frac{R_f}{(R_s + h_{ie})} i_f - \frac{h_{re}}{(R_s + h_{ie})} V_{ce} \qquad \text{... (10)}$$

This can be simulated by signal flow graph as



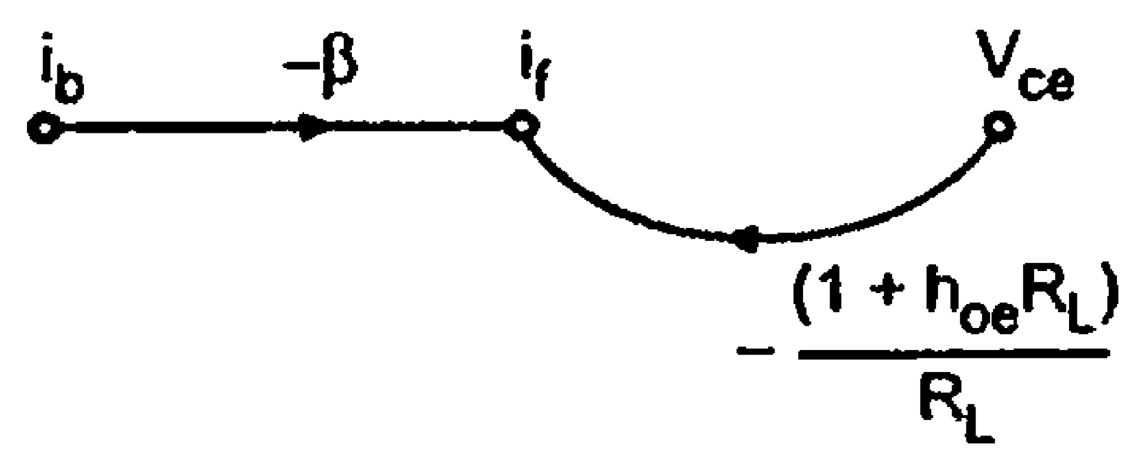
From (8) we can write equation for i_f as,

$$i_f = \frac{- h_{oe} R_L V_{ce} - \beta R_L i_b - V_{ce}}{R_L}$$

i.e.

$$i_f = \frac{-(1 + h_{oe} R_L)}{R_L} V_{ce} - \beta i_b \qquad \text{... (11)}$$

This can be simulated as



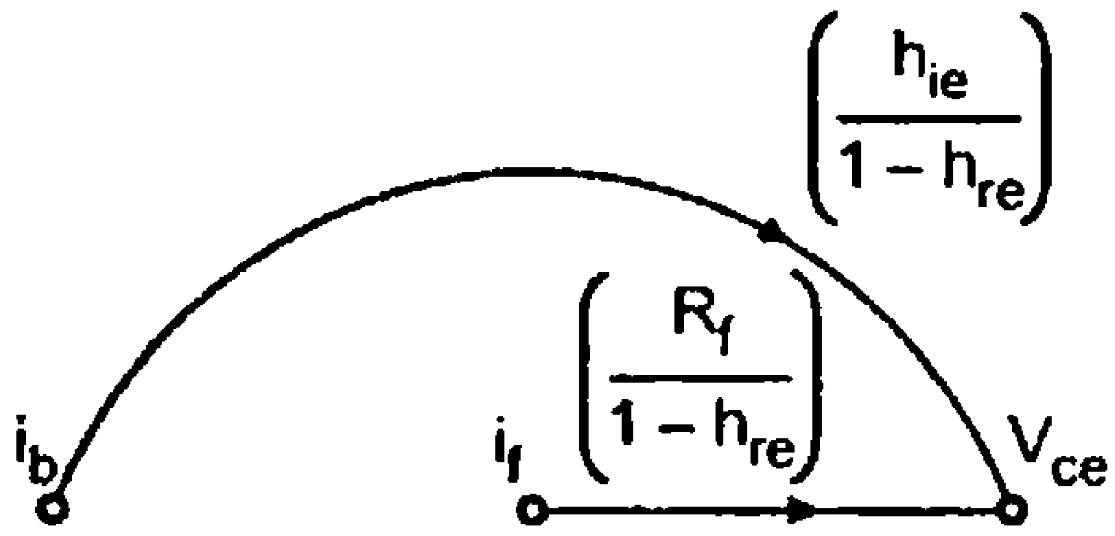
Equation (9) is directly the output equation

$$V_{ce} = i_f R_f + i_b h_{ie} + h_{re} V_{ce}$$

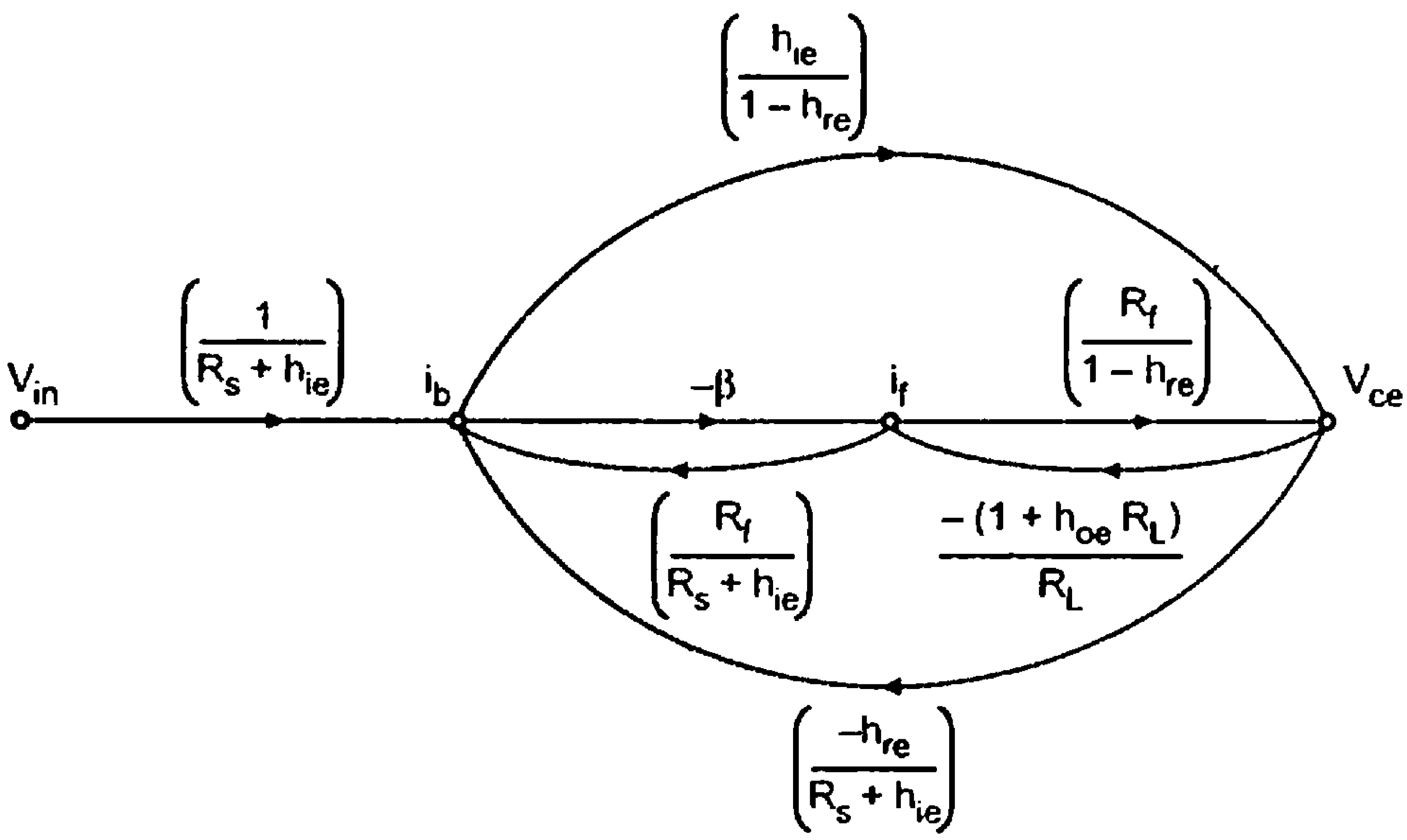
i.e.

$$V_{ce} = \left(\frac{R_f}{1 - h_{re}}\right) i_f + \left(\frac{h_{ie}}{1 - h_{re}}\right) i_b \qquad \text{... (12)}$$

This can be simulated as,



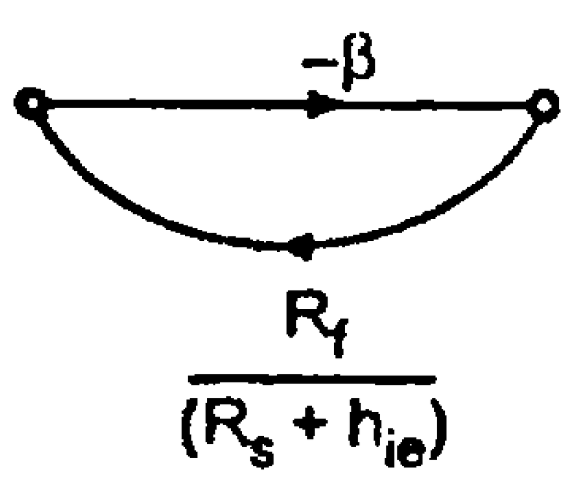
Combining all three signal flow graphs, the complete signal flow graph can be obtained as,

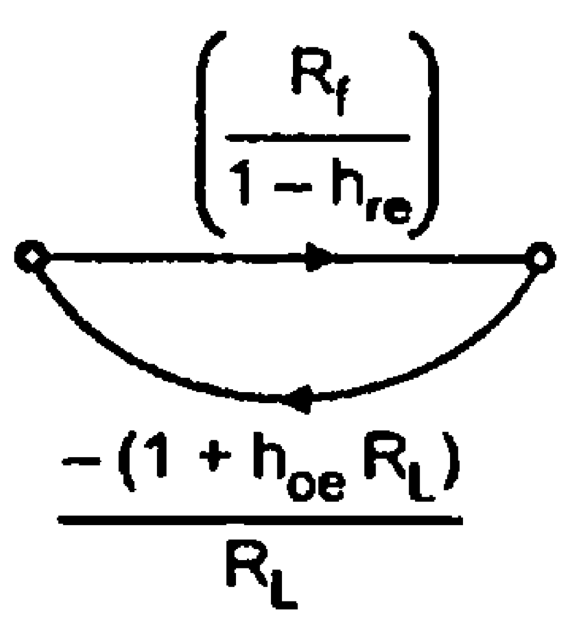


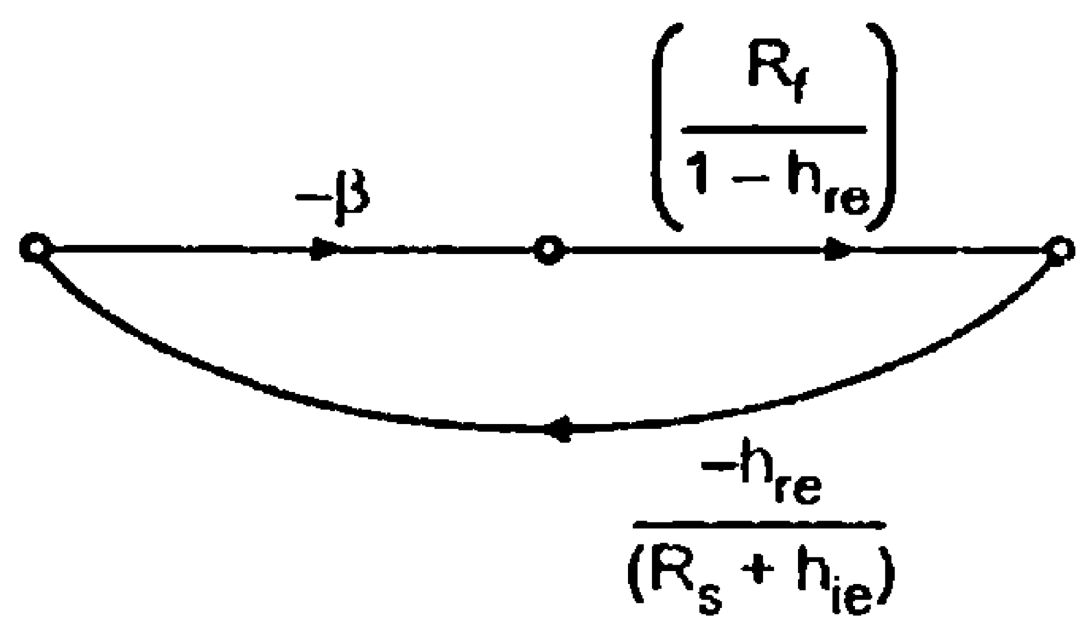
The forward paths are K = 2 Forward path gains are,

$$T_1 = \frac{-\beta R_f}{(R_s+h_{ie})(1-h_{re})} \quad T_2 = \frac{h_{ie}}{(R_s+h_{ie})(1-h_{re})}$$

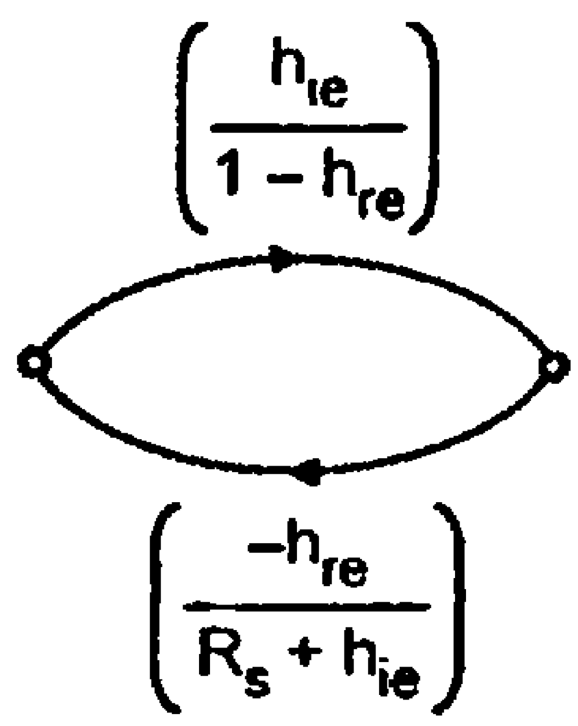
Individual feedback loops are,



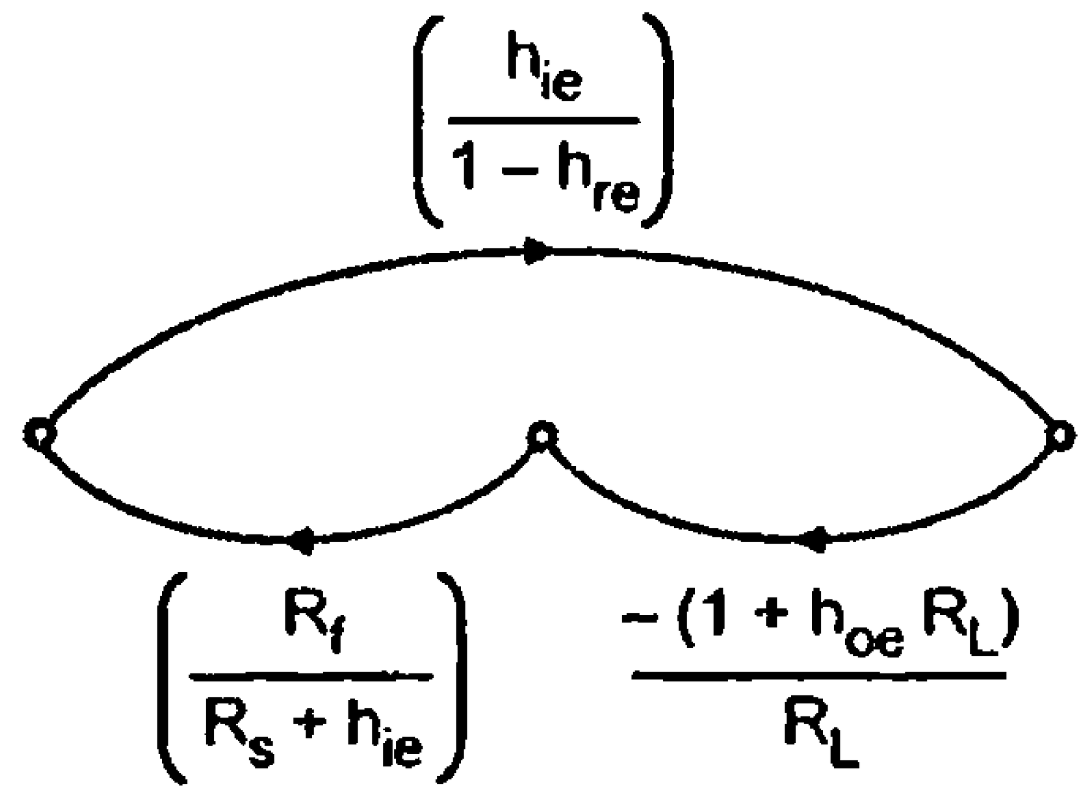




$$L_1 = \frac{-\beta R_f}{(R_s+h_{ie})} \qquad L_2 = \frac{-R_f(1+h_{oe}R_L)}{(1-h_{re})R_L} \qquad L_3 = \frac{+\beta R_f h_{re}}{(R_s+h_{ie})(1-h_{re})}$$



$$L_4 = \frac{-h_{ie}\,h_{re}}{(R_s+h_{ie})(1-h_{re})}$$



$$L_5 = \frac{-R_f\,h_{ie}(1+h_{oe}R_L)}{R_L\,(R_s+h_{ie})(1-h_{re})}$$

$$\therefore \quad \Delta = 1-[L_1+L_2+L_3+L_4+L_5]$$

As there are no combinations of two nontouching loops.

All the loops are touching to both the forward paths $\therefore\ \Delta_1 = \Delta_2 = 1$

$\therefore$ According to Mason's gain formula

$$\frac{V_{ce}}{V_{in}} = \frac{T_1\,\Delta_1 + T_2\,\Delta_2}{\Delta}$$

Simplifying Δ ,

$$\Delta = 1+\frac{\beta R_f}{(R_s+h_{ie})}+\frac{R_f(1+h_{oe}R_L)}{R_L(1-h_{re})} - \frac{\beta R_f\,h_{re}}{(R_s+h_{ie})(1-h_{re})}+\frac{h_{ie}\,h_{re}}{(R_s+h_{ie})(1-h_{re})}+\frac{R_f\,h_{ie}(1+h_{oe}R_L)}{R_L\,(R_s+h_{ie})(1-h_{re})}$$

$$\therefore\ \Delta = \frac{[R_L\,(R_s+h_{ie})(1-h_{re})+\beta R_f\,R_L\,(1-h_{re})+R_f\,(1+h_{oe}R_L)(R_s+h_{ie}) -\beta R_f\,h_{re}\,R_L + h_{ie}\,h_{re}\,R_L + R_f\,h_{ie}(1+h_{oe}R_L)]}{[R_L\,(R_s+h_{ie})(1-h_{re})]}$$

$$\therefore\ \Delta = \frac{R_f\,(1+h_{oe}R_L)(R_s+2\,h_{ie})+R_L\,[R_s\,(1-h_{re})+h_{ie}]+\beta R_f\,R_L\,(1-2\,h_{re})}{R_L\,(R_s+h_{ie})(1-h_{re})}$$

$$\therefore\ \frac{V_{ce}}{V_{in}} = \frac{\dfrac{-\beta R_f + h_{ie}}{(R_s+h_{ie})(1-h_{re})}}{\Delta}$$

$$= \frac{h_{ie}\,R_L - \beta R_f\,R_L}{R_f\,(1+h_{oe}R_L)(R_s+2\,h_{ie})+R_L\,[R_s\,(1-h_{re})+h_{ie}]+\beta R_f\,R_l\,(1-2\,h_{re})}$$

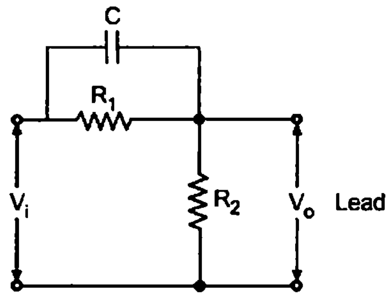
In practice for analysis of such network the values of h_{oe} and h_{re} are neglected.

$\therefore$ Simplified ratio $\frac{V_{ce}}{V_{in}}$ neglecting h_{oe} and h_{re} is

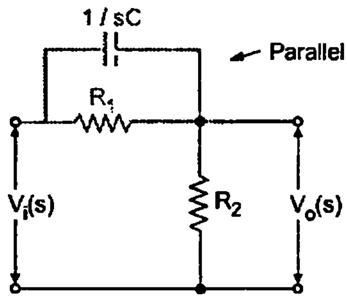
$$\therefore \quad \frac{V_{ce}}{V_{in}} = \frac{h_{ie} R_L - \beta R_f R_L}{R_f (R_s + 2h_{ie}) + R_L (R_s + h_{ie}) + \beta R_f R_L}$$

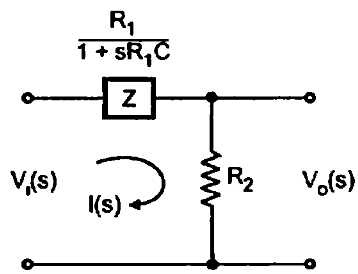
Example 6.26 : *Find T.F. for the given network, using Mason's gain formula*

(M.U. : May-97)



Solution : Laplace transform of the given network is,



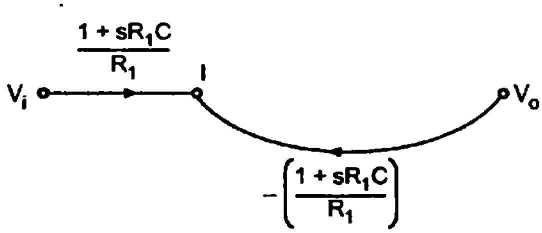


$$Z_{eq} = \frac{R_1 \times \frac{1}{sC}}{R_1 + \frac{1}{sC}} = \frac{R_1}{1 + R_1 Cs}$$

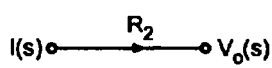
$$I(s) = \frac{(V_i - V_o)}{Z} = (V_i - V_o) \cdot \left(\frac{1 + sR_1 C}{R_1}\right) \qquad \text{... (1)}$$

$$V_o(s) = I(s) R_2 \qquad \text{... (2)}$$

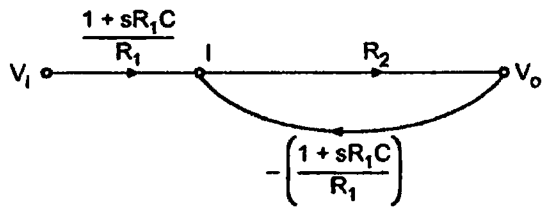
For equation (1) : S.F.G. is,



For equation (2) : S.F.G. is,



Combined S.F.G.



Use Mason's gain formula. Number of forward path = K = 1

$$T_1 = \frac{R_2(1+sR_1C)}{R_1}$$

Individual loop $L_1 = -R_2\left(\frac{(1+sR_1C)}{R_1}\right)$

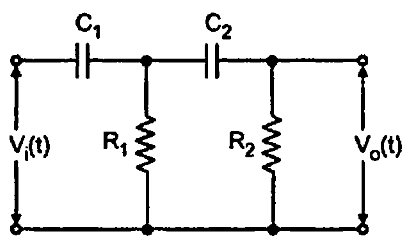
$$\therefore \quad \Delta = 1-[L_1] = 1+\frac{R_2(1+sR_1C)}{R_1} = \frac{R_1+R_2(1+sR_1C)}{R_1}$$

As L_1 is touching to T_1, $\Delta_1 = 1$

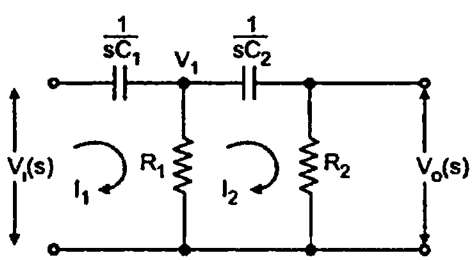
$$\therefore \quad \frac{V_o(s)}{V_i(s)} = \frac{T_1\,\Delta_1}{\Delta}$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{R_2(1+sR_1C)}{R_1+R_2(1+sR_1C)}}$$

Example 6.27 : *Draw the signal flow graph for the following network and find the transfer function* $\frac{V_o(s)}{V_i(s)}$.



Solution : Convert the given network in its laplace form and assume different loop currents and node voltages as shown.



Writing down equations for I_1 , V_1 , I_2 , V_o we get,

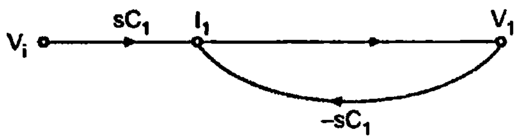
$$I_1 = \frac{(V_i - V_1)}{\frac{1}{sC_1}} = sC_1 (V_i - V_o) \qquad \text{... (I)}$$

$$V_1 = (I_1 - I_2) R_1 \qquad \text{... (II)}$$

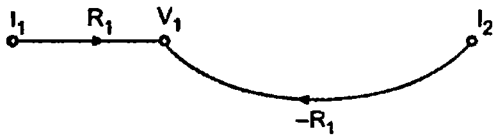
$$I_2 = \frac{(V_1 - V_o)}{\frac{1}{sC_2}} = sC_2 (V_1 - V_o) \qquad \text{... (III)}$$

$$V_o = I_2 R_2 \qquad \text{... (IV)}$$

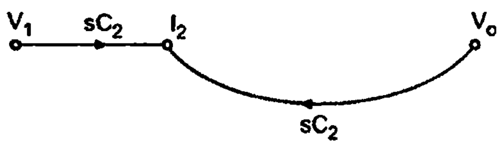
Simulating above equations by signal flow graph.



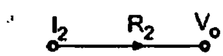
For equation (I)



For equation (II)

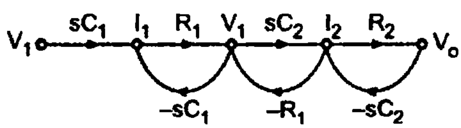


For equation (III)



For equation (IV)

Combining we get signal flow graph for given network.



To find T.F. apply Mason's gain formula

$$\text{T.F.} = \frac{\Sigma T_K \Delta_1}{\Delta} \qquad \text{Number of forward path} = K = 1$$

$$\therefore \quad \text{T.F.} = \frac{T_1 \Delta_1}{\Delta}$$

$$T_1 = sC_1 R_1 sC_2 R_2 = s^2 R_1 R_2 C_1 C_2$$

Individual loops :

$L_1 = -R_1 C_1 s, \quad L_2 = -sR_1 C_2 \quad L_3 = -R_2 sC_2$

Out of three, L_1 and L_3 are nontouching

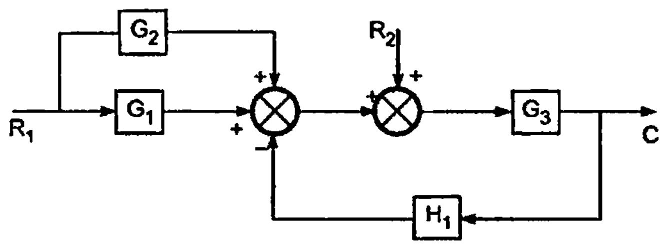
$$\begin{aligned}\Delta &= 1-[L_1+L_2+L_3]+[L_1 L_3] \\ &= 1-[-sR_1 C_1 - sR_1 C_2 - sR_2 C_2]+[s^2 R_1 C_1 R_2 C_2] \\ &= 1+s[R_1 C_1 + R_1 C_2 + R_2 C_2] + s^2 [R_1 C_1 R_2 C_2]\end{aligned}$$

$$\therefore \quad \frac{V_o(s)}{V_i(s)} = \frac{T_1 \Delta_1}{\Delta}$$

All loops are touching to T_1, $\therefore \Delta_1 = 1$

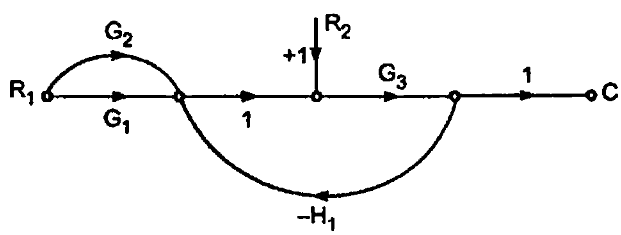
$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{s^2 R_1 C_1 R_2 C_2}{1+s[R_1 C_1 + R_1 C_2 + R_2 C_2]+s^2[R_1 C_1 R_2 C_2]}}$$

Example 6.28 : *Determine the transfer functions C/R_1 and C/R_2 from the block diagram shown by drawing its signal flow graph and using Mason's gain formula.* **(M.U. : May-99)**



Solution : Let us draw the signal flow graph to use the Mason's gain formula.

The signal flow graph is,



For C/R_1, let $R_2 = 0$

$\therefore \quad T_1 = G_1 G_3$

and $\quad T_2 = G_2 G_3$

$L_1 = -G_3 H_1$

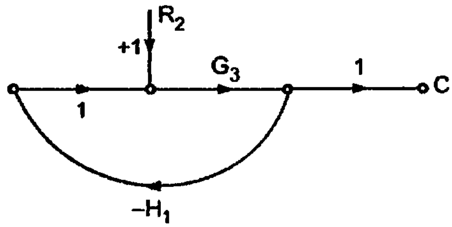
$\Delta = 1-[L_1] = 1-[-G_3 H_1] = 1+G_3 H_1$

$\Delta_1 = 1$ and $\Delta_2 = 1$

$$\therefore \quad \frac{C}{R_1} = \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} = \frac{G_1 G_3 + G_2 G_3}{1+G_3 H_1} = \frac{G_3(G_1+G_2)}{1+G_3 H_1}$$

For C/R_2, let $R_1 = 0$

Hence signal flow graph reduces to,



$$T_1 = G_3$$

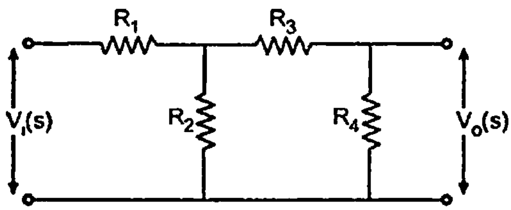
$$L_1 = -G_3 H_1$$

$$\Delta = 1 - L_1 = 1 + G_3 H_1$$

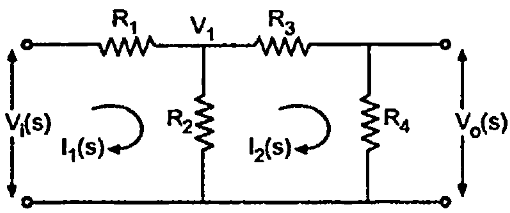
$$\Delta_1 = 1$$

$$\therefore \quad \frac{C}{R_2} = \frac{T_1 \Delta_1}{\Delta} = \frac{G_3}{1 + G_3 H_1}$$

Example 6.29 : *Find the T.F. of the given network.* **(M.U. : May-2003)**

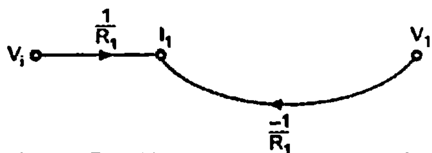


Solution : Laplace Transform of the given network is,



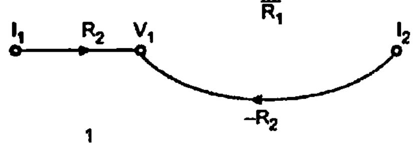
Equations for different currents and voltages are

S.F.G. (I)



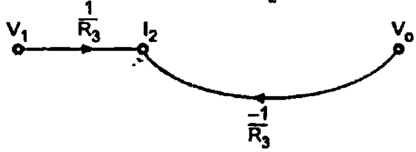
$$I_1 = (V_i - V_1) \times \frac{1}{R_1} \quad \text{... (I)}$$

S.F.G. (II)



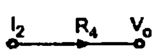
$$V_1 = (I_1 - I_2) R_2 \quad \text{... (II)}$$

S.F.G. (III)



$$I_2 = (V_1 - V_o) \times \frac{1}{R_3} \quad \text{... (III)}$$

S.F.G. (IV)



$$V_o = I_2 R_4 \quad \text{... (IV)}$$

Solution : For the given current distributions,

For branch r_1, $$i_1 = \frac{V_1 - V_2}{r_1} = \frac{1}{r_1} V_1 - \frac{1}{r_1} V_2 \quad \ldots (1)$$

For branch r_3, $$V_2 = (i_1 - i_2)\, r_3 = i_1 r_3 - i_2 r_3 \quad \ldots (2)$$

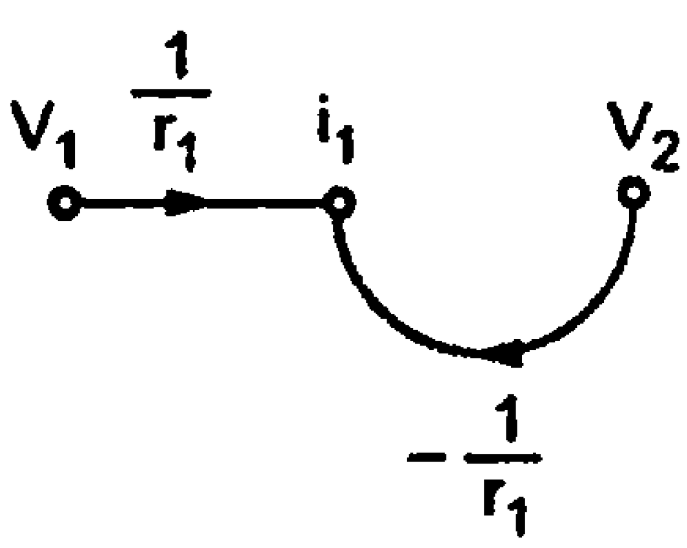
Now current through r_2 is $(i_2 - a\, i_1)$. Hence according to Ohm's law,

$$i_2 - a\, i_1 = \frac{V_2 - V_3}{r_2}$$

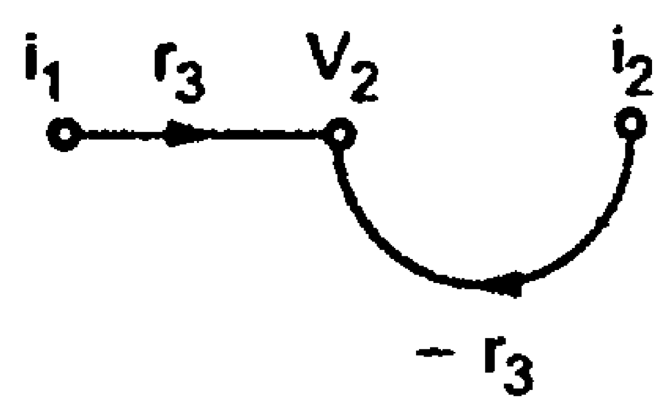
$$\therefore \quad i_2 = \frac{1}{r_2} V_2 - \frac{1}{r_2} V_3 + a\, i_1 \quad \ldots (3)$$

And $$V_3 = i_2\, r_4 \quad \ldots (4)$$

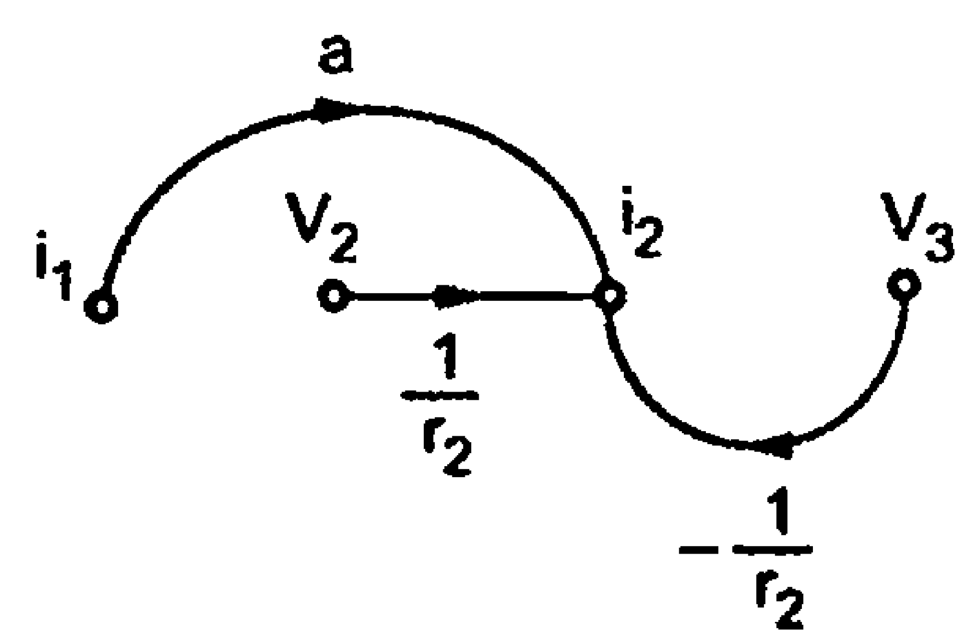
The simulations of all the equations are,



Equation (1)



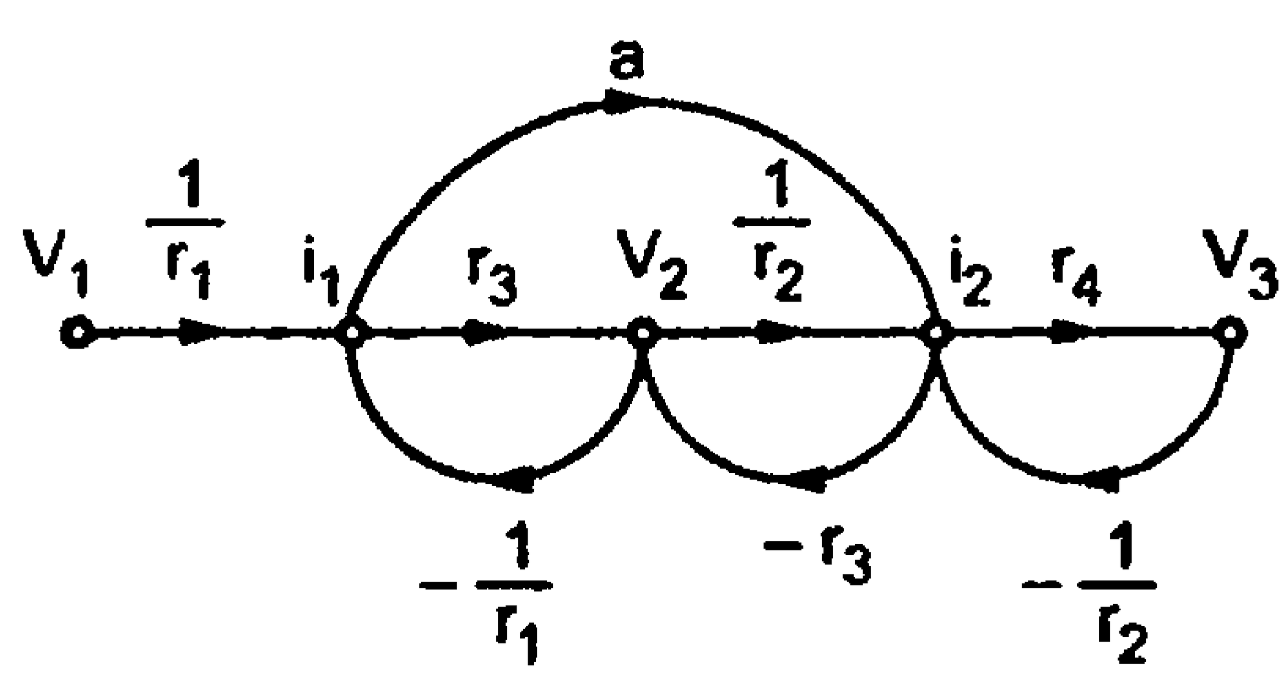
Equation (2)



Equation (3)

Equation (4)

Thus the overall signal flow graph is as shown,

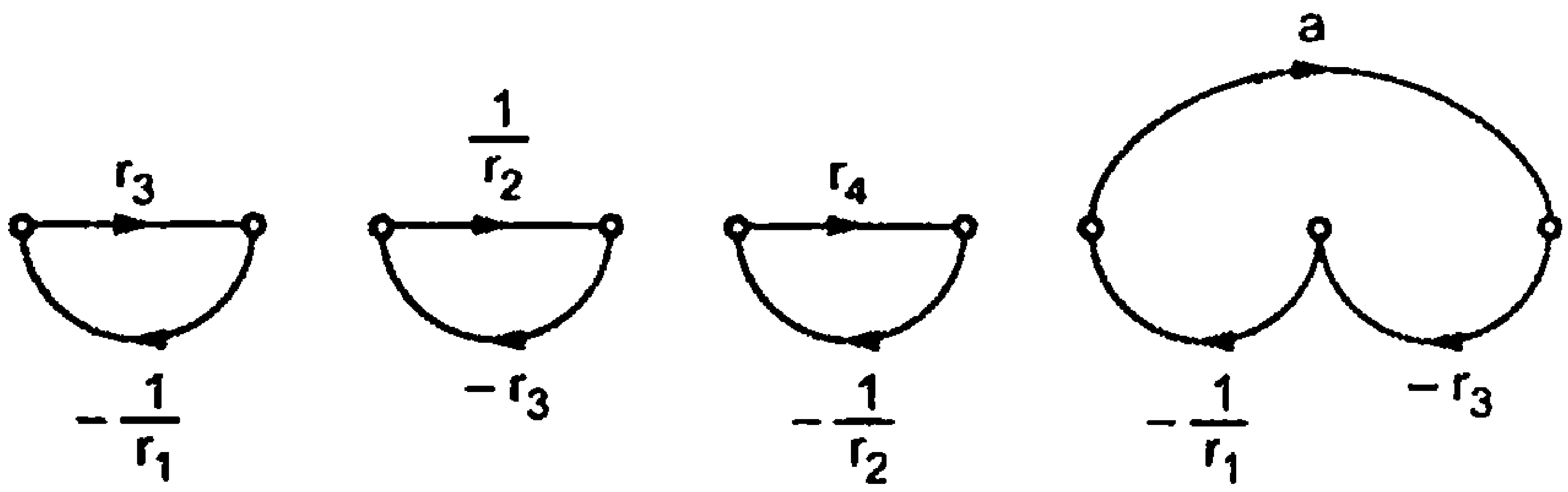


There are two forward paths,

$$T_1 = \frac{r_3\, r_4}{r_1\, r_2}$$

$$T_2 = \frac{a\, r_4}{r_1}$$

The various loops and loop gains are,



$$L_1 = \frac{-r_3}{r_1} \qquad L_2 = \frac{-r_3}{r_2} \qquad L_3 = \frac{-r_4}{r_2} \qquad L_4 = \frac{a\, r_3}{r_1}$$

One combination of two non touching loops is $L_1\ L_3$.

$$\therefore \quad \Delta = 1-[L_1+L_2+L_3+L_4]+[L_1\ L_3]$$

All loop are touching to all the forward paths,

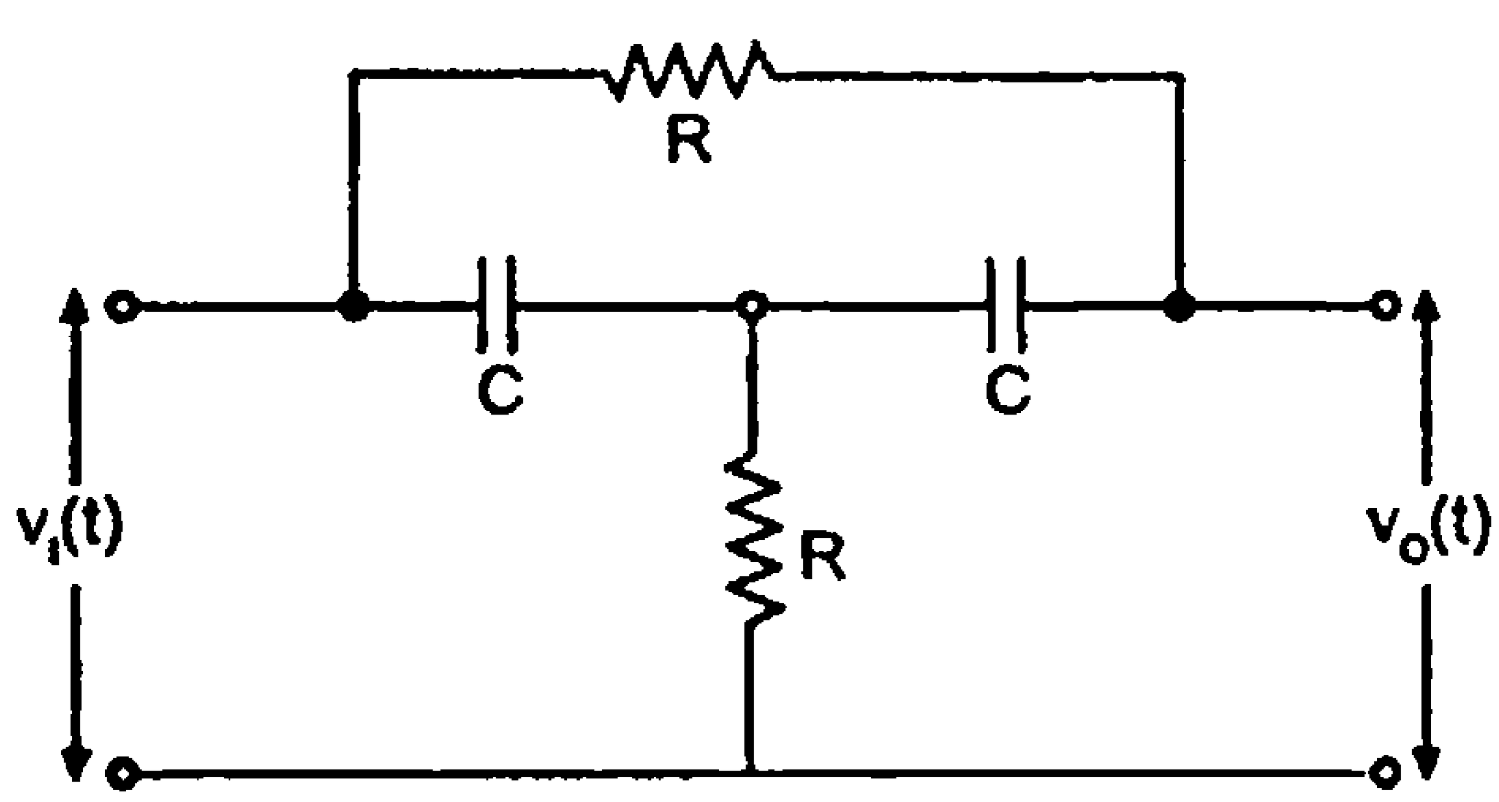
$$\therefore \quad \Delta_1 = \Delta_2 = 1$$

$$\therefore \quad \frac{V_3}{V_1} = \frac{T_1\,\Delta_1+T_2\,\Delta_2}{\Delta} = \frac{\frac{r_3\, r_4}{r_1\, r_2}+\frac{a\, r_4}{r_1}}{1-[L_1+L_2+L_3+L_4]+[L_1\ L_3]}$$

$$= \frac{\frac{r_3\, r_4}{r_1\, r_2}+\frac{a\, r_4}{r_1}}{1+\frac{r_3}{r_1}+\frac{r_3}{r_2}+\frac{r_4}{r_2}-\frac{a\, r_3}{r_1}+\frac{r_3\, r_4}{r_1\, r_2}}$$

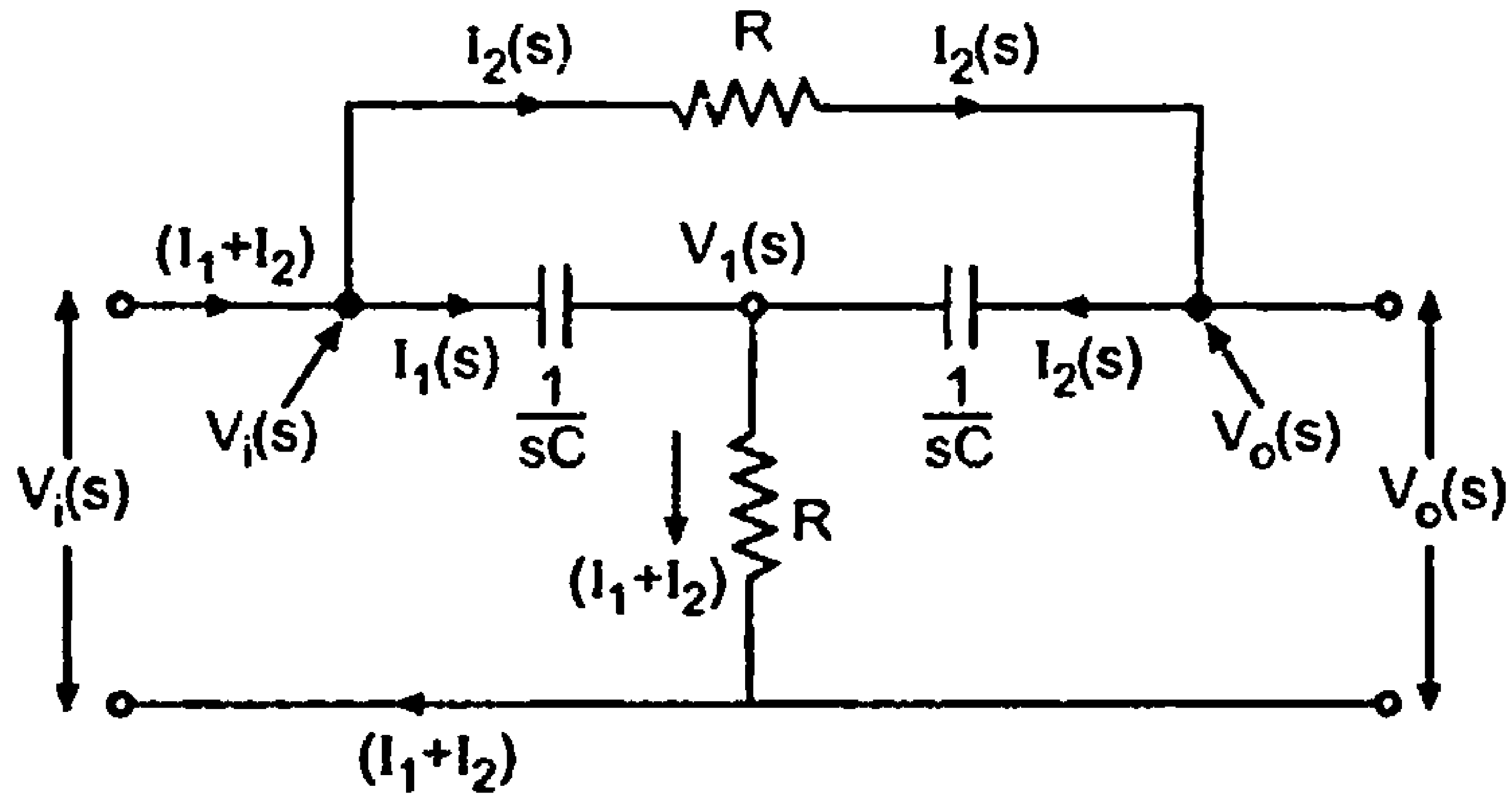
$$\therefore \quad \boxed{\frac{V_3}{V_1} = \frac{r_3\, r_4+a\, r_4\, r_2}{r_1\, r_2+r_3\, r_2+r_3\, r_1+r_1\, r_4-a\, r_2\, r_3+r_3\, r_4}}$$

Example 6.31 : *Obtain the transfer function* $\frac{V_o(s)}{V_i(s)}$ *for the network shown in the figure below.* **(M.U. : May-2003)**



Solution : The Laplace domain representation of the given network is shown below.

The various branch currents are shown,



$$\therefore \quad I_1(s) = \frac{V_i(s)-V_1(s)}{\left(\frac{1}{sC}\right)} = sC\,V_i(s) - sC\,V_1(s) \quad \text{... (1)}$$

Then,
$$I_2(s) = \frac{V_i(s)-V_o(s)}{R} = \frac{1}{R}V_i(s) - \frac{1}{R}V_o(s) \quad \text{... (2)}$$

$$V_1(s) = [I_1(s)+I_2(s)]R = R\,I_1(s) + R\,I_2(s) \quad \text{... (3)}$$

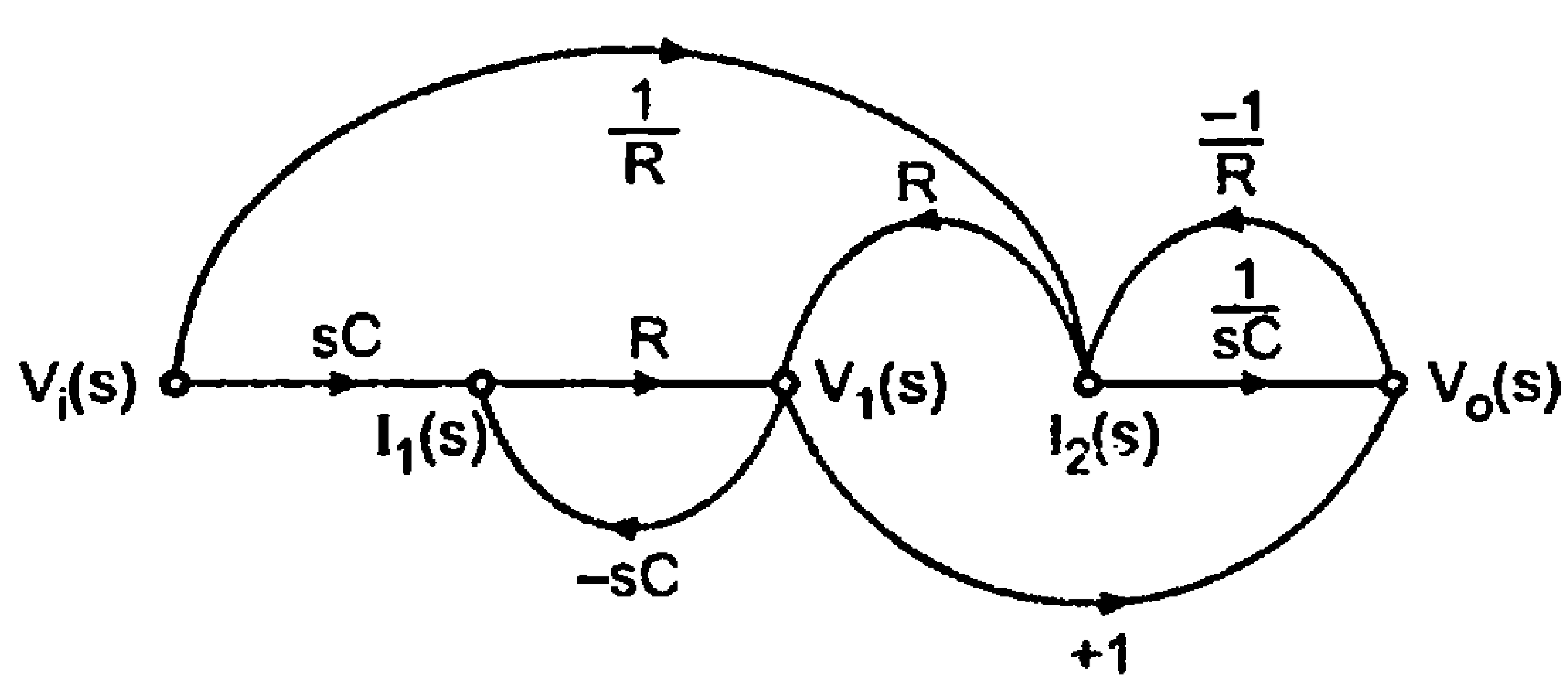
and also,
$$I_2(s) = \frac{V_o(s)-V_1(s)}{\frac{1}{sC}} = sC\,V_o(s) - sC\,V_1(s)$$

From this obtain the equation for $V_o(s)$ as $I_2(s)$ equation is already obtained.

Note : Write the separate equation for separate branch and **each element must be considered at least once.**

$$\therefore \quad V_o(s) = V_i(s) + \frac{1}{sC} I_2(s) \quad \text{... (4)}$$

Hence the signal flow graph is,



The forward path gains are,

$$T_1 = sCR \qquad T_2 = \frac{1}{sCR} \qquad T_3 = \frac{1}{R} \times R \times 1 = 1$$

The various loop gains are,

$$L_1 = -sCR \qquad L_2 = -\frac{1}{sCR} \qquad L_3 = -\frac{1}{R}\times R\times 1 = -1$$

The loops L_1 and L_2 are non touching

$$\therefore \quad L_1L_2 = 1$$

Hence system determinant is,

$$\Delta = 1-[L_1+L_2+L_3]+[L_1L_2]$$

$$= 1+sCR+\frac{1}{sCR}+1+1 = \frac{3sCR+s^2C^2R^2+1}{sCR}$$

For T_1, $\Delta_1 = 1$ all loops touching to T_1

For T_2, $\Delta_2 = 1 - L_1$ as L_1 is non touching to T_2

$= (1 + sCR)$

For T_3, $\Delta_3 = 1$ all loops touching to T_3

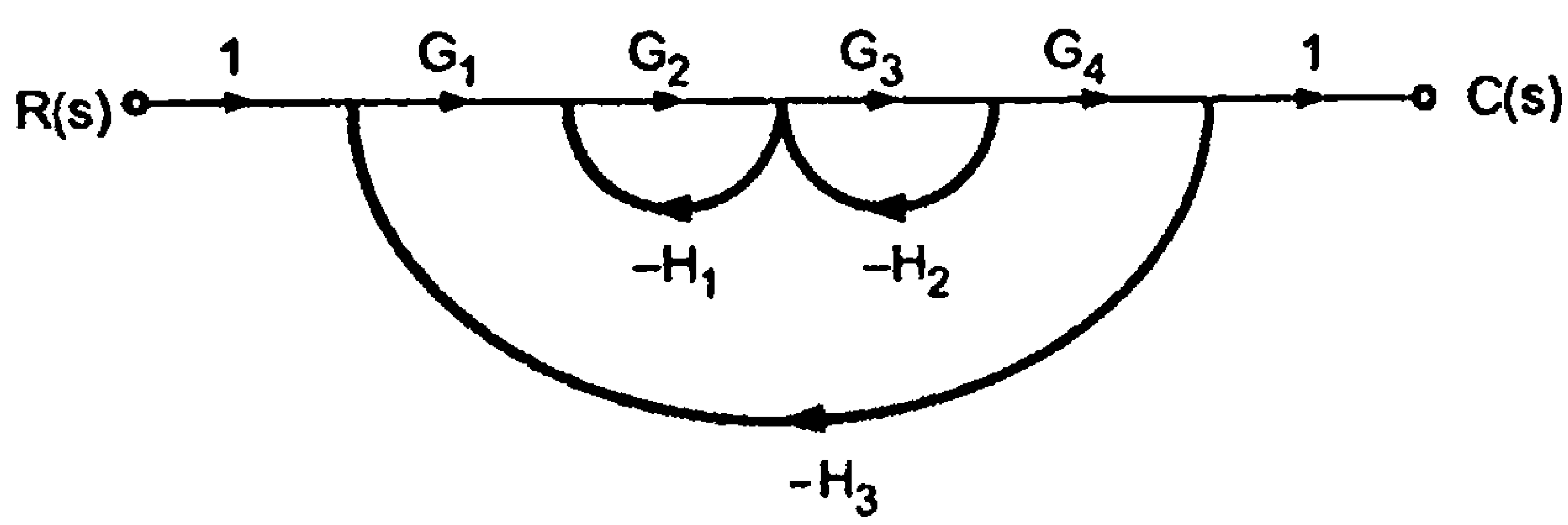
According to Mason's gain formula,

$$\frac{V_o(s)}{V_i(s)} = \frac{T_1\Delta_1+T_2\Delta_2+T_3\Delta_3}{\Delta}$$

$$= \frac{sCR+\left(\frac{1}{sCR}\right)(1+sCR)+1}{\left(\frac{3sCR+s^2C^2R^2+1}{sCR}\right)} = \frac{\left(\frac{s^2C^2R^2+1+sCR+sCR}{sCR}\right)}{\left(\frac{3sCR+s^2C^2R^2+1}{sCR}\right)}$$

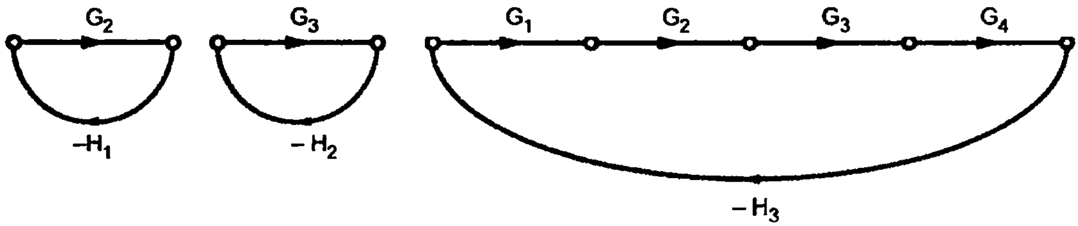
$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{s^2C^2R^2+2sCR+1}{s^2C^2R^2+3sCR+1}}$$

Example 6.32 : *Obtain overall transfer function.*



(M.U. : May - 2005)

Solution : $T_1 = G_1 G_2 G_3 G_4$ ... Only one forward path



$L_1 = -G_2H_1$ $\qquad L_2 = -G_3H_2$ $\qquad L_3 = -G_1G_2G_3G_4H_3$

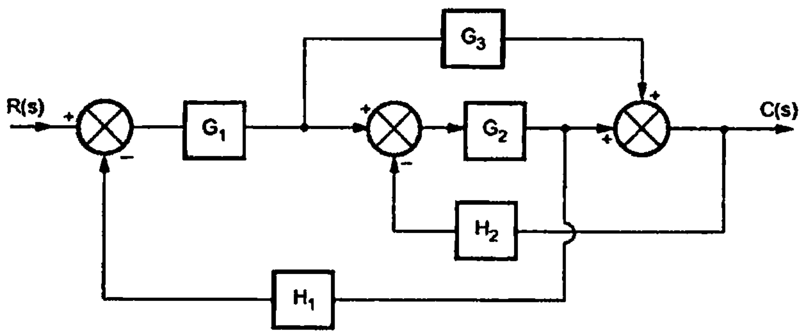
No combination of non touching loops.

$$\therefore \quad \Delta = 1-[L_1+L_2+L_3] = 1+G_2H_1+G_3H_2+G_1G_2G_3G_4H_3$$

$$\Delta_1 = 1 \qquad \text{... All loops touching to } T_1$$

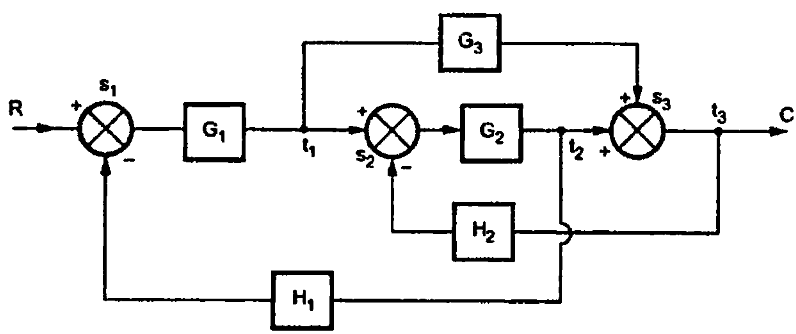
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\Delta_1}{\Delta} = \frac{G_1G_2G_3G_4}{1+G_2H_1+G_3H_2+G_1G_2G_3G_4H_3}$$

Example 6.33 : *Draw the signal flow graph and derive the transfer function using Mason's gain formula.*

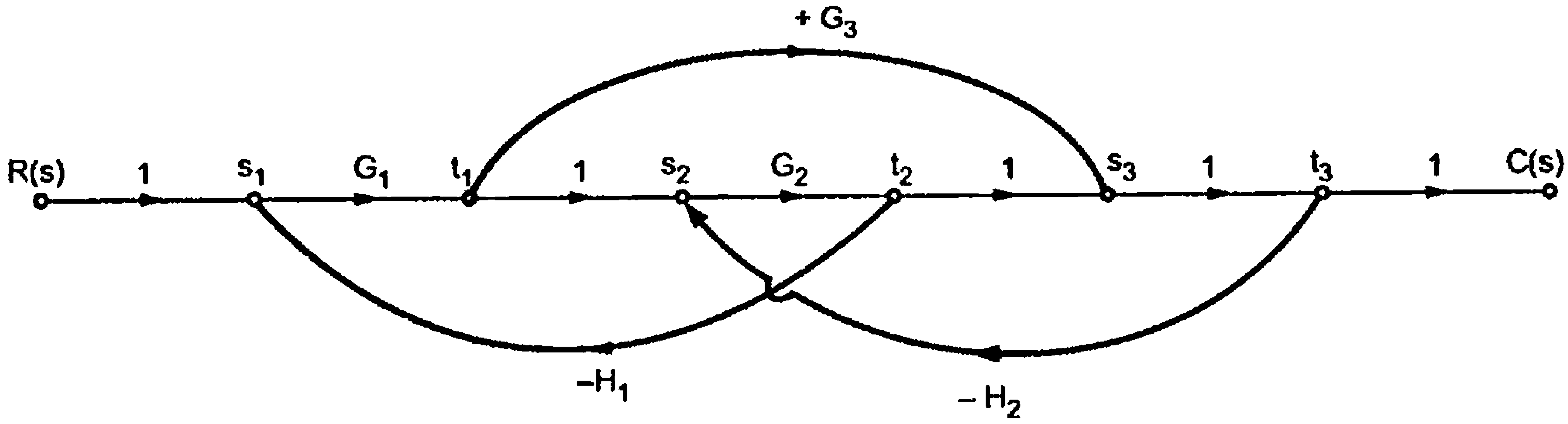


(M.U. : May - 2006)

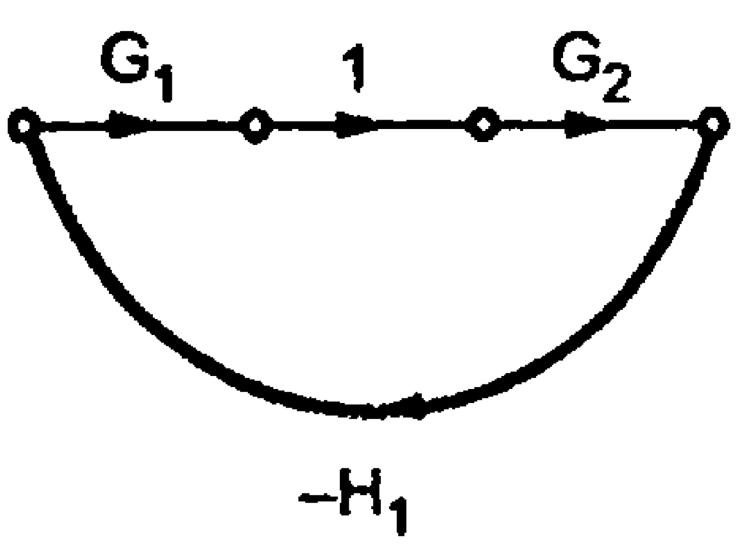
Solution : Name the summing and take off points.

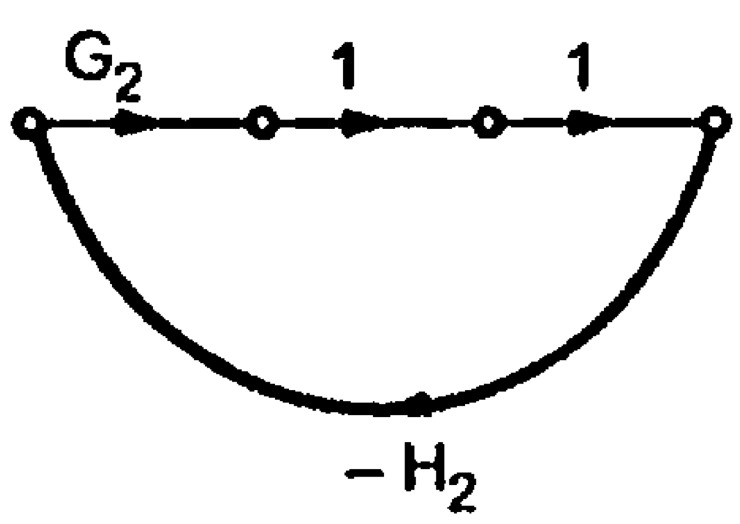


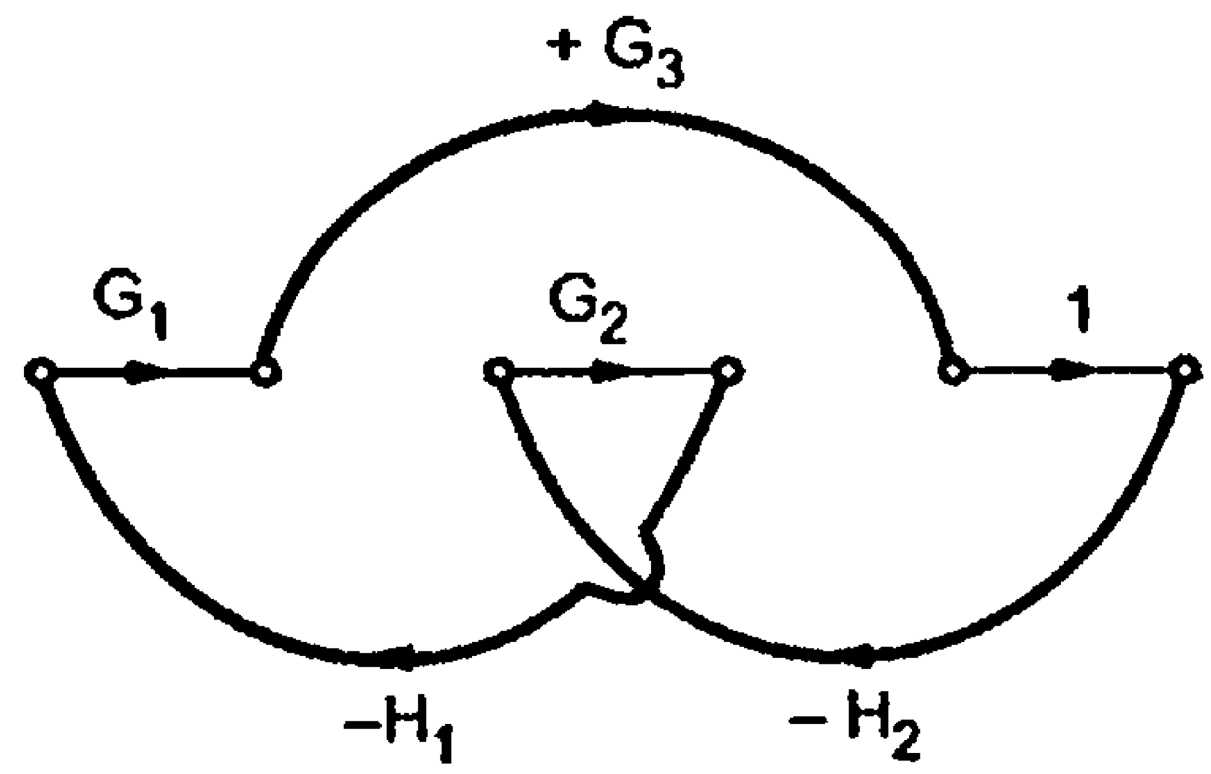
The signal flow graph is,



$T_1 = G_1G_2 \quad T_2 = G_1G_3$ ... K = 2 forward paths







$L_1 = G_1G_2H_1 \qquad L_2 = -G_2H_2 \qquad L_3 = G_1G_2G_3H_1H_2$

All loops are touching to each other, so no combination of nontouching loops.

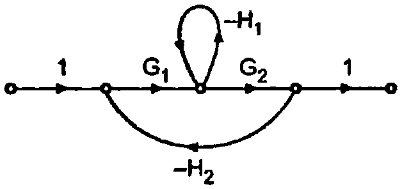
$$\therefore \quad \Delta = 1-[L_1+L_2+L_3] = 1+G_1G_2H_1+G_2H_2-G_1G_2G_3H_1H_2$$

$$\Delta_1 = 1, \quad \Delta_2 = 1$$ ... All loops touching to both forward paths

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{T_1\Delta_1+T_2\Delta_2}{\Delta} = \frac{G_1G_2+G_1G_3}{1+G_1G_2H_1+G_2H_2-G_1G_2G_3H_1H_2}}$$

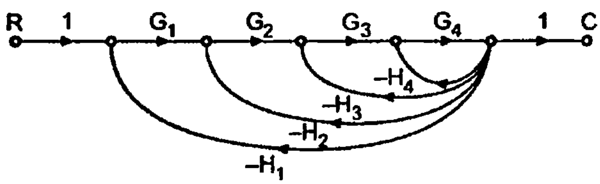
Review Questions

1. *Define signal flow graph.*
2. *Define following terms related to signal flow graph*
   (i) Source node (ii) Sink node (iii) Chain node (iv) Forward path and its gain
   (v) Feedback loop and its gain (vi) Self loop (vii) Nontouching loops.
3. *Explain the various properties of signal flow graph representation.*
4. *Explain how to construct signal flow graph from*
   (i) Set of equations (ii) Block diagram. Give suitable example.
5. *State and explain Mason's Gain Formula.*
6. *Compare block diagram representation with signal flow graph representation.*
7. *Construct the signal flow graph for the following set of simultaneous equations.*
   $X_2 = A_{21} X_1 + A_{23} X_3$ $\quad X_3 = A_{31} X_1 + A_{32} X_2 + A_{33} X_3$
   $X_4 = A_{42} X_2 + A_{43} X_3$
8. *Find transfer functions for signal flow graphs given below*



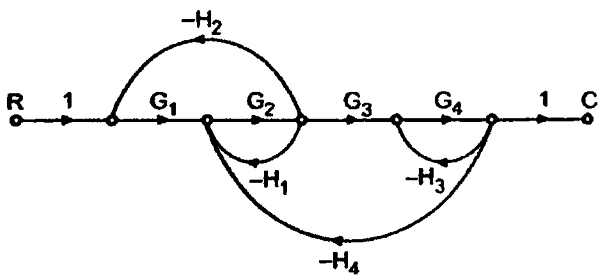
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2 H_2 + H_1}\right)$$

9.



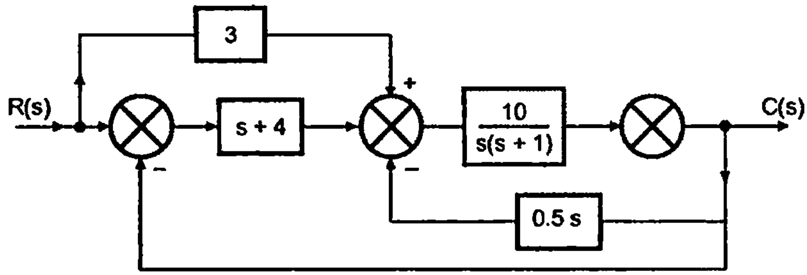
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 G_4 H_4 + G_3 G_4 H_3 + G_2 G_3 G_4 H_2 + G_1 G_2 G_3 G_4 H_1}\right)$$

10.



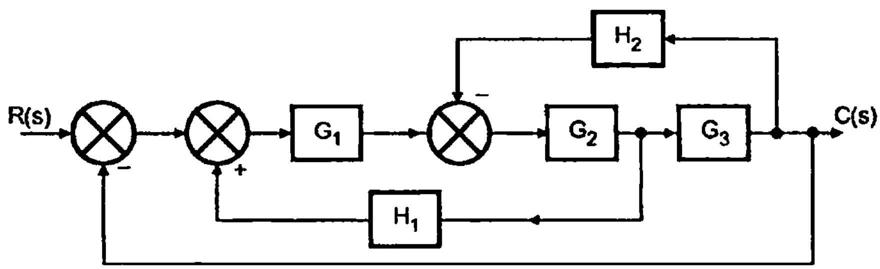
$$\left(\text{Ans.: } \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_2 + G_2 H_1 + G_2 G_3 G_4 H_4 + G_4 H_3 + G_2 G_4 H_1 H_3 + G_1 G_2 G_4 H_2 H_3}\right)$$

11. *Draw signal flow graph for the system shown below.*
Find overall transfer function using Mason's Gain Formula.



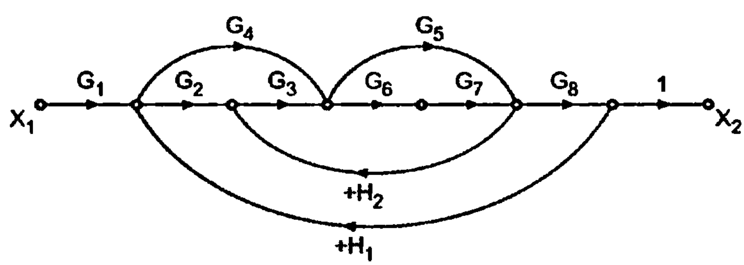
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{10\,(s+7)}{s^2+16\,s+40}\right)$$

12. *For system shown, obtain the closed loop transfer function by Mason's Gain Formula*



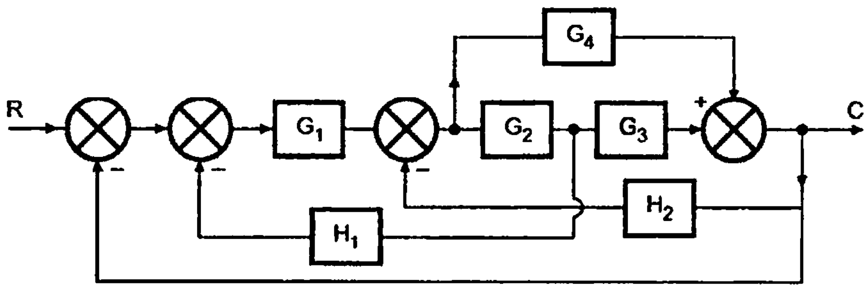
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1\ G_2\ G_3}{1-G_1\ G_2\ H_1+G_2\ G_3\ H_2+G_1\ G_2\ G_3}\right)$$

13. *Use Mason's Gain Formula to find* $\frac{X_2}{X_1}$.



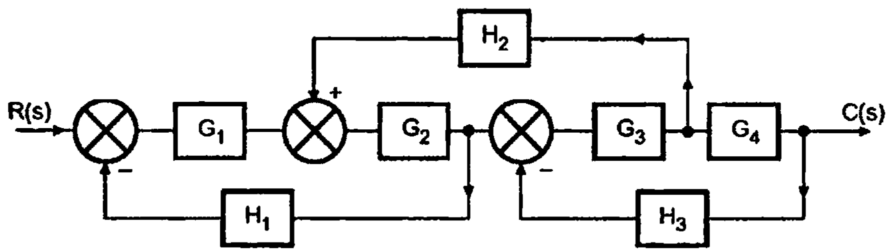
$$\left(\text{Ans. : } \frac{X_2}{X_1} = \frac{G_1G_2G_3G_6G_7G_8+G_1G_4G_6G_7G_8+G_1G_2G_3G_5G_8+G_1G_4G_5G_8}{1-G_2G_3G_6G_7G_8H_1-G_4G_6G_7G_8H_1-G_2G_3G_5G_8H_1-G_4G_5G_8H_1-G_3G_6G_7H_2-G_3G_5H_2}\right)$$

14. *Use Mason's gain formula to calculate C/R of the system shown below.*



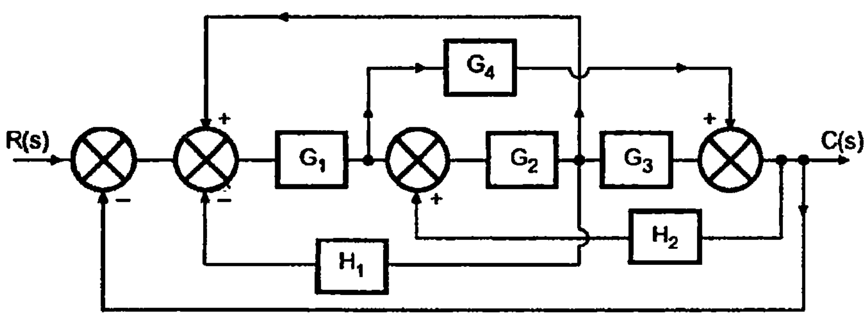
$$\left(\text{Ans. : } \frac{C}{R} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_4 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_4 H_2}\right)$$

15. *Draw the signal flow graph and obtain the transfer function of the system shown below.*



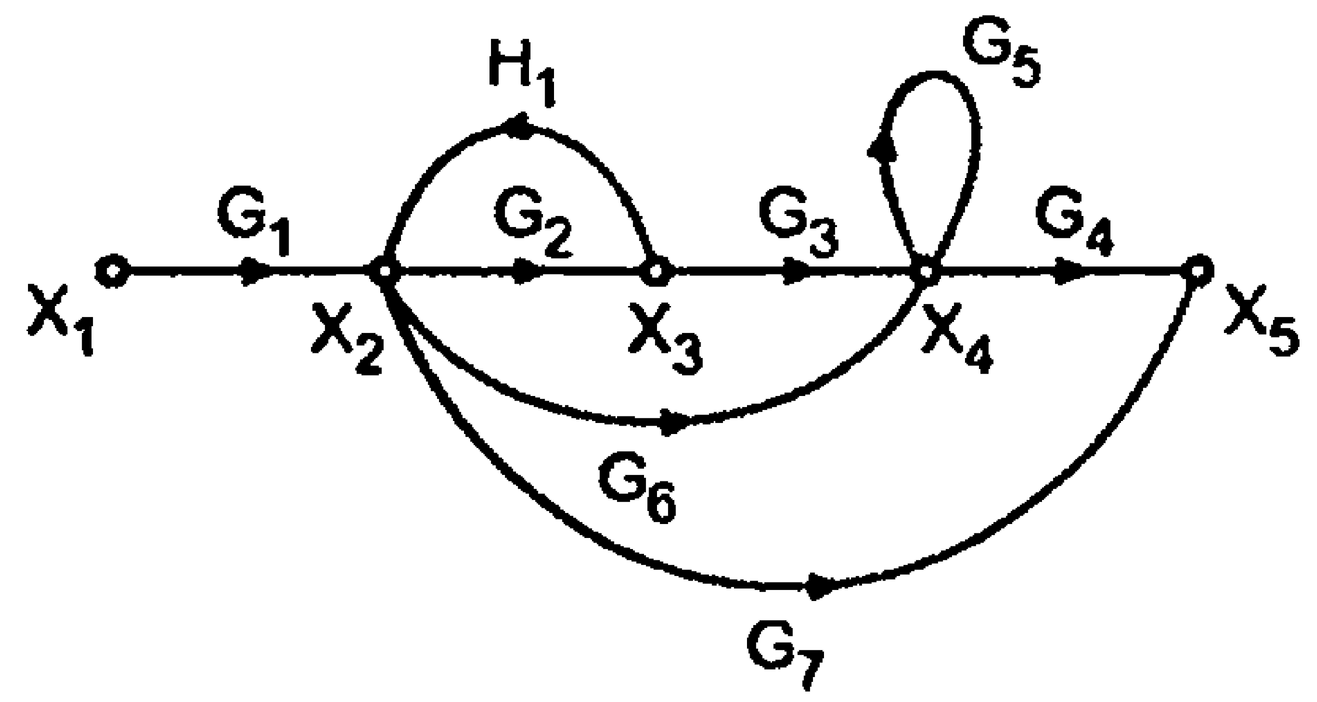
$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_1 - G_2 G_3 H_2 + G_3 G_4 H_3 + G_1 G_2 G_3 G_4 H_1 H_3}\right)$$

16. *Use Mason's Gain Formula to obtain C(s)/R(s) of the system shown below.*



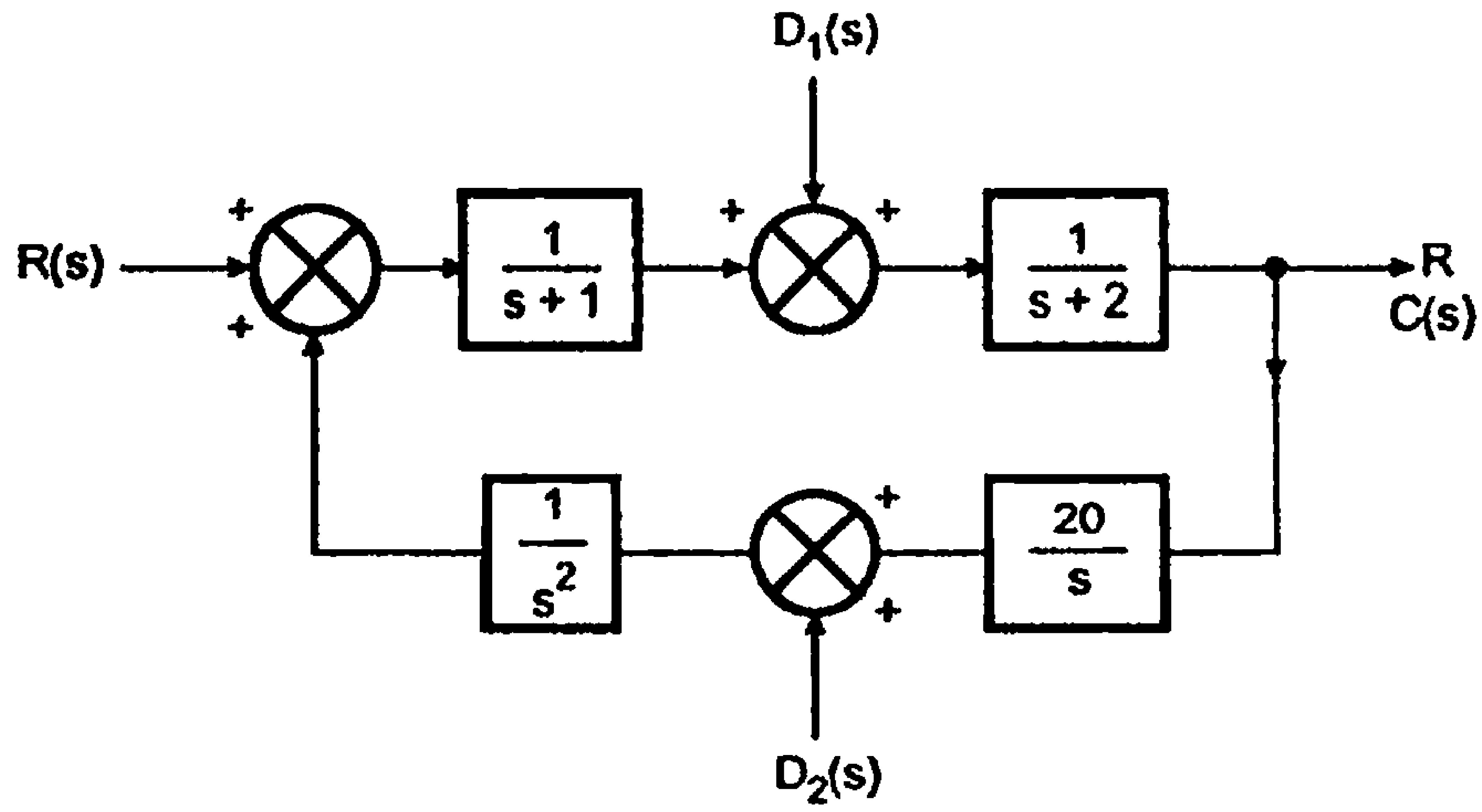
$$\left(\text{Ans.: } \frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 - G_1G_2 - G_2G_3H_2 + G_1G_2G_3 + G_1G_4 + G_1G_2G_4H_1H_2 + G_1G_2H_1 - G_1G_2G_4H_2}\right)$$

17. *For the signal flow graph shown in following figure determine the ratio x_5/x_1. Use Mason's gain formula for signal flow graphs.*



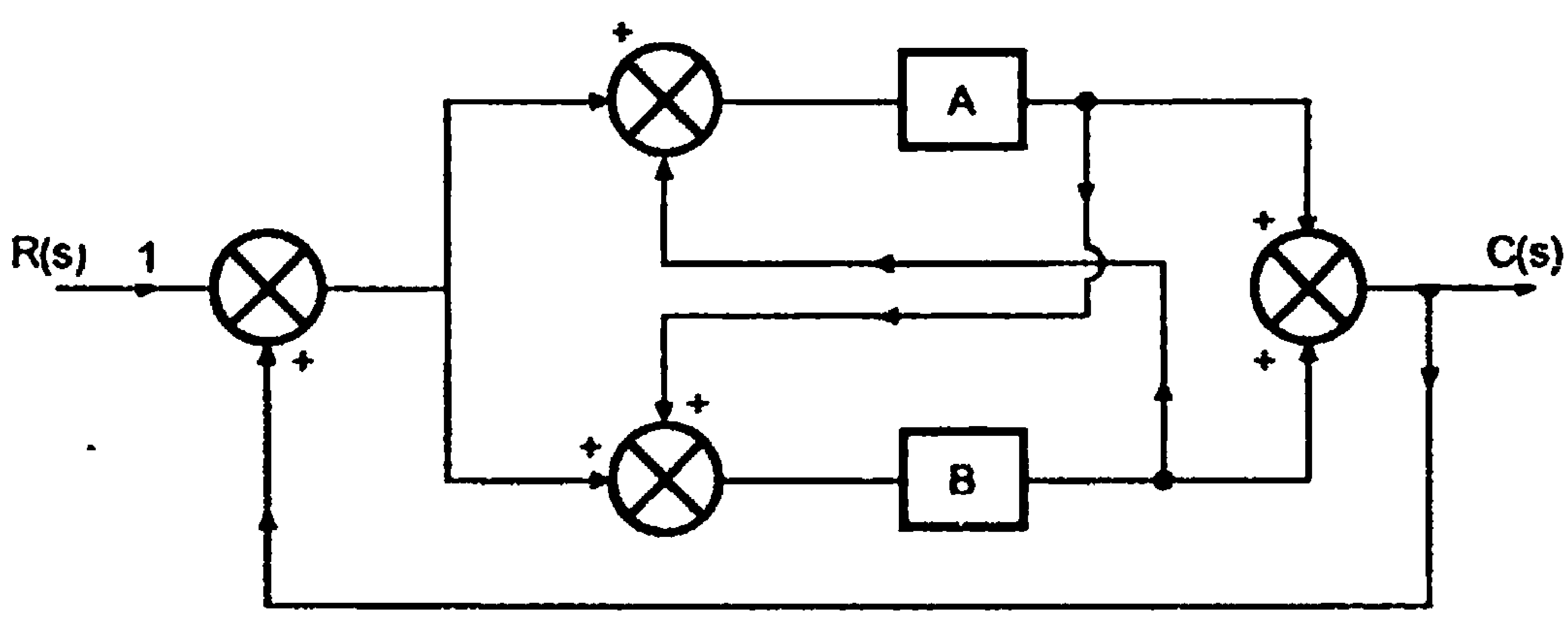
$$\left(\text{Ans. : } \frac{x_5}{x_1} = \frac{G_1 G_2 G_3 G_4 + G_1 G_4 G_6 + G_1 G_7 (1 - G_5)}{1 - G_1 H_1 + G_2 G_5 H_1 - G_5}\right)$$

18. *Using Mason's gain formula determine the system output for input R and disturbances* D_1 and D_2 *for the system described in the block diagram.*



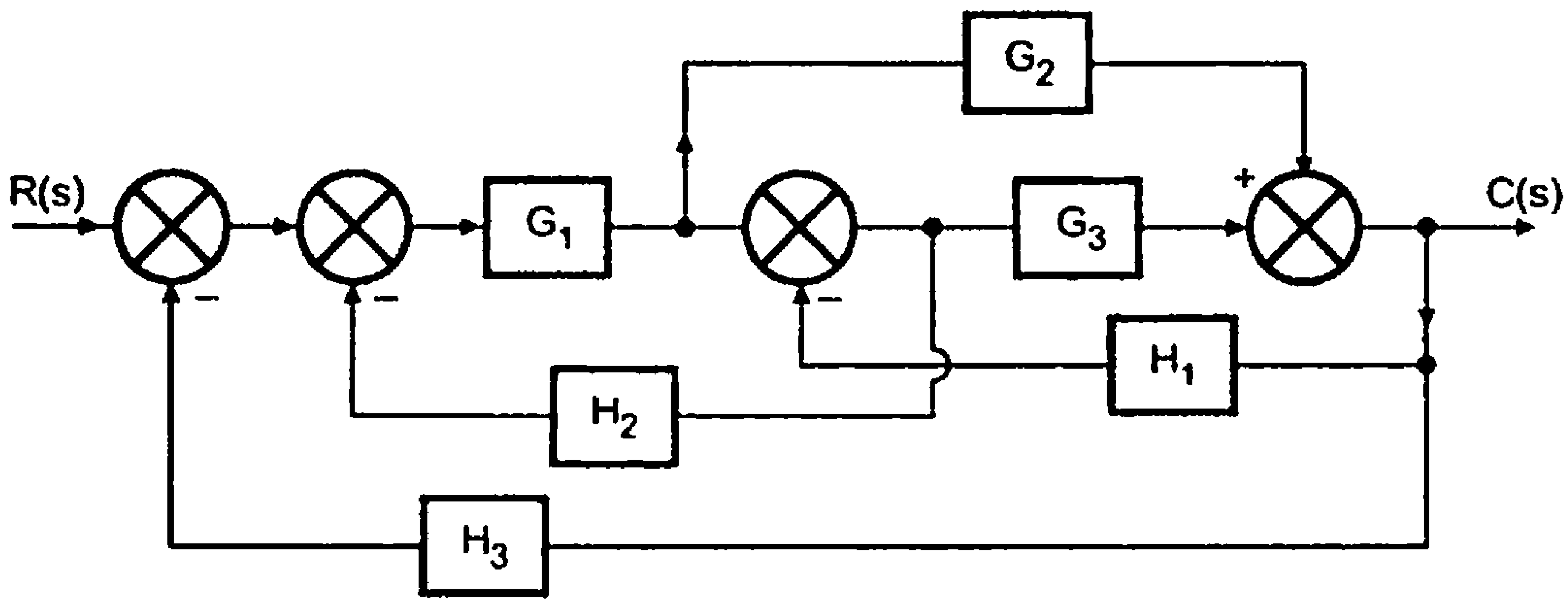
$$\left(\text{Ans. : } C(s) = \frac{s^3 [R(s) + (s+1)D_1(s) + D_2(s)s]}{[s^3(s+1)(s+2) - 20]}\right)$$

19. *Draw the signal flow graph and obtain the transfer function.*



$$\left(\text{Ans. : } \frac{A + B + 2AB}{1 - A - B - 3AB}\right)$$

20. *Draw the signal flow graph and obtain the transfer function.*



$$\left(\text{Ans. : } \frac{C(s)}{R(s)} = \frac{G_1(G_2 + G_3)}{1 + G_1 H_2 + G_3 H_1 + G_1 G_3 H_3 + G_1 G_2 H_3 - G_1 G_2 H_1 H_2}\right)$$

□□□

7 Time Response Analysis of Control Systems

7.1 Background

Most of the control systems, use time as its independent variable, so it is important to analyze the response given by the system for the applied excitation which is function of time. Analysis of response means to see the variation of output with respect to time. The evaluation of system is based on the analysis of such response. This output behaviour with respect to time should be within specified limits to have satisfactory performance of the system. The complete base of stability analysis lies in the time response analysis. The system stability, system accuracy and complete evaluation is always based on the time response analysis and corresponding results.

This chapter explains the concept of time response, steady state analysis, transient response and derives the various transient response specifications.

7.2 Definition and Classification of Time Response

Time response of a control system means, how output behaves with respect to time. So it can be defined as below.

Definition : Time Response : *The response given by the system which is function of the time, to the applied excitation is called time response of a control system.*

In any practical system, output of the system takes some finite time to reach to its final value. This time varies from system to system and is dependent on different factors. Similarly final value achieved by the output also depends on the different factors like friction, mass or inertia of moving elements, some nonlinearities present etc.

For example consider a simple ammeter as a system. It is connected in a system so as to measure current of magnitude 5 A. Ammeter pointer hence must deflect to show us 5 A reading on it. So 5 A is its ideal value that it must show. Now pointer will take some finite time to stabilise to indicate some reading and after stabilising also, it depends on various factors like friction, pointer inertia etc. whether it will show us accurate 5 A or not.

Based on this example, we can classify the total output response into two parts. First is the part of output during the time, it takes to reach to its final value. And second is the final value attained by the output which will be near to its desired value if system is stable and accurate.

This can be further explained by considering another practical example. Suppose we want to travel from city A to city B. So our final desired position is city B. But it will take some finite time to reach to city B. Now this time depends on whether we travel by a bus or a train or a plane. Similarly whether we will reach to city B or not depends on number of factors like vehicle condition, road condition, weather condition etc. So in short we can classify the output as,

i) Where to reach ?

ii) How to reach ?

Successfulness and accuracy of system depends on the final value reached by the system output which should be very close to what is desired from that system. While reaching to its final value, in the mean time, output should behave smoothly.

Thus final state achieved by the output is called **steady state** while output variations within the time it takes to achieve the steady state is called **transient response** of the system.

Definition : Transient Response

The output variation during the time, it takes to achieve its final value is called as ***transient response****. The time required to achieve the final value is called* ***transient period****.*

This can also be defined as that part of the time response which decays to zero after some time as system output reaches to its final value.

Key Point : *The transient response may be exponential or oscillatory in nature. Symbolically it is denoted as $c_t(t)$.*

To get the desired output, system must pass satisfactorily through transient period. Transient response must vanish after some time to get the final value closer to the desired value. Such systems in which transient response dies out after some time are called **Stable Systems.**

Mathematically for stable operating systems,

$$\lim_{t \to \infty} c_t(t) = 0$$

From transient response we can get following information about the system,

i) When the system has started showing its response to the applied excitation ?

ii) What is the rate of rise of output? From this, parameters of system can be designed which can withstand such rate of rise. It also gives indication about speed of the system.

iii) Whether output is increasing exponentially or it is oscillating.

iv) If output is oscillating, whether it is over shootting its final value.

v) When it is settling down to its final value ?

All this information matters much at the time of designing the systems.

Definition : Steady State Response

It is that part of the time response which remains after complete transient response vanishes from the system output.

This also can be defined as **response of the system as time approaches infinity from the time at which transient response completely dies out**. The steady state response is generally the final value achieved by the system output. Its significance is that it tells us how far away the actual output is from its desired value.

Key Point : *The steady state response indicates the accuracy of the system. The symbol for steady state output is C_{ss}.*

From steady state response we can get following information about the system :

i) How much away the system output is from its desired value which indicates error?

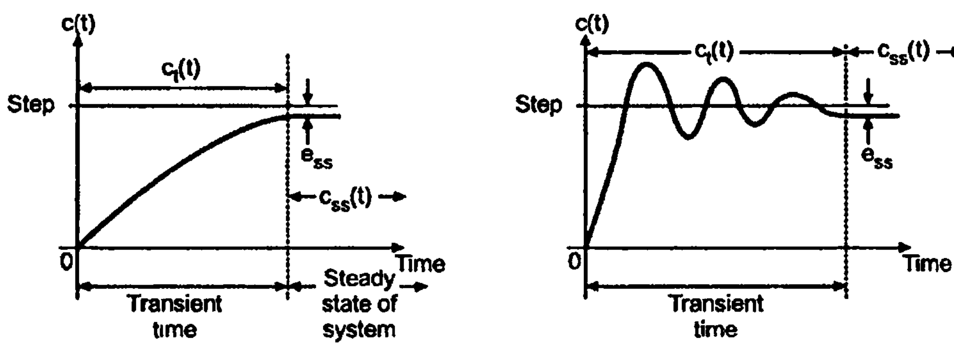
ii) Whether this error is constant or varying with time? So the entire information about system performance can be obtained from transient and steady state response.

Hence total time response c(t) we can write as,

$$c(t) = C_{ss} + c_t(t)$$

The difference between the desired output and the actual output of the system is called ***steady state error*** *which is denoted as e_{ss}.* This error indicates the accuracy and plays an important role in designing the system.

The above definitions can be shown in the waveform as in the Fig. 7.1 (a), (b) where input applied to the system is step type of input.



(a) $c_t(t)$ is exponential (b) $c_t(t)$ is oscillatory

Fig. 7.1

7.3 Standard Test Inputs

In practice, many signals are available which are the functions of time and can be used as reference inputs for the various control systems. These signals are step, ramp, sawtooth type, square wave, triangular etc. But while analysing the systems it is highly impossible to consider each one of it as an input and study the response. Hence from the analysis point of view, those signals which are most commonly used as reference inputs are defined as **Standard Test Inputs.** The evaluation of the system can be done on the basis of the response given by the system to the standard test inputs. Once system behaves satisfactorily to a test input, its time response to actual input is assumed to be upto the mark.

These standard test signals are,

i) Step Input (Position function) :

It is the sudden application of the input at a specified time as shown in the Fig. 7.2.

Mathematically it can be described as,

$$r(t) = A \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

If A = 1, then it is called **unit step function** and denoted by u(t).

Laplace transform of such input is $\frac{A}{s}$.

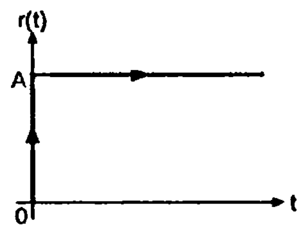


Fig. 7.2 Step

ii) Ramp Input (Velocity function) :

It is constant rate of change in input i.e. gradual application of input as shown in the Fig. 7.3.

Magnitude of Ramp input is nothing but its slope. Mathematically it is defined as,

$$r(t) = At \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

If A = 1, it is called **Unit Ramp input.** It is denoted as r(t). Its Laplace transform is $\frac{A}{s^2}$.

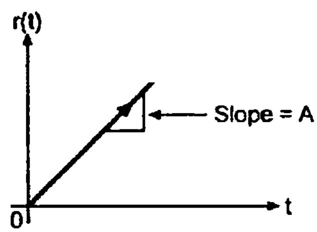


Fig. 7.3 Ramp

iii) Parabolic Input (Acceleration function) :

This is the input which is one degree faster than a ramp type of input as shown in the Fig. 7.4.

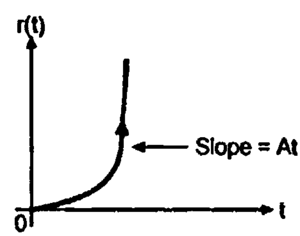


Fig. 7.4 Parabolic

Mathematically this function is described as,

$$\therefore \quad r(t) = \frac{A}{2}t^2, \quad \text{for } t \geq 0$$
$$= 0, \quad \text{for } t < 0$$

where A is called magnitude of the parabolic input.

Key Point : *Parabolic function is expressed as* $\frac{A}{2}t^2$ *so that in Laplace transforms of different standard inputs, similarity will get maintained.*

If A = 1, i.e. $r(t) = \frac{t^2}{2}$ it is called **unit parabolic input.** Its Laplace transform is $\frac{A}{s^3}$.

iv) Impulse Input :

It is the input applied instantaneously (for short duration of time) of very high amplitude as shown in the Fig. 7.5

It is the pulse whose magnitude is infinite while its width tends to zero i.e. $t \rightarrow 0$, applied momentarily .

Area of the impulse is nothing but its magnitude. If its area is unity it is called **Unit Impulse Input,** denoted as $\delta(t)$.

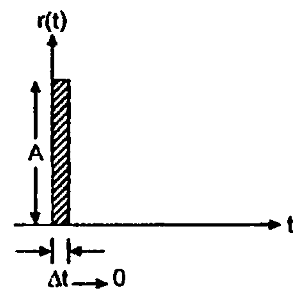


Fig. 7.5 Impulse

Mathematically it can be expressed as,

$$\therefore \quad r(t) = A, \quad \text{for } t = 0$$
$$= 0, \quad \text{for } t \neq 0$$

The Laplace transform of unit impulse input is always 1. (Refer Chapter-2). The unit impulse is denoted as $\delta(t)$.

r(t)	Symbol	R(s)
Unit step	u(t)	1/s
Unit ramp	r(t)	$1/s^2$
Unit parabolic	-	$1/s^3$
Unit impulse	δ(t)	1

Table 7.1

7.4 Steady State Analysis

As discussed earlier, steady state is that part of the output which remains after transients completely vanish from the output.

Mainly the steady state response has following two specifications,

i) How much time system takes to reach its steady state which is called **settling time** which is discussed later in connection with transient response. It is related to transient response also because same time will be required by the transients to die out completely from the system output.

ii) How far away actual output is reached from its desired value which is called **steady state error** (e_{ss}).

Out of the two specifications, the steady state error is the most important specification which is related only to the steady state. So let us see on which factors it depends, how to calculate it and how to reduce it.

Definition : Steady State Error : *It is the difference between the actual output and the desired output.*

Now reference input tells us the level of desired output and actual output is fed back through feedback element to compare it with the reference input. Hence to be precise it can be defined as the difference between reference input and the feedback signal (actual output).

Mathematically it is defined in Laplace domain as,

$L\{e(t)\} = E(s) = R(s) - C(s)H(s)$, for non unity feedback systems

and $L\{e(t)\} = E(s) = R(s) - C(s)$, for unity feedback systems.

7.5 Derivation of Steady State Error

Consider a simple closed loop system using negative feedback as shown in the Fig. 7.6

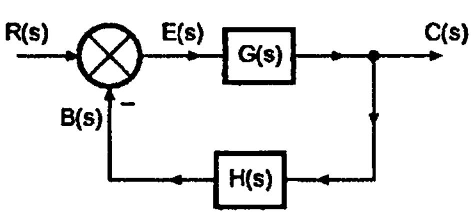


Fig. 7.6

where $E(s)$ = Error signal, and $B(s)$ = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s)H(s)$

$\therefore \quad E(s) = R(s) - C(s)H(s)$

and $\quad C(s) = E(s)G(s)$

$\therefore \quad E(s) = R(s) - E(s)\, G(s)H(s)$

$\therefore \quad E(s) + E(s)G(s)H(s) = R(s)$

$$\therefore \quad E(s) = \frac{R(s)}{1+G(s)H(s)} \quad \text{for nonunity feedback}$$

$$E(s) = \frac{R(s)}{1+G(s)} \quad \text{for unity feedback}$$

This E(s) is the error in Laplace domain and is expression in 's'. We want to calculate the error value. In time domain, corresponding error will be e(t). Now steady state of the system is that state which remains as $t \to \infty$.

$$\therefore \quad \text{Steady state error, } e_{ss} = \lim_{t \to \infty} e(t)$$

Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s) \qquad \text{where } F(s) = L\{ F(t) \}$$

Therefore, $$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \quad \text{where } E(s) \text{ is } L\{ e(t) \}.$$

Substituting E(s) from the expression derived, we can write

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

For negative feedback systems use positive sign in denominator while use negative sign in denominator if system uses positive feedback.

From the above expression it can be concluded that **steady state error depends on,**

i) R(s) i.e. reference input, its type and magnitude.

ii) G(s)H(s) i.e. open loop transfer function.

iii) Dominant nonlinearities present if any.

Now we will study the effect of change in input and product G(s)H(s) on the value of steady state error. As transfer function approach is applicable to only linear systems, the effect of nonlinearities is not discussed.

7.6 Effect of Input (Type and Magnitude) on Steady State Error (Static Error Coefficient Method)

Consider a system having open loop T.F. G(s)H(s) and excited by,

a) Reference input is step of magnitude A :

$$R(s) = \frac{A}{s}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{A}{1+G(s)H(s)}$$

$$\therefore \quad e_{ss} = \frac{A}{1+\lim\limits_{s \to 0} G(s)H(s)}$$

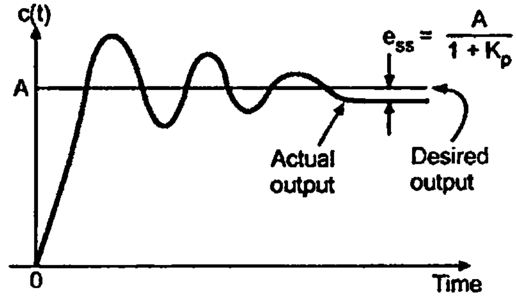


Fig. 7.7

For a system selected, $\lim\limits_{s \to 0} G(s)H(s)$ is constant and called **Positional Error Coefficient** of the system denoted as K_p.

$$K_p = \lim_{s \to 0} G(s)H(s) = \text{Positional error coefficient}$$

And corresponding error is,

$$e_{ss} = \frac{A}{1+K_p}$$

So whenever step input is selected as a reference input, positional error coefficient K_p will control the error in the system along with the magnitude of the input applied.

b) Reference input is ramp of magnitude 'A' :

$$R(s) = A/s^2$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^2}{1+G(s)H(s)}$$

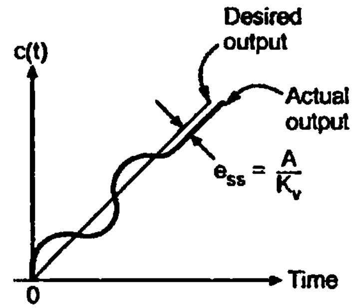


Fig. 7.8

$$= \lim_{s \to 0} \frac{A}{s[1 + G(s)H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s + s\,G(s)H(s)}$$

$$\therefore \quad e_{ss} = \frac{A}{\lim_{s \to 0} s\,G(s)H(s)}$$

For a selected system $\lim_{s \to 0} s\,G(s)H(s)$ is constant and called **Velocity Error Coefficient** as K_v.

$$\boxed{K_v = \lim_{s \to 0} s\,G(s)H(s) = \text{Velocity error coefficient}}$$

And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{K_v}}$$

So whenever ramp input is selected as a reference input, velocity error coefficient K_v will control the error in the system alongwith the magnitude of input applied.

c) Reference input is parabolic of magnitude 'A' :

i.e. $$R(t) = \frac{A}{2} t^2$$

$$\therefore \quad R(s) = \frac{A}{s^3}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^3}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{A}{s^2[1 + G(s)H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)H(s)}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{A}{s^2 G(s)H(s)}$$

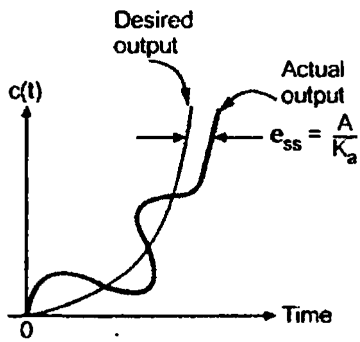


Fig. 7.9

So for a selected system $\lim_{s \to 0} s^2 G(s)H(s)$ is constant and called Acceleration Error **Coefficient** as K_a.

$$\therefore \quad \boxed{K_a = \lim_{s \to 0} s^2 G(s)H(s) = \text{Acceleration error coefficient}}$$

And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{K_a}}$$

So whenever parabolic input is selected as a reference input, acceleration error coefficient K_a will control the error in the system along with magnitude of input applied. So static error coefficients are given in Table 7.2.

Static Error Coefficient	Corresponding S.S. error e_{ss}
$K_p = \lim_{s \to 0} G(s)H(s)$	$\frac{A}{1+K_p}$
$K_v = \lim_{s \to 0} sG(s)H(s)$	$\frac{A}{K_v}$
$K_a = \lim_{s \to 0} s^2 G(s)H(s)$	$\frac{A}{K_a}$

Note : 'A' denotes the magnitude of the input applied

Table 7.2

7.7 Effect of Change in G(s)H(s) on Steady State Error (TYPE of a System)

This can be studied by focusing on to the dominant elements of G(s)H(s) from error point of view. Such elements are constant of system 'K' and poles of G(s)H(s) at origin if G(s)H(s) is expressed in a particular form called **time constant form.** This is as shown below,

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)....}{s^j (1+T_a s)(1+T_b s).......}$$

where K = Resultant system gain and j = TYPE of the system

'TYPE' of the system means number of poles at origin of open loop T.F. G(s)H(s) of the system.

So $j = 0$, TYPE zero system

$j = 1$, TYPE one system

$j = 2$, TYPE two system

:

:

$j = n$, TYPE 'n' system

> **Key Point :** *Thus 'TYPE' is the property of open loop T.F. G(s)H(s) while 'Order' is the property of closed loop T.F.* $\frac{G(s)}{1 \pm G(s)H(s)}$.

This is because, as defined earlier, order is the highest power of s present in the denominator polynomial of closed loop T.F. of the system.

7.8 Analysis of TYPE 0, 1 and 2 Systems

Note : A popular method to assess steady state performance of servomechanisms or unity feedback systems is to find their error co-efficients K_p, K_v and K_a.

where, K_p = Position error constant,

K_v = Velocity error constant and

K_a = Acceleration error constant.

Obviously in order to find these error constants the system must be stable, because for an unstable system there is no steady state and K_p, K_v and K_a are undefined.

Hence before we proceed to find K_p, K_v and K_a we must ensure (either by pole location or by Routh table of the closed loop system) that it is stable.

Thus the concept of K_p, K_v and K_a is applicable only if,

i) System is represented in its simple form.

ii) Only if the system is stable.

Consider the input selected as step of magnitude 'A'.

i) Let us assume that the system is of TYPE '0'.

i.e. $$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a s)(1+T_b s)\ldots\ldots}$$

For step input $$K_p = \lim_{s \to 0} G(s)H(s) = K$$ ... using above G(s)H(s)

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = \frac{A}{1+K}$$

i.e. TYPE '0' systems follow the step type of input with finite error $\frac{A}{1+K}$ which can be reduced by change in 'A' or 'K' or both as per requirement.

Now 'K' can be increased by introducing a variable gain amplifier in the forward path and error can be reduced. But there is limitation on the increase in value of 'K' form stability point of view which is discussed later. But increase in 'K' is one way to reduce the error. Corresponding response will be as shown in the Fig. 7.10 (a) or (b).

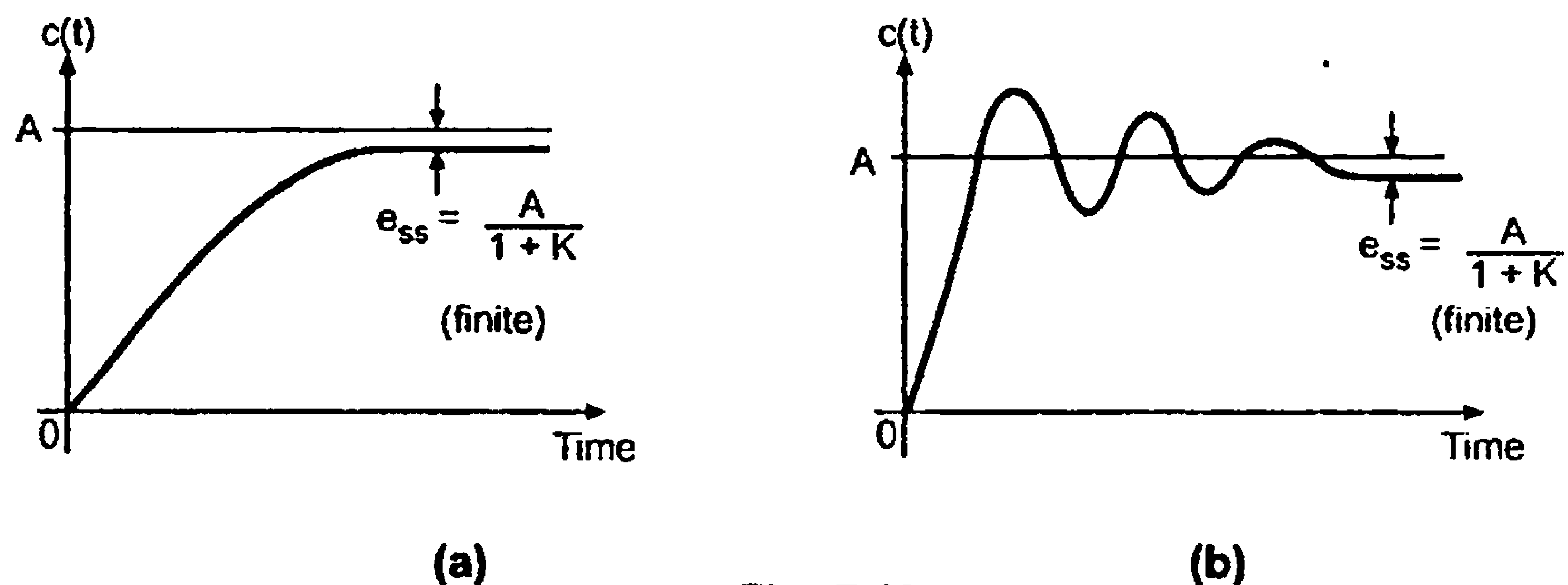


Fig. 7.10

ii) If for the same input now TYPE is increased to one' by adding pole at origin in G(s)H(s).

$$\text{TYPE 1 :} \quad G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s(1+T_a s)(1+T_b s)\ldots\ldots}$$

$$\text{As input is step,} \quad K_p = \lim_{s \to 0} G(s)H(s) = \infty$$

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

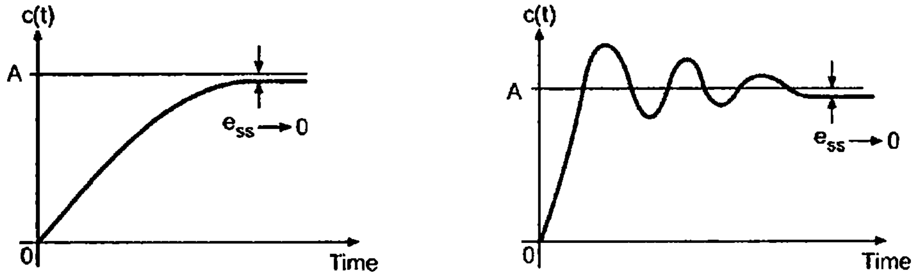
iii) Similarly if now TYPE is further increased to 'two' i.e. G(s)H(s) with 2 poles at origin,

$$\text{TYPE 2 :} \quad G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^2(1+T_a s)(1+T_b s)\ldots\ldots}$$

$$\text{As input is step,} \quad K_p = \lim_{s \to 0} G(s)H(s) = \infty$$

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

In general, for any TYPE of system more than zero, K_p will be infinite (∞) and error will be zero. Though mathematically answer for error is zero, practically small error will be present but it will be negligibly small. Such type of responses may take one of the forms shown in the Fig. 7.11 (a) and (b).



(a) (b)

Fig. 7.11

Thus TYPE 1 and above systems follow a step type of reference input of any magnitude, successfully, with negligibly small error.

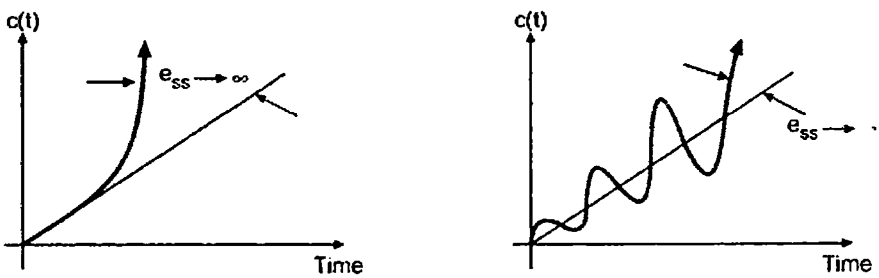
Let us now change the selected input from step to ramp of magnitude 'A' so K_v will control the error.

iv) Let the system be of TYPE 0.

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ ....}{(1+T_a s)(1+T_b s)\ ....}$$

$$\therefore \quad K_v = \lim_{s \to 0} s\,G(s)H(s) = 0 \qquad \therefore \quad e_{ss} = \frac{A}{K_v} = \frac{A}{0} = \infty$$

i.e. TYPE 0 systems will not follow ramp input of any magnitude and will give large error in the output which may damage the parameters of system or may cause the saturation in parameters. Hence ramp input should not be applied to TYPE '0' systems. The output may take the form as shown in the Fig. 7.12 (a) and (b).



(a) (b)

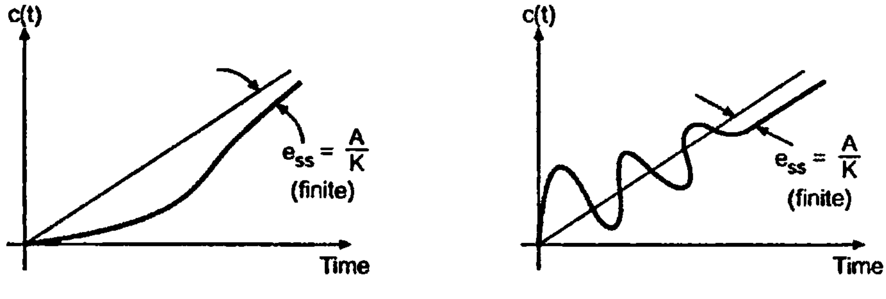
Fig. 7.12

v) If TYPE 1 System is subjected to Ramp input then,

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s(1+T_a s)(1+T_b s)\ldots\ldots}$$

$$\therefore \quad K_v = \lim_{s \to 0} s\,G(s)H(s) = K \qquad \therefore\ e_{ss} = \frac{A}{K_v} = \frac{A}{K} \text{ finite}$$

i.e. TYPE 1 systems follow the ramp type of input of magnitude `A' with finite error A/K which can be reduced as discussed earlier. The output may take the form as shown in the Fig. 7.13 (a) and (b).



(a) (b)

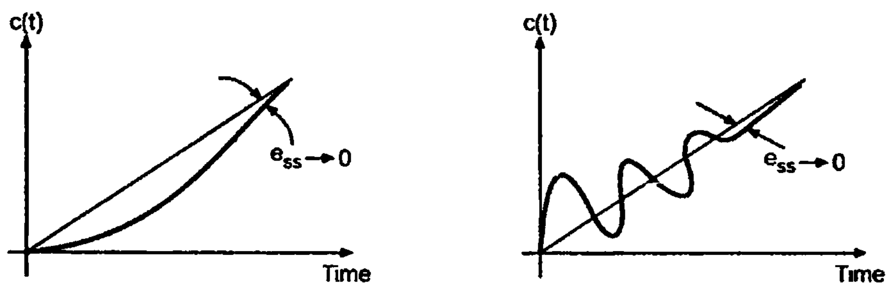
Fig. 7.13

vi) If TYPE 2 system is excited by Ramp input then,

$$\text{i.e.} \quad G(s)H(s) = \frac{K(1+T_1 s)\ (1+T_2 s)\ldots\ldots\ldots}{s^2(1+T_a s)\ (1+T_b s)\ldots\ldots\ldots}$$

$$\therefore \quad K_v = \lim_{s \to 0} s\,G(s)H(s) = \infty \qquad \therefore\ e_{ss} = \frac{A}{K_v} = \frac{A}{\infty} = 0$$

This is true for any system of TYPE more than one. Hence all systems of TYPE 2 and more than two follow ramp type of input with negligible small error and may take the form as shown in the Fig. 7.14 (a) and (b).



(a) (b)

Fig. 7.14

Let us now change the selected input from ramp to parabolic input of magnitude A hence coefficient K_a will control the error.

vii) Consider TYPE 0 system :

$$G(s)H(s) = \frac{K(1+T_1 s)\,(1+T_2 s)\,......}{(1+T_a s)\,(1+T_b s)\,......}$$

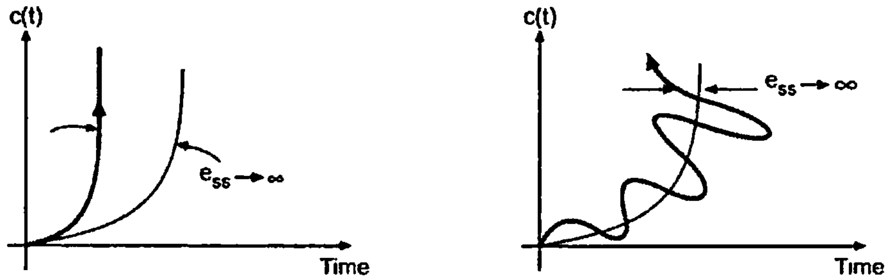
$$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = 0 \qquad \therefore\ e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$

viii) Consider TYPE 1 system :

$$G(s)H(s) = \frac{K(1+T_1 s)\,(1+T_2 s)\,.......}{s(1+T_a s)\,(1+T_b s)\,........}$$

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = 0 \qquad \therefore\ e_{ss} = \frac{A}{K_a} = \infty$$

For both TYPE '0' and '1' systems, error will be very large and uncontrollable if parabolic input is used. Hence parabolic input should not be used as a reference to excite TYPE '0' and TYPE '1' systems. The output may take the form as shown in the Fig. 7.15 (a) and (b) if excited by such input.



(a) (b)

Fig. 7.15

ix) If TYPE 2 system is used i.e.

$$G(s)H(s) = \frac{K(1+T_1 s)\,(1+T_2 s)\,.......}{s^2(1+T_a s)\,(1+T_b s)\,........}$$

Then $$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = K \qquad \therefore\ e_{ss} = \frac{A}{K_a} = \frac{A}{K} \text{ finite}$$

Hence TYPE 2 systems will follow Parabolic input with finite error A/K which can be controlled by change in A or K or both and output may take form as shown in the Fig.7.16 (a) and (b).

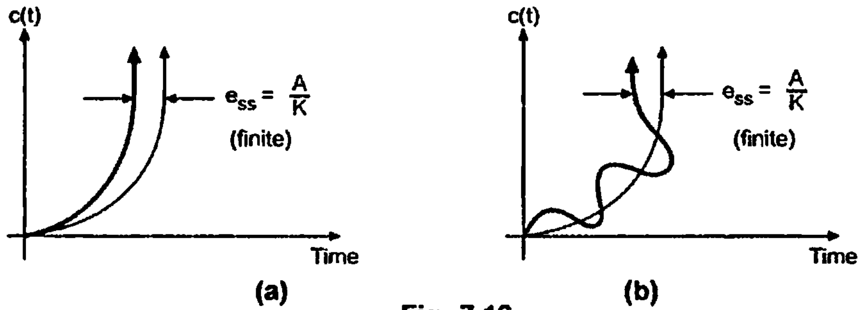


Fig. 7.16

And for any system of TYPE 3 or more if parabolic input is used, error will be negligibly small. The results can be summarised as shown in the Table 7.3.

Type of System	Error Coefficients			Error e_{ss} for		
	K_p	K_v	K_a	Step input	Ramp input	Parabolic input
0	K	0	0	$\frac{A}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{A}{K}$	∞
2	∞	∞	K	0	0	$\frac{A}{K}$

Table 7.3

7.9 Disadvantages of Static Error Coefficient Method

The disadvantages of Static Error Coefficient Method are :

1) Method cannot give the error if inputs are other than the three standard test inputs.
2) Most of the times, method gives mathematical answer of the error as '0' or 'infinite' and hence does not provide precise value of the error.
3) Method does not provide variation of error with respect to time, which will be otherwise very useful from design point of view.

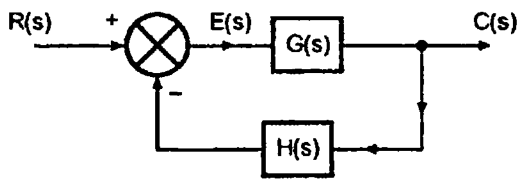


Fig. 7.17

4) Error constants are defined for the loop transfer function G(s)H(s), strictly, hence the method is applicable to only the system configuration shown in the Fig. 7.17.
5) As final value theorem is used to calculate steady state error so before applying it is necessary to check if sE(s) has any poles on the jω axis or in the right half of the s-plane. This means before applying this method, the system must be checked for its stability. The method can not be applied to unstable systems.
6) When the system configuration is different than as shown above then it is necessary to establish the expression for the error signal and apply the final value theorem directly, without the use of error coefficients.
7) The method is applicable only to stable systems.

Because of these disadvantages Dynamic error coefficient method (Error series method) is developed which eliminates above said disadvantages.

7.10 Generalised Error Coefficient Method (or Dynamic Error Coefficients)

As explained earlier,

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Let us assume that this is the product of two polynomials of 's'.

$$E(s) = F_1(s)\cdot F_2(s)$$

Where $F_1(s) = \frac{1}{1+G(s)H(s)}, \quad F_2(s) = R(s)$

Now if, $F(s) = F_1(s)\cdot F_2(s)$ then using convolution integral,

$$L^{-1}\{F(s)\} = F(t) = \int_0^t F_1(\tau)\, F_2(t-\tau)\, d\tau$$

Similarly $e(t) = \int_0^t F_1(\tau)\, F_2(t-\tau)\, d\tau = \int_0^t F_1(\tau)\, R(t-\tau)\, d\tau$

R (t – τ) can be expanded by using Taylor series form as,

$$R(t-\tau) = R(t) - \tau R'(t) + \frac{\tau^2}{2!} R''(t) - \frac{\tau^3}{3!} R'''(t) + \ldots$$

Substituting $e(t) = \int_0^t F_1(\tau)\left[R(t) - \tau R'(t) + \frac{\tau^2}{2!} R''(t) - \frac{\tau^3}{3!} R'''(t) + \ldots\right] d\tau$

$$= \int_0^t R(t) F_1(\tau) d\tau - \int_0^t \tau R'(t) F_1(\tau) d\tau + ...$$

Now $$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{t \to \infty} \left[\int_0^t R(t) F_1(\tau) d\tau - \int_0^t \tau R'(t) F_1(\tau) d\tau + ... \right]$$

$$= R(t) \int_0^\infty F_1(\tau) d\tau - R'(t) \int_0^\infty \tau F_1(\tau) d\tau + R''(t) \int_0^\infty \frac{\tau^2}{2!} F_1(\tau) d\tau ....$$

Where $$K_0 = \int_0^\infty F_1(\tau) d\tau, \quad K_1 = -\int_0^\infty \tau F_1(\tau) d\tau, \quad K_2 = \int_0^\infty \tau^2 F_1(\tau) d\tau .....$$

Substituting these values we have,

$$e_{ss} = K_0 R(t) + K_1 R'(t) + \frac{K_2}{2!} R''(t) + ...$$

where K_0 , K_1 , K_2 .... are called **dynamic error coefficients.**

To calculate these coefficients use the following method :

According to definition of Laplace transform,

$$F(s) = \int_0^\infty F_1(\tau) e^{-s\tau} d\tau$$

Now $$K_0 = \int_0^\infty F_1(\tau) d\tau$$

Multiplying by $e^{-s\tau}$ to both sides,

$$\therefore \quad K_0 e^{-s\tau} = \int_0^\infty F_1(\tau) d\tau \, e^{-s\tau} = F_1(s)$$

Taking limit as $s \to 0$ of both sides,

$$\lim_{s \to 0} K_0 e^{-s\tau} = \lim_{s \to 0} F_1(s)$$

$$\therefore \quad K_0 = \lim_{s \to 0} F_1(s) \qquad \text{where} \quad F_1(s) = \frac{1}{1 + G(s)H(s)}$$

Taking derivative of $K_0 e^{-s\tau}$ w.r.t. 's' we get,

$$-\tau K_0 e^{-s\tau} = \frac{d F_1(s)}{ds}$$

Substituting $K_0 = \int_0^{\infty} F_1(\tau)\, d\tau$

$$-\tau \int_0^{\infty} F_1(\tau)\, d\tau\; e^{-s\tau} = \frac{d F_1(s)}{ds}$$

i.e. $K_1 e^{-s\tau} = \frac{d F_1(s)}{ds}$

Taking limit as $s \to 0$ of both sides,

$$K_1 = \lim_{s \to 0} \frac{d F_1(s)}{ds}$$

In general, $$\boxed{K_n = \lim_{s \to 0} \frac{d^n F_1(s)}{ds}}$$

This method eliminates the disadvantages of static error coefficient method.

Advantages of this method is,

i) It gives variation of error as a function of time and

ii) For any input, other than standard input error can be determined.

Example 7.1 : *A unity feedback system has* $G(s) = \frac{40\,(s+2)}{s\,(s+1)\,(s+4)}$.

Determine (i) Type of the system, (ii) All error coefficients and (iii) Error for ramp input with magnitude 4.

Solution : To determine type of system arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{40(s+2)}{s(s+1)(s+4)} = \frac{40(2)(1+0.5s)}{s(1+s)(4)(1+0.25s)}$$

$$= \frac{20\,(1+0.5s)}{s(1+s)(1+0.25\,s)}$$

Comparing this with standard form,

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^j\,(1+T_a s)(1+T_b s)\ldots\ldots}$$

where j = Type of system

$\therefore$ j = 1 so given system is type 1 system.

$$\therefore \quad e_{ss}(t) = K_0 R(t) + K_1 R'(t) + \frac{K_2}{2!} R''(t)$$

$$R(t) = 6 + 5t + \frac{6t^2}{2}, \; R'(t) = 5 + 6t, \; R''(t) = 6$$

$$\therefore \quad e_{ss}(t) = 0.1428[6 + 5t + 3t^2] + 0.0857[5 + 6t] + \frac{7.346 \times 10^{-3}}{2!} \times 6$$

$$= 0.8568 + 0.714t + 0.4284\,t^2 + 0.4285 + 0.5142\,t + 0.02203$$

$$\therefore \quad e_{ss}(t) = 0.4284\,t^2 + 1.2282\,t + 1.3073$$

7.11 Transient Response Analysis

Transient response is the part of the total output which can not be separated out and hence to analyse it, it is necessary to calculate total output c(t) for the applied input. This contains steady state value along with transient response. From the expression of c(t) only, the transient response $c_t(t)$ can be analysed.

7.11.1 Method to Determine Total Output c(t)

1. Determine the closed loop transfer function of the system $\frac{C(s)}{R(s)}$.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

2. Find the expression for R(s) from the information of reference input r(t) to be applied to the system.
3. Substitute R(s) in the closed loop T.F. to obtain expression for C(s).
4. Take Laplace inverse of C(s) by using partial fraction method to obtain total c(t) of the system to the applied input r(t).

As $\quad c(t) = C_{ss} + c_t(t)$

The transient output can be studied from the above expression. Transient response may be exponential or oscillatory in nature.

Key Point : *Taking* $\lim_{t \to \infty}$ *of c(t), the final steady state value* C_{ss} *of the output also can be obtained.*

7.12 Analysis of First Order System

Order : Order of system is the highest power of 's' in the denominator of a *closed loop transfer function.*

Consider a simple system shown in the Fig. 7.18 (a).

Find $v_o(t)$ i.e. response if it is excited by unit step input.

$$v_i(t) = 1, \qquad t \geq 0$$

$$= 0, \qquad t < 0$$

$$\therefore \quad V_i(s) = 1/s$$

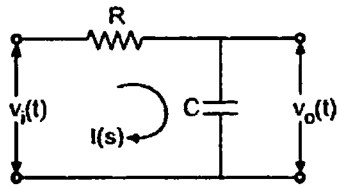


Fig. 7.18 (a)

Now first calculate system T.F. The Laplace network is shown in the Fig. 7.18 (b).

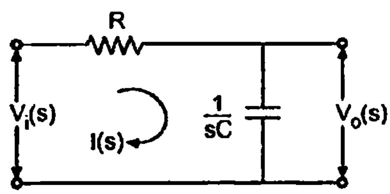


Fig. 7.18 (b)

$$V_i(s) = I(s)\,R + \frac{1}{sC} I(s) \qquad \text{... (1)}$$

$$V_o(s) = \frac{1}{sC} I(s) \qquad \text{... (2)}$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{1+sRC} = \frac{1}{1+Ts}} \qquad \text{where } T = RC$$

7.12.1 Unit Step Response of First Order System

Let input applied $v_i(t)$ is unit step voltage.

Substituting $V_i(s) = 1/s$ in the transfer function

$$V_o(s) = \frac{1}{s(1+sRC)} = \frac{A'}{s} + \frac{B'}{1+sRC} \qquad A' = 1 \text{ and } B' = -RC$$

$$\therefore \quad V_o(s) = \frac{1}{s} - \frac{RC}{1+sRC} = \frac{1}{s} - \frac{1}{s+(1/RC)}$$

Taking Laplace inverse,

$$v_o(t) = 1 - e^{-t/RC} \Rightarrow C_{ss} + c_t(t) \text{ form}$$

So $$\boxed{C_{ss} = 1 \text{ and } c_t(t) = e^{-t/RC}}$$

Now transient term is totally dependent on the values of R and C and its rate of exponential decay will get controlled by '−1/RC' which is pole of the system.

Key Point : *The values of R and C will not affect the steady state part.*

The response will be as shown in the Fig. 7.18 (c).

t	$v_o(t)$
0	0
RC	0.632
2RC	0.860
3RC	0.950
4RC	0.982
∞	1

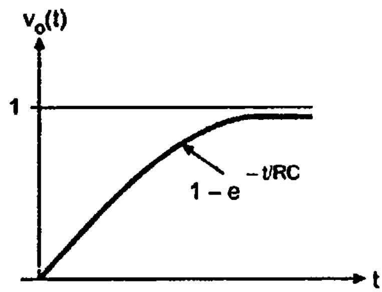


Fig. 7.18 (c) Unit step response of first order system

The response is purely exponential.

Now suppose input is changed to step of 'A' units.

Then $$V_i(s) = \frac{A}{s}$$

$$\therefore \quad V_o(s) = \frac{A}{s(1+sRC)} = \frac{A'}{s} + \frac{B'}{1+sRC}$$

$$\therefore \quad A = A'(1+sRC) + sB'$$

$$\therefore \quad A'RC + B' = 0 \text{ and } A' = A$$

$$\therefore \quad B' = -ARC$$

$$\therefore \quad V_o(s) = \frac{A}{s} - \frac{ARC}{1+sRC}$$

$$= \frac{A}{s} - \frac{A}{s+(1/RC)}$$

$$\therefore \quad v_o(t) = A\left[1 - e^{-t/RC}\right]$$

So rate of decay is not changed but the steady state value has changed. The corresponding response can be shown as in the Fig. 7.19.

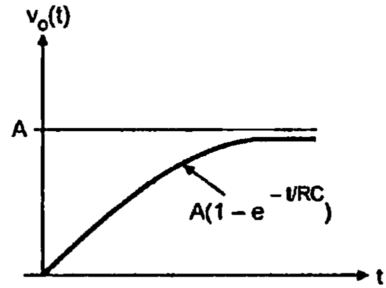


Fig. 7.19

But if values of R and C are changed i.e. location of pole $s = -1/RC$ is changed, the transient output will behave in a different way as rate of decay will get affected without change in its steady state.

The transient term is vanishing as it contains **exponential term of negative index** which is only because pole of the system is having real negative part.

Key Point : *In general transient response depends on,*

i) Poles of the closed loop T.F.

ii) Location of the poles of the closed loop T.F. and it is independent of the magnitude of the input applied. Any change in the magnitude of the selected input will not have any effect on the transient response of the system.

7.12.2 Closed Loop Poles of First Order System

The closed loop transfer function of a system is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

The equation which gives poles of system is called **characteristic equation** which is,

$$\boxed{1 + G(s)H(s) = 0}$$

For first order system this equation is also first order in general of the form.

$$\boxed{1 + Ts = 0}$$

As closed loop poles are the roots of characteristic equation, so for first order system there is only one closed loop pole i.e.

$$\boxed{s = -\frac{1}{T}}$$

The pole-zero plot is as shown in the Fig. 7.20.

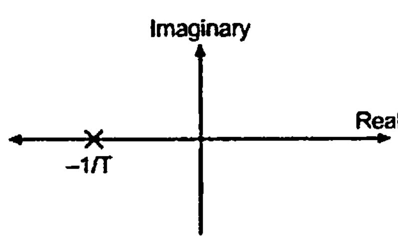


Fig. 7.20

Key Point : *The value of closed loop pole appears as an exponential index in the transient output.*

e.g. in the system considered above the closed loop pole is $s = -\frac{1}{RC}$ and in the output $c(t) = 1 - e^{-t/RC}$, the exponential index is $-\frac{1}{RC}$ which is nothing but the closed loop pole.

Consider two systems, one with closed loop pole at $s_1 = -2$ while other having closed loop pole at $s_2 = -4$. In the output of two systems we will get the terms e^{-2t} in first system and e^{-4t} in second system. Now as $t \to \infty$, e^{-4t} will vanish earlier than e^{-2t}. Hence second system transients will vanish quickly and will stabilise to its final value earlier than the first system.

From this we can conclude that locations of closed loop poles affect the transient behaviour.

> **Key Point :** *As closed loop pole moves in the left half away from the imaginary axis in the s-plane, transients die out more quickly making system more stable.*

Example 7.4 : *For a first order system, find out the output of the system when the input applied to the system is unit ramp input. Sketch the r(t) and c(t) and show the steady state error.*

Solution : Let the first order system has transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{(s+T)}$$

Input $\quad r(t) = \text{unit ramp input} = t$

$$\therefore \quad R(s) = \frac{1}{s^2}$$

$$\therefore \quad C(s) = \frac{1}{s^2(s+T)} = \frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+T}$$

$$\therefore \quad A(s+T) + Bs(s+T) + Cs^2 = 1$$

$$\therefore \quad (B+C)s^2 + (A+BT)s + AT = 1$$

$$\therefore \quad AT = 1 \qquad \therefore \quad A = \frac{1}{T}$$

$$A + BT = 0 \qquad \therefore \quad B = -\frac{1}{T^2}$$

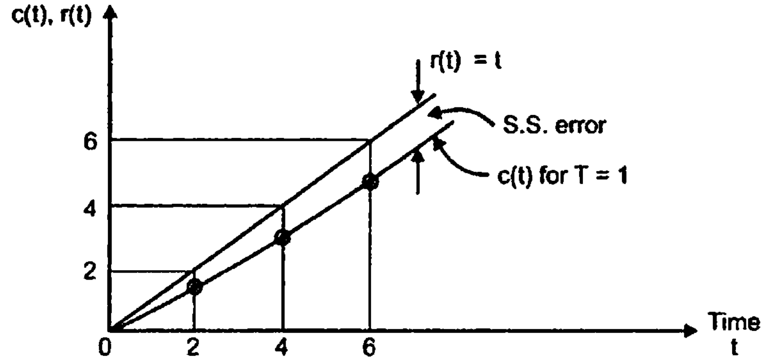
$$B + C = 0 \qquad \therefore \quad C = \frac{1}{T^2}$$

$$\therefore \quad C(s) = \frac{(1/T)}{s^2} - \frac{1/T^2}{s} + \frac{1/T^2}{s+T}$$

Taking Laplace inverse,

$$\therefore \quad c(t) = \left(\frac{1}{T}\right)t - \frac{1}{T^2} + \frac{1}{T^2}e^{-Tt}$$

The sketch of c(t) and r(t) is shown below. Take T = 1 for plotting c(t).



7.13 Analysis of Second Order System

Every practical system takes finite time to reach to its steady state and during this period, it oscillates or increases exponentially. The behaviour of system gets decided by type of closed loop poles and locations of closed loop poles in s-plane. The closed loop poles are dependent on selection of the parameters of the system. Every system has a tendency to oppose the oscillatory behaviour of the system which is called **damping.** Now this tendency controls the type of closed loop poles and hence the nature of the response.

This damping is measured by a factor or a ratio called **damping ratio** of the system. This factor explains us, how much dominant the opposition from the system is to the oscillations in the output. In some systems it will be low in which case system will oscillate but slowly i.e. with damped frequency. In some systems it may be so high that system output will not oscillate at all and not only that it will be exponential, so slow that it will take very long time to reach the steady state. This damping ratio, explaining such behaviour is denoted by a greek symbol (Zeta) ξ. Now as this measures the opposition by the system to the oscillatory behaviour, if it is made zero, ($\xi = 0$) system will oscillate with maximum frequency. As there is no opposition from system, system naturally and freely oscillates under such condition. Hence this frequency of oscillations under $\xi = 0$ condition is called **natural frequency of oscillations** of the system and denoted by the symbol ω_n rad / sec .

For a second order system the denominator of closed loop T.F. is quadratic and the coefficients of this equation are directly related to ξ and ω_n as explained below.

The C.L.T.F. (closed loop transfer function) for a standard second order system takes the form as,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\, s + \omega_n^2} \quad \text{... standard } 2^{nd} \text{ order system}$$

Where characteristic equation is, $s^2 + 2\,\xi\,\omega_n\, s + \omega_n^2 = 0$

The standard second order system is that where in C.L.T.F. numerator is ω_n^2.

Key Point: *In practice it is not necessary that numerator must be always ω_n^2. It may be other constant or polynomial of 's' but denominator middle term coefficient and last term coefficient always reflect '$2\,\xi\,\omega_n$' and 'ω_n^2' of the system respectively.*

Hence always denominator of a T.F. must be compared with the standard form $s^2 + 2\xi\,\omega_n\, s + \omega_n^2 = 0$ to decide the values of ξ and ω_n of the system. The numerator should not be used for comparison to obtain the values of ξ and ω_n.

e.g. :

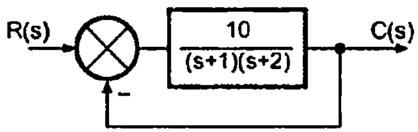


Fig. 7.21

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{10}{(s+1)(s+2)}}{1 + \frac{10}{(s+1)(s+2)}} = \frac{10}{s^2 + 3s + 12}$$

This C.L.T.F. is not standard as numerator term is not ω_n^2 but denominator always reflects ξ and ω_n. The values can be decided by comparing the denominator with the standard characteristic equation $s^2 + 2\xi\,\omega_n\, s + \omega_n^2 = 0$.

$\therefore \quad \omega_n^2 = 12 \qquad$ i.e. $\omega_n = \sqrt{12}$ rad / sec

While $\quad 2\xi\,\omega_n = 3 \qquad \therefore\ \xi = \frac{3}{2\sqrt{12}} = 0.433$

7.14 Effect of ξ on Second Order System Performance

Consider input applied to the standard second order system is unit step.

$\therefore \quad R(s) = 1/s$

While $\quad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\, s + \omega_n^2}$

$$\therefore \qquad C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Finding the roots of the equation $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

i.e. $\dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$

i.e. $$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

We can write, $$C(s) = \frac{\omega_n^2}{s\left(s + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)\left(s + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)}$$

Now nature of these roots is dependent on damping ratio ξ. Consider the following cases,

Case 1 : $1 < \xi < \infty$

The roots are,

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

i.e. **real, unequal and negative,** say $-K_1$ and $-K_2$

$$\therefore \qquad C(s) = \frac{\omega_n^2}{s(s+K_1)(s+K_2)} = \frac{A}{s} + \frac{B}{s+K_1} + \frac{C}{s+K_2}$$

Taking Laplace inverse, c(t) will take the following form,

$$c(t) = C_{ss} + Be^{-K_1 t} + Ce^{-K_2 t},$$

where C_{ss} = Steady state output = A

The output is purely exponential. This means damping is so high that there are no oscillations in the output and is purely exponential. Hence such systems are called **'Overdamped'**.

Hence nature of response will be as shown in the Fig. 7.22.

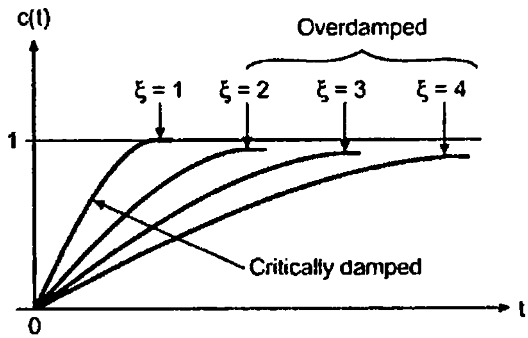


Fig. 7.22 $\xi \geq 1$

The partial fraction can be calculated for the Laplace inverse as below,

$$C(s) = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\therefore \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{a_1(s^2 + 2\xi\omega_n s + \omega_n^2) + s(a_2 s + a_3)}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

equating numerators on both sides,

$$\omega_n^2 = s^2(a_1 + a_2) + s(a_1 2\xi\omega_n + a_3) + a_1\omega_n^2$$

$\therefore \quad a_1\omega_n^2 = \omega_n^2$ equating constant

$a_1 + a_2 = 0$ equating coefficients of s^2

$a_1 2\xi\omega_n + a_3 = 0$ equating coefficients of s

$\therefore \; a_1 = 1, \quad a_2 = -1, \quad a_3 = -2\xi\omega_n$

As $\xi\omega_n = \alpha$ assumed earlier for ease of computations.

$\therefore \quad a_1 = 1, \quad a_2 = -1, \quad a_3 = -2\alpha$

$$\therefore \quad C(s) = \frac{1}{s} + \frac{-s - 2\alpha}{s^2 + 2\alpha s + \omega_n^2}$$

$$\therefore \quad C(s) = \frac{1}{s} - \left\{\frac{s + 2\alpha}{s^2 + 2\alpha s + \omega_n^2}\right\}$$ .... (Taking negative sign outside.)

Now consider $s^2 + 2\alpha s + \omega_n^2$,

For completing square,

$$\text{Last term} = \frac{(\text{middle term})^2}{4 \times \text{first term}} = \frac{4\alpha^2}{4 \times 1} = \alpha^2$$

So adjusting denominator as $s^2 + 2\alpha s + \alpha^2 + \omega_n^2 - \alpha^2$

i.e. denominator $= (s+\alpha)^2 + \omega_n^2 - \alpha^2$

but $\alpha = \xi\omega_n \quad \therefore \quad \alpha^2 = \xi^2\omega_n^2$

Substituting in above we get, $(s+\alpha)^2 + \omega_n^2 - \xi^2\omega_n^2$

i.e. denominator $= (s+\alpha)^2 + \omega_n^2(1 - \xi^2)$

Now $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\therefore \quad \omega_d^2 = \omega_n^2(1-\xi^2)$

Substituting this in the expression of C(s) we get,

$$\therefore \quad C(s) = \frac{1}{s} - \left\{\frac{s + 2\alpha}{(s+\alpha)^2 + \omega_d^2}\right\}$$

$$\theta = \tan^{-1}\left\{\frac{\sqrt{1-\xi^2}}{\xi}\right\} \text{ radians}$$

Key Point : *Substituting $\xi = 0$ in this expression, we can find out expression for the output for undamped systems.*

Important Remarks

1) The result derived is applicable for **standard** second order systems which is **underdamped** and excited by **unit step input**.

2) If in the problem, the input is given as step of A units rather than unit step then each term of the final expression of c(t) gets multiplied by A as,

$$c(t) = A\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)\right] \quad \text{... For step of A units}$$

3) If the system is not in the standard form i.e. numerator of closed loop transfer function is not ω_n^2 but some other constant as,

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+2\xi\omega_n s+\omega_n^2} \qquad K = \text{constant}$$

Then the above result can be applied by expressing C(s)/R(s) as,

$$\frac{C(s)}{R(s)} = \frac{K}{\omega_n^2}\left[\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right]$$

And in such case, the standard expression of c(t) gets multiplied by the constant K/ω_n^2.

$$\therefore \quad c(t) = \frac{K}{\omega_n^2}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)\right]$$

The values of ξ and ω_n must be obtained by comparing denominator of C(s)/R(s) with the standard form.

4) If numerator of C(s) / R(s) has some polynomial in s as,

$$\frac{C(s)}{R(s)} = \frac{P(s)}{s^2+2\xi\omega_n s+\omega_n^2} \text{ i.e., } \frac{C(s)}{R(s)} = \frac{2s+4}{s^2+10s+64}$$

Then the above result **cannot be** applied to get c(t). In such case, though ξ and ω_n values are to be obtained by comparing denominator with the standard form, the expression for c(t) must be obtained by actually calculating partial fraction coefficients, after substituting the proper value of R(s).

5) For any other input, other than step the derivation is not applicable but the steps of the deriv.:tion can be used as a guide line to obtain expression for c(t) in any other condition.

7.16 Transient Response Specifications

The actual output behaviour according to the expression derived can be shown as in the Fig. 7.26.

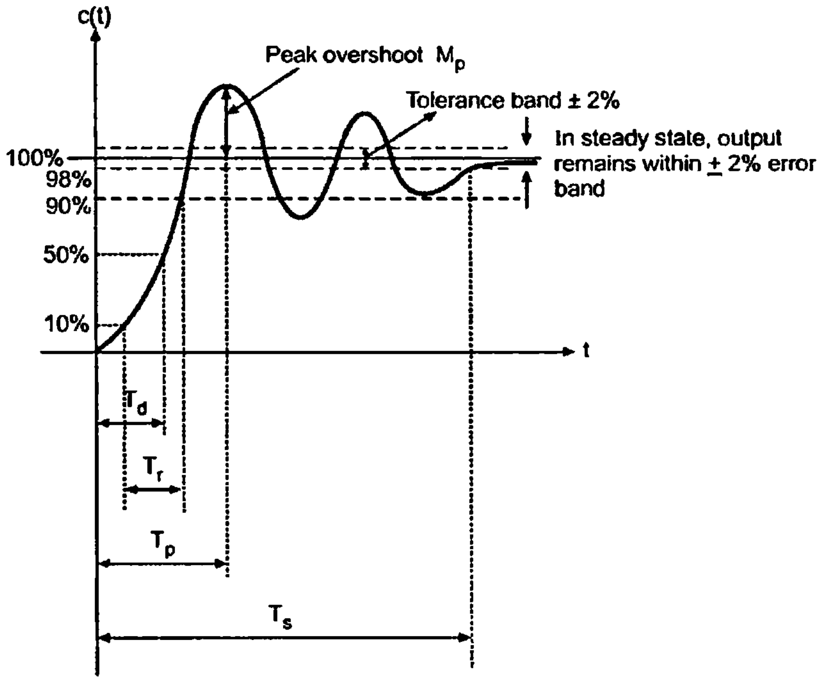


Fig. 7.26 Transient response specifications

Let us define the various time response specifications referring to the Fig. 7.26.

1) ***Delay Time T_d : It is the time required for the response to reach 50% of the final value in the first attempt. It is given by,***

$$T_d = \frac{1 + 0.7\,\xi}{\omega_n}$$

2) ***Rise Time T_r : It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems. The rise time is reci -ocal of the slope of the response at the instant, the response is equal to 50% of the final value. It is given by,***

$$T_r = \frac{\pi - \theta}{\omega_d} \text{ sec where } \theta \text{ must be in radians.}$$

3) ***Peak Time T_p : It is the time required for the response to reach its peak value. It is also defined as the time at which response undergoes the first overshoot which is always peak overshoot.***

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} \text{ sec}$$

4) ***Peak Overshoot M_p : It is the largest error between reference input and output during the transient period.***

It also can be defined as the amount by which output overshoots its reference steady state value during the first overshoot.

$$M_p = \left\{ c(t)\,\Big|_{t = T_p} \right\} - 1 \quad \text{1 is for unit step input}$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

5) ***Settling Time T_s : This is defined as the time required for the response to decrease and stay within specified percentage of its final value (within tolerance band).***

$$\text{Time constant of system} = \frac{1}{\xi \omega_n} = T$$

$$T_s = 4 \times \text{Time constant}$$

Practically the setting time is assumed to be 4 times, the time constant of the system.

$$T_s = \frac{4}{\xi \omega_n} \quad \text{..... for a tolerance band of } \pm 2\% \text{ of steady state}$$

Key Point : *1 Time constant 'T' is the time required by the system output to reach 63.2 % of its final value during the first attempt.*

7.17 Derivations of Time Domain Specifications

7.17.1 Derivation of Peak Time T_p

Transient response of second order underdamped system is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Where $\theta = \tan^{-1} \dfrac{\sqrt{1-\xi^2}}{\xi}$

As at $t = T_p$, c(t) will achieve its maxima. According to maxima theorem,

$$\left.\frac{dc(t)}{dt}\right|_{t=T_p} = 0$$

So differentiating c(t) w.r.t. 't' we can write,

i.e. $$-\frac{e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d\cos(\omega_d t+\theta) = 0$$

Substituting $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_n\sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

$$\therefore \quad \xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

$$\therefore \quad \tan(\omega_d t+\theta) = \frac{\sqrt{1-\xi^2}}{\xi}$$

Now $\theta = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$

$$\therefore \quad \frac{\sqrt{1-\xi^2}}{\xi} = \tan\theta$$

$$\therefore \quad \tan(\omega_d t+\theta) = \tan\theta$$

From trignometric formula,

$$\tan(n\pi+\theta) = \tan\theta$$

$\therefore \quad \omega_d t = n\pi$ where n = 1, 2, 3

But T_p, time required for first peak overshoot. $\therefore$ n = 1

$$\omega_d T_p = \pi$$

$$\therefore \quad \boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \text{ sec}}$$

7.17.2 Derivation of M_p

From the Fig. 7.27, $M_p = C(T_p) - 1$

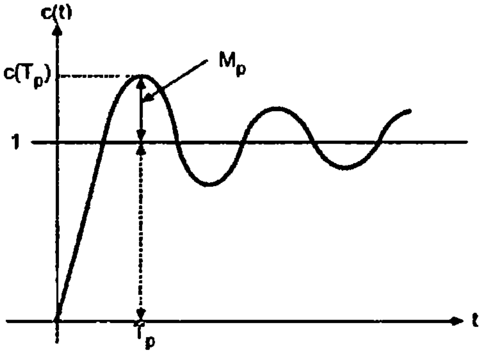


Fig. 7.27

$$M_p = \left\{1 - \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \theta)\right\} - 1$$

$$\therefore \quad M_p = -\frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \theta)$$

But $T_p = \dfrac{\pi}{\omega_d}$, substituting above we get,

$$\therefore \quad M_p = \frac{-e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

Now, $\sin(\pi+\theta) = -\sin(\theta)$

$$\therefore \quad M_p = \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin\theta$$

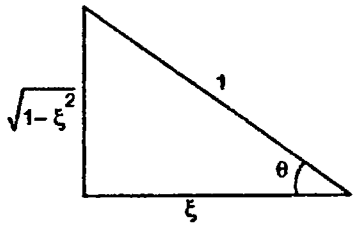


Fig. 7.28

We know, $\tan\theta = \dfrac{\sqrt{1-\xi^2}}{\xi}$ as shown in the Fig. 7.28.

$$\therefore \quad \sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \text{ but hypotenuse} = \sqrt{(\sqrt{1-\xi^2})^2 + \xi^2} = 1$$

$$\therefore \quad \sin\theta = \sqrt{1-\xi^2}$$

$$\therefore \quad M_p = \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \cdot \frac{\sqrt{1-\xi^2}}{1}$$

$$= e^{-\xi\omega_n T_p}$$

Substitute $T_p = \frac{\pi}{\omega_d}$

$$\therefore \quad M_p = e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

$$= e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\therefore \quad \boxed{\% M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}$$

7.17.3 Derivation of T_r

Time required by output to achieve 100% of its final value, starting from zero during the first attempt is the rise time.

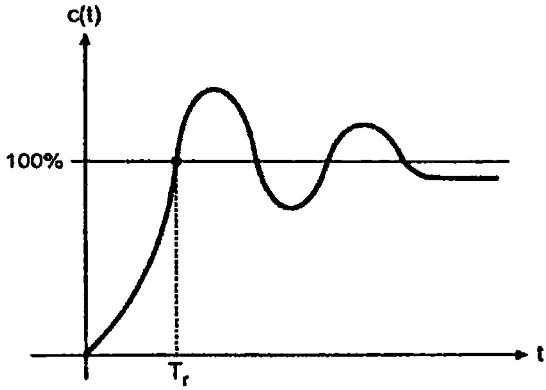


Fig. 7.29

i.e. $\{c(t)|_{t=T_r}\} = 1$ for unit step input

$$\therefore \quad 1 = 1 - \frac{e^{-\xi\omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta)$$

$$\therefore \quad -\frac{e^{-\xi\omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 0$$

Equation will get satisfied only if,

$$\sin(\omega_d T_r + \theta) = 0$$

Trigonometrically this is true only if,

$$\omega_d T_r + \theta = n\pi \quad \text{where } n = 1, 2 \ldots.$$

defining the term sensitivity for the system. Let us study the effect of parameter variations in an open loop control system.

7.18.1 Effect of Parameter Variations in an Open Loop Control System

Consider an open loop control system shown in the Fig. 7.30. The overall transfer function of the system is given by,

$$G(s) = \frac{C(s)}{R(s)}$$

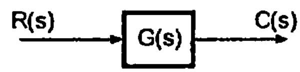


Fig. 7.30

$$\therefore \quad C(s) = G(s) \cdot R(s)$$

Let Δ G(s) be the change in G(s) due to the parameter variations. The corresponding change in the output be Δ C(s).

$$\therefore \quad C(s) + \Delta C(s) = [G(s) + \Delta G(s)] \; R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = G(s) \cdot R(s) + \Delta G(s) \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = C(s) + \Delta G(s) \cdot R(s)$$

$$\therefore \quad \boxed{\Delta C(s) = \Delta G(s) \cdot R(s)} \qquad \ldots (1)$$

The equation (1), gives the effect of change in transfer function, due to the parameter variations, on the system output, in an open loop control system. Let us discuss now, the effect of such parameter variations in a closed loop control system.

7.18.2 Effect of Parameter Variations in a Closed Loop System

Consider a closed loop system as shown in Fig.7.31. The signal E(s) is the Laplace transform of the error signal e(t). The overall transfer function of the system is given by,

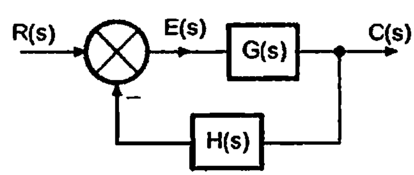


Fig. 7.31

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)}$$

Let Δ G(s) be the change in G(s) which is due to the parameter variations in the system. The corresponding change in the output be Δ C(s).

$$C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + [G(s) + \Delta G(s)] H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + G(s)H(s) + \Delta G(s)H(s)} \cdot R(s)$$

The term Δ G(s)H(s) is negligibly small as compared to G(s)H(s), as the change Δ G(s) is very small compared to G(s). Neglecting the term Δ G(s)H(s) from the denominator, we get,

$$C(s)+\Delta C(s) = \frac{G(s)+\Delta G(s)}{1+G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s)+\Delta C(s) = \frac{G(s)}{1+G(s)H(s)} \cdot R(s)+\frac{\Delta G(s)}{1+G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s)+\Delta C(s) = C(s)+\frac{\Delta G(s)}{1+G(s)H(s)} \cdot R(s)$$

$$\therefore \quad \boxed{\Delta C(s) = \frac{\Delta G(s)}{1+G(s)H(s)} \cdot R(s)} \qquad \text{... (2)}$$

The equation (2), gives the change in the output due to the parameter variations in G(s), in a closed loop system.

In practice, the magnitude of 1 + G(s)H(s) is very much greater than unity.

$$|G(s)H(s)| >> 1$$

Hence it can be observed from the equations (1) and (2), that in a closed loop system, due to the feedback, the change in the output, due to the parameter variations in G(s), is reduced by the factor $[1+G(s)H(s)]$. In an open loop system, such a reduction does not exist, due to the absence of the feedback.

7.18.3 Sensitivity of a Control System

The parameters of any control system cannot be constant throughout its entire life. There are always changes in the parameters due to environmetal changes and other disturbances. These changes are called parameter variations. Such parameter variations affect the system performance adversely. Such an effect, in the system performance due to parameter variations can be studied mathematically defining the term sensitivity of a control system. The change in a particular variable due to the parameter variations can be expressed in terms of sensitivity as below :

Let the variable in a system which is varying be 'T', due to the variations in the parameter 'K' of the system. The sensitivity of the system parameter T to the parameter K is expressed as,

$$\boxed{S = \frac{\%\text{ change in T}}{\%\text{ change in K}}} \qquad \text{... (3)}$$

Mathematically, it can be expressed as

$$S_K^T = \frac{d\, ln(T)}{d\, ln(K)}$$

$$S_K^T = \frac{\left(\frac{1}{T}\right)}{\left(\frac{1}{K}\right)} \cdot \frac{\partial T}{\partial K}$$

$$\therefore \quad \boxed{S_K^T = \frac{(\partial T / T)}{(\partial K / K)}} \qquad \dots (4)$$

The symbolic representation S_K^T represents the sensitivity of a variable T with respect to the variations in the parameter K. In practice, the variable T may be an output variable while the parameter K may be the gain, the feedback factor etc. The representation S_K^T is also called the sensitivity function of a system. For a good control system, the value of such a sensitivity function should be as minimum as possible.

Let T(s) be the overall transfer function of a control system. The forward path transfer function G(s) is varying. Then the sensitivity of overall transfer function with respect to the variations in G(s) is defined as,

$$S_G^T = \frac{\partial T(s) / T(s)}{\partial G(s) / G(s)}$$

$$\therefore \quad S_G^T = \frac{G(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial G(s)}$$

For the open loop system,

$$T(s) = G(s)$$

$$\therefore \quad S_G^T = \frac{G(s)}{G(s)} \cdot \frac{\partial G(s)}{\partial G(s)}$$

$$= 1 \qquad \dots (5)$$

Thus the sensitivity of T(s) with respect to G(s) for an open loop system is unity.

Let us find out the sensitivity function for a closed loop system. For a closed loop system,

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\therefore \quad \frac{\partial T(s)}{\partial G(s)} = \frac{[1+G(s)H(s)] \cdot [1] - [G(s)][H(s)]}{[1+G(s)H(s)]^2} = \frac{1}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_G^T = \frac{G(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial G(s)}$$

$$= \frac{G(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} \cdot \frac{1}{[1+G(s)H(s)]^2}$$

$$\therefore \quad \boxed{S_G^T = \frac{1}{1+G(s)H(s)}} \quad \text{... (6)}$$

Comparing the two equations (5) and (6), it can be observed that due to the feedback the sensitivity function gets reduced by the factor $1/[1+G(s)H(s)]$ compared to an open loop system. And less the value of sensitivity function, less sensitive is the system to the variations in the forward path transfer function G(s).

Sensitivity of T(s) with respect to H(s)

Let us calculate the sensitivity function which indicates the sensitivity of the overall transfer function T(s) with respect to the feedback path transfer function H(s). Such a function can be expressed as,

$$S_H^T = \frac{H(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial H(s)}$$

For a closed loop system,

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$\therefore \quad \frac{\partial T(s)}{\partial H(s)} = \frac{[1+G(s)H(s)][0]-[G(s)][G(s)]}{[1+G(s)H(s)]^2} = \frac{-[G(s)]^2}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_H^T = \frac{H(s)}{T(s)} \cdot \frac{-[G(s)]^2}{[1+G(s)H(s)]^2}$$

$$= \frac{H(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} \cdot \frac{-[G(s)]^2}{[1+G(s)H(s)]^2} = \frac{-G(s)H(s)}{1+G(s)H(s)} \quad \text{... (7)}$$

It can be observed from the equations (6) and (7) that the closed loop system is more sensitive to variations in the feedback path parameters than the variations in the forward path parameters. Thus, the specifications of the feedback elements must be observed strictly as compared to the specifications of the forward path elements.

7.18.4 Effect of Feedback on Time Constant of a Control System

Consider an open loop system with overall transfer function as,

$$G(s) = \frac{K}{1+sT}$$

When this system is subjected to unit step input, its response can be obtained as,

$$\frac{C(s)}{R(s)} = \frac{K}{1+sT} \qquad \text{..... as open loop system}$$

$$R(s) = \frac{1}{s} \quad \text{unit step input}$$

$$\therefore \quad C(s) = \frac{K}{s(1+sT)}$$

$$\therefore \quad c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{K}{s} - \frac{K}{\left(s+\frac{1}{T}\right)}\right] = K\left[1-e^{\,t/T}\right]$$

So T is the time constant of the open loop system.

Now the feedback is introduced in the system with feedback transfer function as H(s) = h. This is shown in the Fig. 7.32.

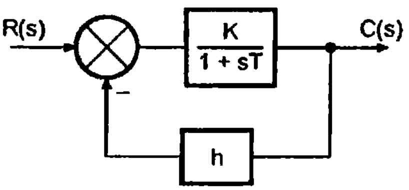


Fig. 7.32

Let us obtain the response of this closed loop system for unit step input. The overall transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{1+sT}}{1+\frac{K}{1+sT}\cdot h}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{K}{1+sT+Kh} = \frac{K}{1+Kh+sT} = \frac{K/T}{\left[s+\left(\frac{1+Kh}{T}\right)\right]}$$

$$R(s) = \frac{1}{s}$$

$$\therefore \quad C(s) = \frac{K/T}{s\left[s+\left(\frac{1+Kh}{T}\right)\right]}$$

$$\therefore \quad c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{K/T}{s\left[s+\left(\frac{1+Kh}{T}\right)\right]}\right] = L^{-1}\left[\frac{(K/1+Kh)}{s} - \frac{(K/1+Kh)}{s+\left(\frac{1+Kh}{T}\right)}\right]$$

$$= \frac{K}{1+Kh} - \frac{K}{1+Kh}\cdot e^{-\frac{t(1+Kh)}{T}} = \frac{K}{1+Kh}\left[1-e^{-\frac{t}{(T/1+Kh)}}\right]$$

Thus it can be observed that the new time constant due to the feedback is $(T/1+Kh)$. Thus for positive value of h and $K>1$, the time constant $(T/1+Kh)$ is less than T. Thus it can be concluded that the time constant of a closed loop system is less than the open loop system.

Key Point : *Less the time constant faster is the response. Hence the feedback improves the time response of a system.*

7.18.5 Effect of Feedback on Overall Gain

Consider an open loop system with overall transfer function as G(s). The overall gain of such system is nothing but G(s).

If the feedback with transfer function H(s) is introduced in such a system, then its overall gain becomes $[G(s)/1 \pm G(s)H(s)]$. The positive or negative sign in the denominator gets decided by the sign of the feedback.

For a negative feedback, the gain G(s) reduces by a factor $[1/(1+G(s)H(s))]$.

Due to the negative feedback overall gain of the system reduces.

7.18.6 Effect of Feedback on Stability

It is discussed earlier that the feedback reduces the time constant and makes the system response more fast. Hence the transient response decays more quickly.

Consider an open loop system with overall transfer function as,

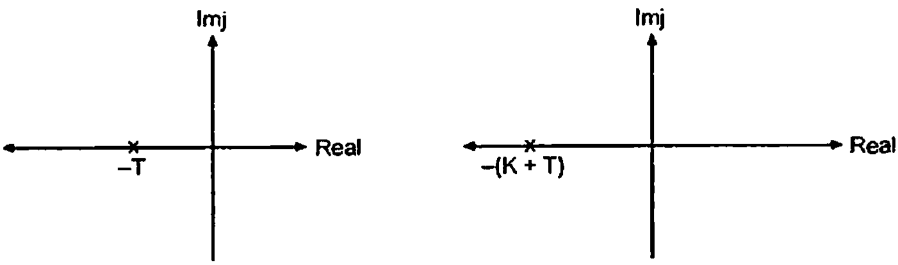
$$G(s) = \frac{K}{s+T}$$

The open loop pole is located at $s=-T$.

Now let a unity negative feedback is introduced in the system. The overall transfer function of a closed loop system becomes,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s+T}}{1+\frac{K}{s+T}} = \frac{K}{s+(K+T)}$$

Thus the closed loop pole is now located at $s=-(K+T)$. This is shown in the Fig. 7.33 (a) and (b).



(a) Open loop system (b) Closed loop system

Fig. 7.33

Thus the feedback controls the time response i.e. dynamics of the system by adjusting location of its poles. The stability of a system depends on the location of poles in s-plane. Thus it can be concluded that the feedback affects the stability of the system. The feedback may improve the stability and also may be harmful to the system from stability point of view. The closed loop system may be unstable though the open loop system is stable.

Key Point : *Thus the stability of the system can be controlled by proper design and application of the feedback.*

7.18.7 Effect of Feedback on Disturbance

Every control system has some nonlinearities present in it. The dominant nonlinearities like friction, dead zone, saturation etc. affect the output of the system adversely. Some external disturbance signals also make the system output inaccurate. The examples of such external disturbances are high frequency noise in electronic applications, thermal noise in amplifier tubes, wind gusts on antenna of radar systems etc. The disturbance may be in the forward path, feedback path or output of a system.

(a) Disturbance in the forward path

Let us assume that there is a disturbance in the forward path of a control system produced due to varying properties of forward path elements or due to effect of surrounding conditions. Fig. 7.34 shows the disturbance signal $T_d(s)$ produced in the forward path.

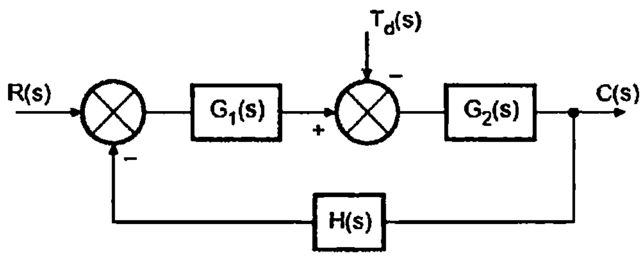


Fig. 7.34

Assuming R(s) to be zero, let us obtain the ratio $C(s) / T_d(s)$ to study the effect of disturbance on output. With R(s) = 0, system becomes.

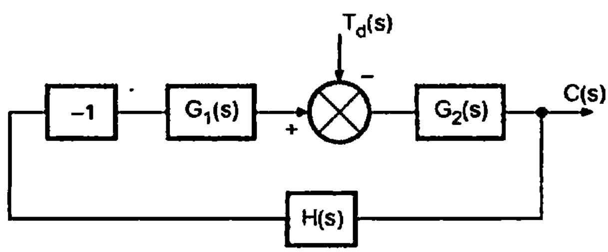


Fig. 7.35

The resultant elements are,

$$G(s) = G_2(s)$$

$$H'(s) = -G_1(s)H(s)$$

Positive feedback
Negative input

$$\therefore \quad \frac{C(s)}{-T_d(s)} = \frac{G_2(s)}{1-\left[G_2(s)(-G_1(s)H(s))\right]}$$

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1\,G_2H(s)}$$

$$\therefore \quad C(s) = \frac{-T_d(s)\,G_2}{1+G_1\,G_2H}$$

In the denominator assume that $1 << G_1\ G_2\ H$ hence we get,

$$C(s) = \frac{-T_d(s)}{G_1\ H(s)}$$

Thus to make the effect of disturbance on the output as small as possible, the $G_1(s)$ must be selected as large as possible.

(b) Disturbance in the feedback path

These are produced due to the nonlinear behaviour of the feedback path elements. The Fig. 7.36 shows the disturbance signal $T_d(s)$ produced in the feedback path.

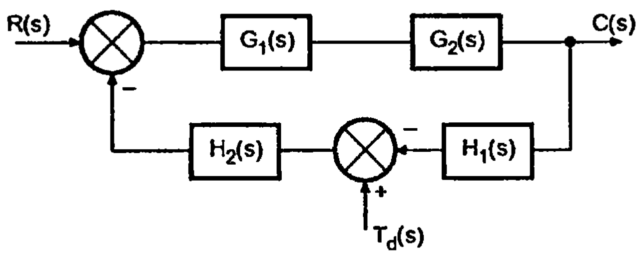


Fig. 7.36

With $R(s) = 0$, the effect of $T_d(s)$ on output can be obtained.

The system becomes,

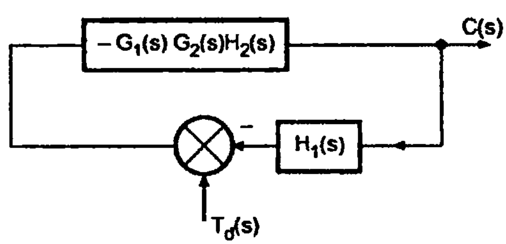


Fig. 7.36(a)

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{-G_1 G_2 H_2}{1+G_1 G_2 H_1 H_2}$$

For large values of G_1, G_2, H_1, H_2, in the denominator 1 can be neglected.

$$\therefore \quad \frac{C(s)}{T_d(s)} = -\frac{1}{H_1(s)}$$

Thus designing proper feedback element $H_1(s)$, the effect of disturbance in feedback path on output can be reduced.

(c) Disturbance at the output

Consider that there is disturbance $T_d(s)$ affecting the output directly as shown in the Fig. 7.37.

with R(s) = 0, we get

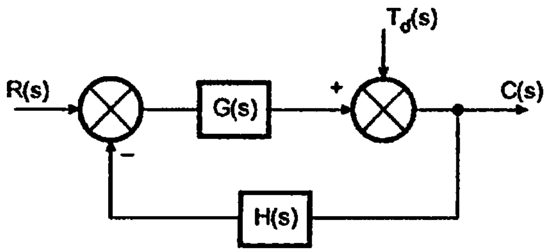


Fig. 7.37

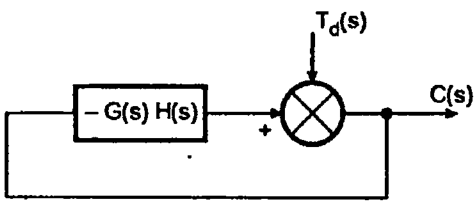


Fig. 7.38

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{1}{1-[-G(s)H(s)]} = \frac{1}{1+G(s)H(s)}$$

For large values of G (s) H (s), 1 in denominator can be neglected.

$$\therefore \quad C(s) = \frac{T_d(s)}{G(s)H(s)}$$

Thus if disturbance is affecting the output directly then by changing the values of G(s), H(s) or both the effect of disturbance can be minimised.

The feedback minimizes the effect of disturbance signals occurring in the control system.

Example 7.5 : *The negative feedback control system has the forward path transfer function as,*

$$G(s) = \frac{10}{s(s+1)}$$

While the feedback path transfer function H(s) *is 5. Determine the sensitivity of the closed loop transfer function with respect to G and H at* $\omega = 1$ *rad/sec.* **(A.M.I.E.-97)**

Solution : For the given closed loop system,

$$G(s) = \frac{10}{s(s+1)}, \quad H(s) = 5$$

$$\therefore \quad T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{10}{s(s+1)}}{1+\frac{10}{s(s+1)}\cdot 5} = \frac{10}{s^2+s+50}$$

The sensitivity of T(s) with respect to G(s) is given by,

$$S_G^T = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{10}{s(s+1)}\cdot 5} = \frac{s(s+1)}{s^2+s+50}$$

To calculate S_G^T at $\omega = 1$, substitute $s = j\omega$ to convert the time domain function to the frequency domain.

$$\therefore \quad S_G^T = \frac{j\omega(1+j\omega)}{(j\omega)^2+j\omega+50} = \frac{-\omega^2+j\omega}{(50-\omega^2)+j\omega}$$

For the value of $\omega = 1$,

$$S_G^T = \frac{-1+j1}{49+j1}$$

$$\therefore \quad |S_G^T| = \frac{\sqrt{1+1}}{\sqrt{49^2+1}}$$

$$= \mathbf{0.02885}$$

The sensitivity of T(s), with respect to H(s) is given by,

$$S_H^T = \frac{-G(s)H(s)}{1+G(s)H(s)} = \frac{-\frac{10}{s(s+1)}\cdot 5}{1+\frac{10}{s(s+1)}\cdot 5}$$

$$= \frac{-50}{s^2+s+50}$$

Replacing s by $j\omega$, to convert the function to the frequency domain we get,

$$S_H^T = \frac{-50}{(j\omega)^2+j\omega+50} = \frac{-50}{(50-\omega^2)+j\omega}$$

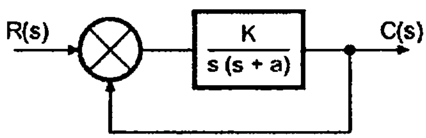
At $\omega=1$, $S_H^T = \dfrac{-50}{49+j1}$

$$\therefore \quad \left|S_H^T\right| = \frac{|-50|}{\sqrt{49^2+(1)^2}}$$

$$= 1.0202$$

It can be observed that S_H^T is more than S_G^T i.e. the system is more sensitive to the variations in H(s) rather than G(s).

Example 7.6 : *A position control system is shown in the figure.*



Evaluate the sensitivities S_K^T, S_a^T.

K = 20 and a = 4

Solution : For a given system,

$$G(s) = \frac{K}{s(s+a)} \text{ and } H(s) = 1$$

Now $S_K^T = \dfrac{K}{T(s)} \cdot \dfrac{\partial T(s)}{\partial K}$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+a)}}{1+\dfrac{K}{s(s+a)}} = \frac{K}{s^2+as+K}$$

$$\therefore \quad \frac{\partial T(s)}{\partial K} = \frac{\left(s^2+as+K\right)(1)-K(1)}{\left(s^2+as+K\right)^2} = \frac{s^2+as}{\left(s^2+as+K\right)^2}$$

$$\therefore \quad S_K^T = \frac{K}{\left(\dfrac{K}{s^2+as+K}\right)} \times \frac{\left(s^2+as\right)}{\left(s^2+as+K\right)^2} = \frac{s(s+a)}{s^2+as+K}$$

$$\therefore \quad \boxed{S_K^T = \frac{s(s+4)}{\left(s^2+4s+20\right)}}$$

$$\therefore \quad 0.005 = \frac{0.1}{K}$$

$$\therefore \quad K = \frac{0.1}{0.005} = 20$$

For any value of K greater than 20, e_{ss} will be less than 0.005. Hence the range of value of K for $e_{ss} \le 0.005$ is,

$$20 \le K < \infty$$

Example 7.11 : *For a unity feedback system* $G(s) = \frac{200}{s(s+8)}$ *and* $r(t) = 2t$ *determine steady state error. If it is desired to reduce this existing error by 5% find new value of gain of the system.*

Solution : The input is 2t i.e. ramp of magnitude 2, so K_v will control the error.

$$\therefore \quad K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s \cdot \frac{200}{s(s+8)} \cdot 1 = 25$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{2}{25} = 0.08$$

This error is to be reduced by 5% of existing value, with new gain of G(s) as K_2 instead of 200.

$$\therefore \quad e_{ss1} = e_{ss} - \left(\frac{5}{100} \times e_{ss}\right) = 0.08 - \frac{5 \times 0.08}{100} = 0.076$$

New error is 0.076.

$$\text{New} \quad G(s) = \frac{K_2}{s(s+8)} \text{ and } H(s) = 1 \text{ with same input}$$

$$\therefore \quad K_v = \lim_{s \to 0} sG(s)H(s) = \frac{s \cdot K_2}{s(s+8)} = \frac{K_2}{8}$$

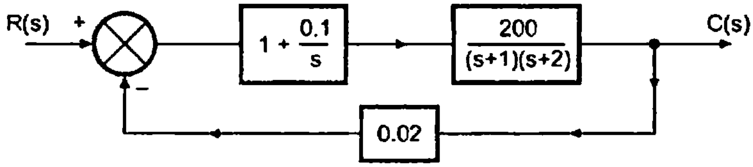
$$\therefore \quad e_{ss1} = \frac{A}{K_v} = \frac{2}{\left(\frac{K_2}{8}\right)} = \frac{16}{K_2}$$

$$\therefore \quad 0.076 = \frac{16}{K_2}$$

$$\therefore \quad K_2 = 210.52$$

So new gain is 210.52.

Example 7.12 : *The control system is shown below. If the input to the system is i) Unit step and ii) Unit ramp, find e_{ss}* **(M.U. : Nov. - 96)**



Solution :

$$G(s) = \left(1+\frac{0.1}{s}\right)\left(\frac{200}{(s+1)(s+2)}\right) = \frac{(s+0.1)200}{s(s+1)\ (s+2)}$$

$$H(s) = 0.2$$

$$\therefore \quad G(s)H(s) = \frac{200(s+0.1)}{s(s+1)\ (s+2)} \times 0.02 = \frac{200 \times 0.02 \times 0.1}{1 \times 2} \times \frac{(1+10s)}{s(1+s)\ (1+0.5s)}$$

$$= \frac{0.2(1+10s)}{s(1+s)(1+0.5s)}$$

i) For unit step input

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{0.2(1+10s)}{s(1+s)(1+0.5s)} = \infty$$

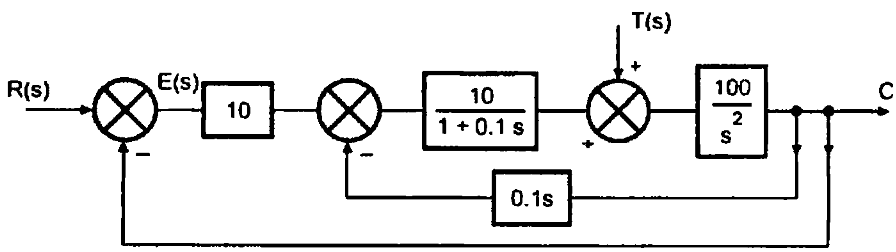
$$\therefore \quad e_{ss} = \frac{1}{1+K_p} = 0$$

ii) For unit ramp input

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{0.2(1+10s)}{s(1+s)(1+0.5s)} = 0.2$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \frac{1}{0.2} = 5$$

Example 7.13 : *Find the steady state error E, if T is unit step input and R = 0.*



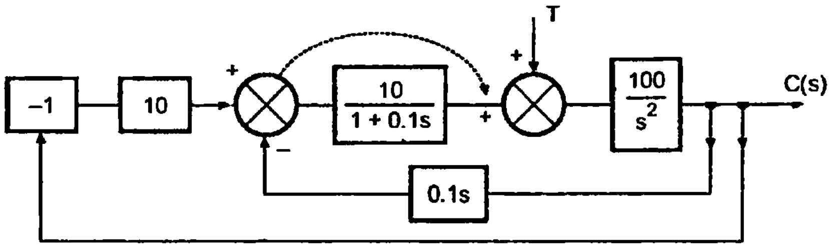
Solution : When system is not in the simple closed loop form then we can not apply the error coefficients.

In such case, we have to use final value theorem

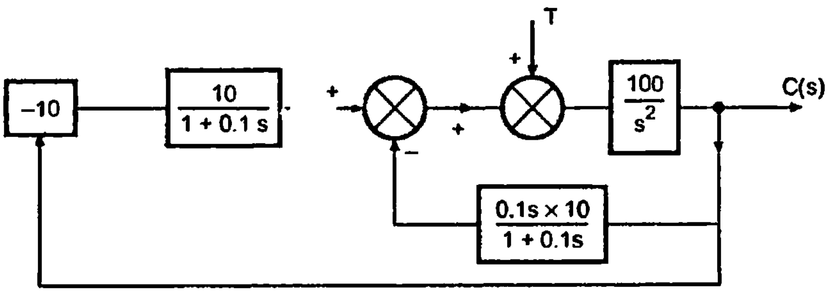
i.e. $$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

In the system given E(s) = – C(s) when R(s) = 0

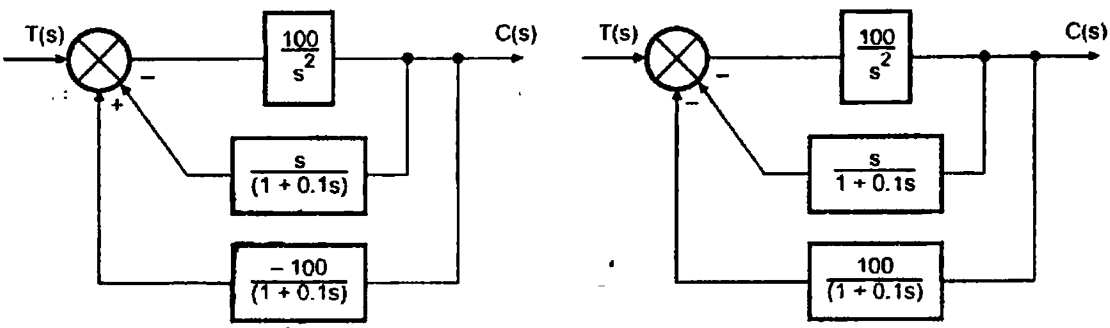
Now let us find out $\frac{C(s)}{T(s)}$, so that for unit step disturbance we can calculate C(s) and hence E(s). When R = 0, summing point at R(s) can be removed and block of '–1' is to be added to consider sign of the signal at that summing point.



Shifting summing point to right.

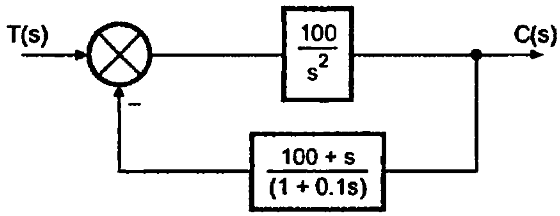


Combining the two summing points and redrawing the diagram.



Negative sign of $\left(\frac{-100}{1+0.1s}\right)$ can be taken out to change sign of the signal at the summing point from positive to negative.

Now the two blocks are in parallel, in the feedback path.



$$\therefore \quad \frac{C(s)}{T(s)} = \frac{\frac{100}{s^2}}{1+\frac{100}{s^2}\cdot\frac{(100+s)}{(1+0.1s)}}$$

$$\therefore \quad \frac{C(s)}{T(s)} = \frac{100\,(1+0.1s)}{(0.1s^3+s^2+100s+10000)}$$

For $T(s) = \frac{1}{s}$, $\quad C(s) = \frac{1}{s} \times \frac{100\,(1+0.1s)}{(0.1s^3+s^2+100s+10000)}$

Now : $\quad E(s) = -C(s) = -\frac{100\,(1+0.1s)}{s(0.1s^3+s^2+100s+10000)}$

$$\therefore \quad e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$\therefore \quad e_{ss} = \lim_{s\to 0} s \times \left\{\frac{-100\,(1+0.1s)}{s(0.1s^3+s^2+100s+10000)}\right\} = -\frac{100}{10000}$$

Steady state error $= -$ **0.01**

Example 7.14 : *Determine K_p, K_v and K_a for a system with*

$$G(s) = \frac{100}{s(s+0.5)\,(4-s)\,(s+1000)},\; H(s) = 1$$

(M.U. : Nov. - 95)

Solution :

$$G(s)H(s) = \frac{100}{s\times 0.5\times\left(1+\frac{s}{0.5}\right)\times 4\times\left(1-\frac{s}{4}\right)\times 100\times\left(1+\frac{s}{100}\right)}$$

$$\therefore \quad G(s)H(s) = \frac{0.5}{s(1+2s)\,(1-0.25s)\,(1+0.01s)}$$

$$\therefore \quad K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{0.5}{s(1+2s)\ (1-0.25s)\ (1+0.01s)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s0.5}{s(1+2s)\ (1-0.25s)\ (1+0.01s)} = 0.5$$

$$K_a = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} \frac{s^2 \times 0.5}{s(1+2s)\ (1-0.25s)\ (1+0.01s)} = 0$$

Example 7.15 : *Find the steady state error for various types of standard test inputs for a unity feedback system with*

$$G(s) = \frac{K}{s(s+5)\ (s+10)}$$

(a) K = 10 (b) K = 200 **(M.U. : May-1995)**

Solution :

$$G(s)H(s) = \frac{K}{s(s+5)\ (s+10)} = \frac{K}{s \times 5 \times \left(1+\frac{s}{5}\right) \times 10 \times \left(1+\frac{s}{10}\right)}$$

$$= \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)}$$

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \frac{K}{50}$$

$$K_a = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} \frac{s^2\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = 0.$$

$\therefore$ For step input of magnitude 1,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0 \quad \text{... for any value of K.}$$

For ramp input of magnitude 1,

$$e_{ss} = \frac{1}{K_v} = \frac{50}{K}$$

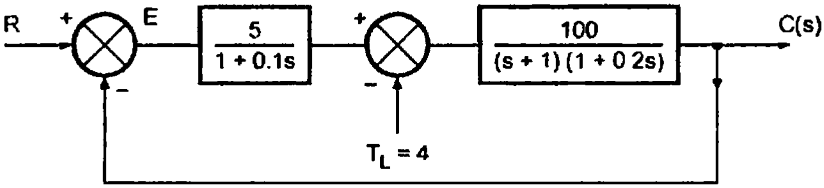
a) For K = 10, $e_{ss} = 5$

b) For K = 200, $e_{ss} = 0.25$

For parabolic input of magnitude 1,

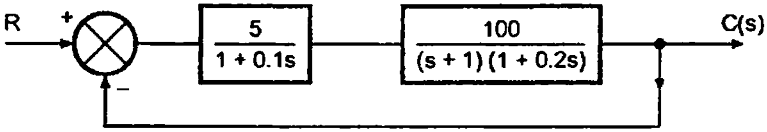
$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty \qquad \text{... for any value of K.}$$

Example 7.16 : *In the system given, the command input is R = 10 and disturbance signal is T_L = 4, what is the steady state error ?* **(M.U. : Nov. - 94)**



Solution : Using superposition principle, consider inputs separately.

a) R acting, $T_L = 0$



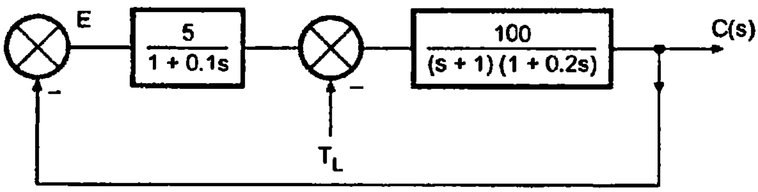
$$\therefore \quad G(s)H(s) = \frac{500}{(1+0.1s)\ (s+1)\ (1+0.2s)}$$

For step input

$$K_p = \lim_{s \to 0} G(s)H(s) = 500$$

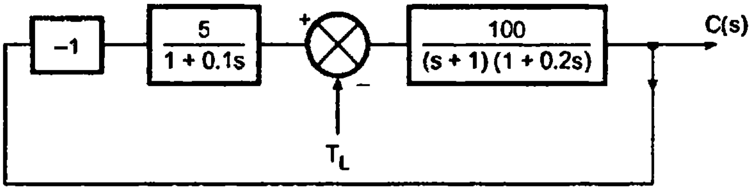
$$\therefore \quad e_{ss1} = \frac{A}{1+K_p} \quad \text{where A = magnitude of step} = \frac{10}{1+500} = \frac{10}{501}$$

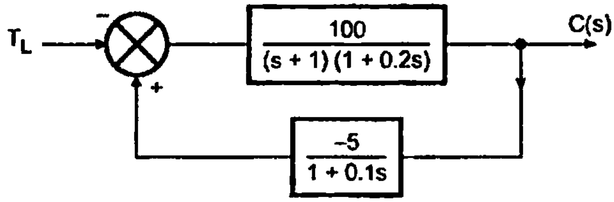
b) T_L acting, $R = 0$



$$E(s) = -C(s)$$

As system is not in standard form, error coefficient method cannot be used.





$$\frac{C(s)}{T(s)} = \frac{\frac{100}{(1+s)\ (1+0.2s)}}{1-\frac{100}{(1+s)\ (1+0.2s)}\times\left(\frac{-5}{1+0.1s}\right)}$$

$$\frac{C(s)}{T(s)} = \frac{100(1+0.1s)}{(1+s)\ (1+0.2s)\times(1+0.1s)+500}$$

Now $T(s) = \frac{-4}{s}$ −ve sign as T_L applied with −ve sign.

$\therefore$ $$C(s) = \frac{-400(1+0.1s)}{s[(1+s)\ (1+0.2s)\times(1+0.1s)+500]}$$

but $$E(s) = -C(s) = \frac{+400(1+0.1s)}{s[(1+s)\ (1+0.2s)\times(1+0.1s)+500]}$$

$\therefore$ $$e_{ss2} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{400(1+0.1s)}{[(1+s)\ (1+0.2s)\times(1+0.1s)+500]}$$

$$= \left[\frac{400}{1+500}\right] = \frac{+400}{501}$$

$\therefore$ Total error $$e_{ss} = e_{ss1} + e_{ss2} = \frac{10}{501} + \frac{400}{501} = \frac{410}{501} = \mathbf{0.8183}$$

Example 7.17 : *A second order system is given by* $\frac{C(s)}{R(s)} = \frac{25}{s^2+6s+25}$. *Find its rise time, peak time, peak overshoot and settling time if subjected to unit step input. Also calculate expression for its output response.*

Solution : Comparing the denominator of T.F. with the standard form $s^2 + 2\xi\omega_n s + \omega_n^2$

$$\omega_n^2 = 25 \quad \text{and} \quad 2\xi\omega_n = 6$$

$$\omega_n = 5 \qquad \therefore \quad \xi = 0.6$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0.9272 \text{ radians}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.6)^2} = 4 \text{ rad/sec}$$

$$T_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-0.9272}{4} = \mathbf{0.5535\ sec}$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = \mathbf{0.785\ sec}$$

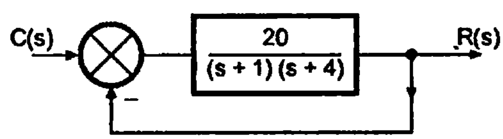
$$\%\, M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{9.48\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{1.33\ sec}$$

and
$$c(t) = 1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) = 1-\frac{e^{-3t}}{\sqrt{1-(0.6)^2}}\sin(4t+0.9272)$$

$\therefore$
$$= \mathbf{1 - 1.5625\, e^{-3t}\ \sin(4t + 0.9272)}$$

Example 7.18 : *For the system shown in the figure obtain the closed loop T.F., damping ratio, natural frequency and expression for the output response if subjected to unit step input.*



Solution :
$$\frac{C(s)}{R(s)} = \frac{\dfrac{20}{(s+1)(s+4)}}{1+\dfrac{20}{(s+1)(s+4)}} = \frac{20}{s^2+5s+24}$$

Key Point : *Now though T.F. is not in standard form, denominator always reflect* $2\xi\,\omega_n$ *and* ω_n^2 *from middle term and the last term respectively.*

$\therefore$ Comparing $s^2+5s+24$ with $s^2+2\xi\,\omega_n\, s+\omega_n^2$

$\therefore$ $\omega_n^2 = 24$ $\quad\therefore\ \omega_n = \mathbf{4.8989\ rad/sec}$

$2\xi\,\omega_n = 5$ $\quad\therefore\ \xi = \mathbf{0.51031}$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \mathbf{4.2129\ rad/sec}$$

Now for c(t) we can use standard expression for $\frac{C(s)}{R(s)}$ in standard form. So writing

$$\frac{C(s)}{R(s)} = \frac{20}{24} \cdot \left\{\frac{24}{s^2+5s+24}\right\}$$

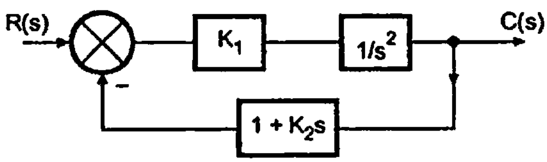
For the bracket term use standard expression, and then c(t) can be obtained by multiplying this expression by constant $\frac{20}{24}$.

$$\therefore \quad c(t) = \frac{20}{24}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)\right]$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \text{ radians} = 1.03 \text{ radians}$$

$$\therefore \quad c(t) = \frac{20}{24}\left[1-1.1628\, e^{-2.5t}\sin(4.2129\,t+1.03)\right]$$

Example 7.19 : *For a control system shown in figure, find the values of K_1 and K_2 so that M_p = 25% and T_p = 4 sec. Assume unit step input.*



Solution : $G(s) = \frac{K_1}{s^2}$, $H(s) = 1 + K_2\, s$

$$\text{T.F.} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K_1}{s^2}}{1+\frac{K_1}{s^2}(1+K_2\, s)} = \frac{K_1}{s^2+K_1 K_2\, s+K_1}$$

$\therefore$ Comparing with standard form,

$$\omega_n = \sqrt{K_1}, \quad 2\xi\omega_n = K_1 \cdot K_2, \quad \therefore \quad \xi = \frac{1}{2}\sqrt{K_1}\cdot K_2$$

Now M_p is function of ξ alone,

$$\therefore \quad \% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$\therefore \quad 25 = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$\therefore \quad 0.25 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

Now the time for 1 cycle is known and if it is known to us that what is the time required by the system to achieve steady state, we can find how many cycles output will perform before reaching steady state.

$$\therefore \quad T_s = \frac{4}{\xi \omega_n} = 2 \text{ sec}$$

So 1.6792 sec for one cycle, how many cycles output will perform in 2 sec.

$$\therefore \text{ Total no. of cycles} = \frac{2}{1.6792} = 1.191$$

Output will perform **1.191** cycles before reaching the steady state.

This can be shown as in Fig. 7.39.

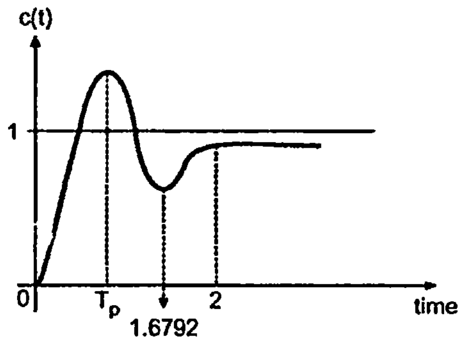


Fig. 7.39

The frequency in Hz if required can be obtained as,

$$f_d = \frac{1}{T} = 0.5955 \text{ Hz}$$

$$f_d = \text{Frequency of damped oscillations in Hz.}$$

Example 7.22 : *A second order system is represented by the transfer function,*

$$\frac{Q(s)}{I(s)} = \frac{1}{Js^2 + fs + K}$$

A step input of 10 Nm is applied to the system and the test results are,

a) Maximum overshoot = 6 %
b) Time at peak overshoot = 1 sec
c) The steady state value of the output is 0.5 radians
Determine the values of J, f and K.

Solution : Arrange the given transfer function as,

$$\frac{Q(s)}{I(s)} = \frac{1}{J\left[s^2+\frac{f}{J}s+\frac{K}{J}\right]} = \frac{\left(\frac{1}{J}\right)}{s^2+\frac{f}{J}s+\frac{K}{J}}$$

Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$$\omega_n^2 = \frac{K}{J} \quad \text{i.e.} \quad \omega_n = \sqrt{\frac{K}{J}} \quad \text{... (1)}$$

and $$2\xi\omega_n = \frac{f}{J} \quad \text{i.e.} \quad \xi = \frac{f}{2\sqrt{KJ}} \quad \text{... (2)}$$

Now $M_p = 6\%$ i.e., 0.06

$$\therefore \quad 0.06 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\therefore \quad \ln(0.06) = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

Solving for ξ, $\xi = 0.667$ ... (3)

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1 \text{ sec}$$

$$\therefore \quad \omega_n = \frac{\pi}{\sqrt{1-(0.667)^2}} = 4.2165 \text{ rad/sec} \quad \text{... (4)}$$

The Laplace transform of output is Q(s).

Now input is step of 10 Nm hence corresponding Laplace transform is,

$$I(s) = \frac{10}{s}$$

$$\therefore \quad \frac{Q(s)}{\left(\frac{10}{s}\right)} = \frac{1}{Js^2+fs+K}$$

$$\therefore \quad Q(s) = \frac{10}{s(Js^2+fs+K)}$$

The steady state of output can be obtained by final value theorem.

$$\text{Steady state output} = \lim_{s\to 0} sQ(s)$$

$$\therefore \quad 0.5 = \lim_{s\to 0}\frac{s\cdot 10}{s(Js^2+fs+K)} = \frac{10}{K}$$

$$\therefore \quad K = 20 \qquad \text{... (5)}$$

Equating (1) and (4), $4.2165 = \sqrt{\frac{K}{J}}$

$$\therefore \quad 4.2165 = \sqrt{\frac{20}{J}}$$

$$\therefore \quad J = 1.1249$$

From equation (2), $0.667 = \frac{f}{2\sqrt{KJ}}$

$$\therefore \quad 0.667 = \frac{f}{2\sqrt{20 \times 1.1249}}$$

$$\therefore \quad f = 6.3274$$

Example 7.23 : *The open loop T.F. of unity feedback system is* $G(s) = \frac{K}{s(1+Ts)}$. *For this system overshoot reduces from 0.6 to 0.2 due to change in 'K' only. Show that* $\frac{TK_1 - 1}{TK_2 - 1} = 43.33$ *where* K_1 *and* K_2 *are values of K for 0.6 and 0.2 overshoot respectively.*

Solution : Closed loop T.F. $= \frac{G(s)}{1+G(s)}$, $\quad H(s) = 1$

$$= \frac{\frac{K}{s(1+Ts)}}{1+\frac{K}{s(1+Ts)}} = \frac{K}{s^2 T + s + K} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

Comparing with standard form $\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$$\omega_n^2 = \frac{K}{T} \qquad \omega_n = \sqrt{\frac{K}{T}}$$

$$2\xi\omega_n = \frac{1}{T} \qquad \xi = \frac{1}{2\sqrt{\frac{K}{T}} \cdot T} = \frac{1}{2\sqrt{KT}}$$

Now for $M_p = 0.6$ Let $\xi = \xi_1$

$$\therefore \quad 0.6 = e^{-\pi\xi_1/\sqrt{1-\xi_1^2}}$$

i.e. $$-0.51 = -\frac{\pi\xi_1}{\sqrt{1-\xi_1^2}}$$

As $\xi > 1$, system is overdamped, hence the output will not contain any oscillations. Hence standard expression for c(t) cannot be used.

Now input is unit step, so X(s) = 1/s Substituting in T.F.

$$Y(s) = \frac{12}{s(s^2+7s+12)} \qquad \text{Use partial fraction method}$$

$$= \frac{12}{s(s+3)(s+4)} = \frac{A}{s}+\frac{B}{s+3}+\frac{C}{s+4}$$

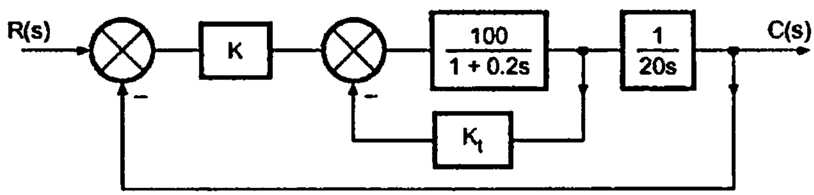
where, A = 1, B = − 4, C = 3

$$Y(s) = \frac{1}{s}-\frac{4}{s+3}+\frac{3}{s+4}$$

Taking Laplace inverse,

$$y(t) = 1-4e^{-3t}+3e^{-4t}$$

Example 7.25 : *For a control system shown in figure, find the values of K and K_t so that the damping ratio of system is 0.6 and settling time is 0.1 sec. Use* $T_s = \frac{3.2}{\xi\omega_n}$. *Assume unit step input.*



Solution : Using block diagram reduction rule, reduction of inner loop is,

$$\frac{\frac{100}{1+0.2s}}{1+\frac{100}{1+0.2s}\cdot K_t} = \frac{100}{1+0.2s+100K_t}$$

$$\therefore \quad \text{Overall } G(s) = K\cdot\frac{100}{1+0.2s+100K_t}\cdot\frac{1}{20s} = \frac{5K}{s[1+100K_t+0.2s]}$$

$$H(s) = 1$$

$$\therefore \quad T.F. = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{5K}{s(1+100K_t+0.2s)}}{1+\frac{5K}{s(1+100K_t+0.2s)}}$$

$$= \frac{5K}{0.2s^2 + s(1+100\,K_t) + 5K}$$

But coefficient of s^2 in the denominator must be 'unity' to compare it with the standard form. So dividing it by 0.2.

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{25K}{s^2 + 5s(1+100\,K_t) + 25K}$$

$$\therefore \quad \omega_n^2 = 25\,K, \qquad \omega_n = 5\sqrt{K}$$

$$2\,\xi\omega_n = 5\,(1 + 100\,K_t)$$

$$\xi = \frac{5(1+100K_t)}{10\sqrt{K}}$$

Now $\quad \xi = 0.6$ and $T_s = 0.1$ sec

Using $\quad T_s = \dfrac{3.2}{\xi\,\omega_n}$

i.e. $\quad 0.1 = \dfrac{3.2}{0.6\,\omega_n}$

$$\therefore \quad \omega_n = 53.33 \text{ rad/sec}$$

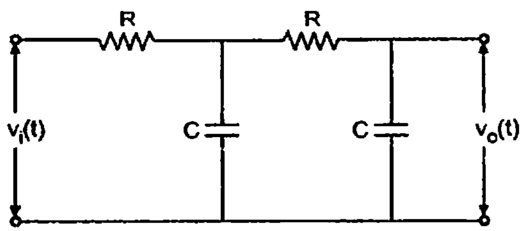
$$\therefore \quad \omega_n = 5\sqrt{K}$$

$$\therefore \quad K = 113.78$$

and $\quad \xi = \dfrac{5(1+100\,K_t)}{10\sqrt{K}} = 0.6$

$$\therefore \quad K_t = \mathbf{0.118}$$

Example 7.26 : *For the system shown in figure show that system is always overdamped, independent of the selection of R and C.*



Solution : To prove that system is overdamped means to prove $\xi > 1$ and not dependent on the values of R and C.

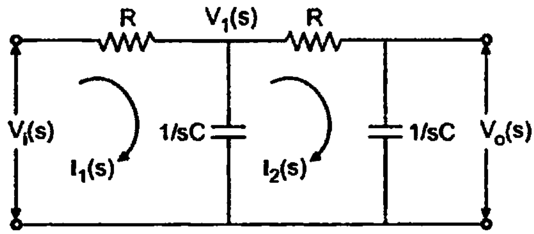
So first to find its C.L.T.F. $\dfrac{V_o(s)}{V_i(s)}$ use signal flow graph method.

$\therefore \quad I_1(s) = \frac{V_i - V_1}{R}$

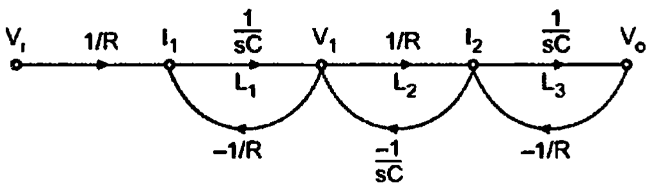
$$V_1(s) = (I_1 - I_2)\cdot\frac{1}{sC}$$

$$I_2(s) = \frac{V_1 - V_o}{R}$$

$$V_o(s) = I_2 \cdot \frac{1}{sC}$$



$\therefore$ Signal flow graph is shown in the following figure.



$$T_1 = \frac{1}{s^2 C^2 R^2}, \quad L_1 = L_2 = L_3 = \frac{-1}{sCR}$$

L_1 and L_3 is combination of 2 nontouching loops.

$\therefore \quad L_1 L_3 = \frac{1}{s^2 C^2 R^2}$

Using Mason's gain formula,

$\therefore \quad \Delta = 1 - [L_1 + L_2 + L_3] + [L_1 L_3] = 1 + \frac{3}{sCR} + \frac{1}{s^2 C^2 R^2}$

$$\Delta_1 = 1 \text{ as all loops are touching to } T_1$$

$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta} = \frac{\frac{1}{s^2 C^2 R^2}}{1 + \frac{3}{sCR} + \frac{1}{s^2 C^2 R^2}}$

$$= \frac{1}{s^2 C^2 R^2 + 3sCR + 1} = \frac{\frac{1}{C^2 R^2}}{s^2 + \frac{3s}{CR} + \frac{1}{C^2 R^2}}$$

Comparing with standard form,

$$\omega_n^2 = \frac{1}{C^2 R^2} \quad \therefore \omega_n = \frac{1}{CR} \text{ rad/sec}$$

$$2\xi\omega_n = \frac{3}{CR} \quad \therefore \quad \xi = \frac{3}{CR}\cdot\frac{1}{2}\cdot CR = \frac{3}{2} = 1.5$$

As $\xi > 1$ system is overdamped and ξ is independent of R and C values.

For this, two resistances must be equal, and two capacitor values must be equal.

Example 7.27 : *A position control system drives a load through a 50 : 1 gear ratio. The inertia of motor shaft is 20×10^{-5} kg·m² and friction is 60×10^{-5} N-m sec. The torque constant is 0.04 N-m per degree of error, if output speed is 20 rpm. Calculate*
i) ξ *ii)* ω_n .

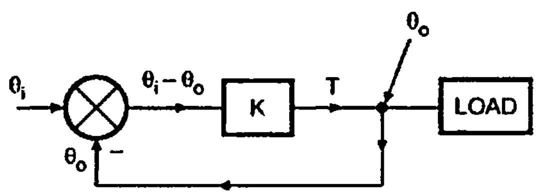
Solution : Let T = Torque produced proportional to '$\theta_i - \theta_o$'

$$\therefore \quad T = K[\theta_i - \theta_o]$$

where, θ_i = Reference input position

θ_o = Output position

Used to drive load of M.I. 'J' and friction 'B'



$$\therefore \quad T = J\frac{d^2\theta_o}{dt^2} + B\frac{d\theta_o}{dt}$$

$$K[\theta_i - \theta_o] = J\frac{d^2\theta_o}{dt^2} + B\frac{d\theta_o}{dt}$$

Taking Laplace we have,

$$K\theta_i(s) = K\theta_o(s) + Js^2\theta_o(s) + Bs\theta_o(s)$$

$$\therefore \quad \frac{\theta_o(s)}{\theta_i(s)} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}}$$

$$\omega_n = \sqrt{\frac{K}{J}} \qquad \xi = \frac{B}{2J}\sqrt{\frac{J}{K}} = \frac{B}{2\sqrt{JK}}$$

$$K = 0.04 \text{ N-m/deg} = 0.04 \times 57.3 \text{ N-m/radian}$$

$$\text{Gear ratio} = 1/50$$

$$\therefore \text{K at motor shaft} = 0.04 \times 57.3/50$$

$$\therefore \quad \omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{0.04 \times 57.3}{50 \times 20 \times 10^{-5}}} = 15.13 \text{ rad/sec}$$

$$\xi = \frac{B}{2\sqrt{JK}} = 0.099$$

Example 7.28 : *A system has the following transfer function*

$$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

Determine its unit impulse, step and ramp response with zero initial conditions. Sketch the responses. **(M.U. : May-96)**

Solution : a) Unit impulse input

$$R(s) = 1$$

as $$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

$$C(s) = \frac{20}{s+10}$$

$$\therefore \quad c(t) = L^{-1}\left\{\frac{20}{s+10}\right\}$$

$$\therefore \quad c(t) = e^{-10t}$$

b) Unit step input

$$\therefore \quad R(s) = \frac{1}{s}$$

$$\therefore \quad C(s) = \frac{20}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

where $A = 2, \quad B = -2$

$$\therefore \quad C(s) = \frac{2}{s} - \frac{2}{s+10}$$

$$\therefore \quad c(t) = L^{-1}\left\{\frac{2}{s} - \frac{2}{s+10}\right\}$$

$$\therefore \quad c(t) = 2 - 2e^{-10t}$$

c) Unit ramp input

$$\therefore \quad R(s) = \frac{1}{s^2}$$

$$\therefore \quad C(s) = \frac{20}{s^2(s+10)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+10}$$

$$A(s+10) + Bs(s+10) + Cs^2 = 20$$

$\therefore \quad B + C = 0, \qquad A + 10B = 0, \qquad 10A = 20$

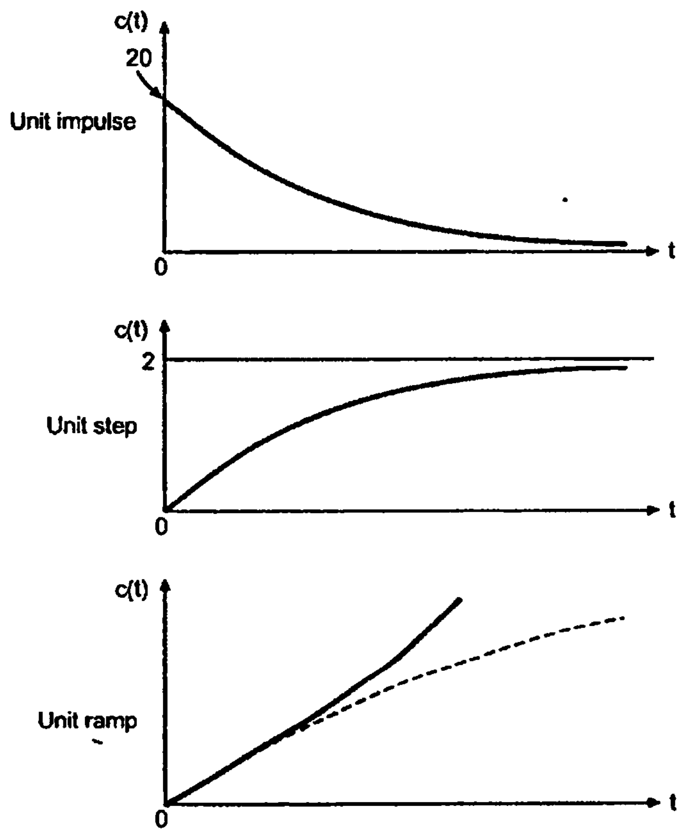
$\therefore \quad A = 2, \qquad B = -0.2, \qquad C = 0.2$

$$\therefore \quad C(s) = \frac{2}{s^2} - \frac{0.2}{s} + \frac{0.2}{s+10}$$

$$\therefore \quad c(t) = L^{-1}\left\{\frac{2}{s^2} - \frac{0.2}{s} + \frac{0.2}{s+10}\right\}$$

$$\therefore \quad \mathbf{c(t) = 2t - 0.2 + 0.2e^{-10t}}$$

The responses are,



Example 7.29 : *Derive an expression for the time response of a second order system subjected to a unit impulse input for $\xi < 1$, $\xi > 1$, where ξ is a damping ratio.*

Solution : The second order system transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The characteristic equation is,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

where ξ = Damping ratio

The roots of this equation are,

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\therefore$ $s_{1,2} = -\xi\omega_n \pm j\omega_d$

Case i) : $\xi < 1$, under damped.

The roots are complex conjugates. The input is unit impulse.

$\therefore$ $R(s) = 1$

$$\therefore \quad C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \xi^2\omega_n^2 + \omega_n^2 - \xi^2\omega_n^2}$$

$$\therefore \quad C(s) = \frac{\omega_n^2}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)} = \frac{\omega_n^2}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}$$

$$\therefore \quad L^{-1}\left\{\frac{\omega}{(s+a)^2 + \omega^2}\right\} = e^{-at}\sin\omega t$$

$$\therefore \quad L^{-1}[C(s)] = L^{-1}\left[\frac{\omega_n^2}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}\right]$$

$$\therefore \quad c(t) = \frac{\omega_n^2}{\omega_n\sqrt{1-\xi^2}}\cdot e^{-\xi\omega_n t}\sin\omega_d t$$

$$\therefore \quad c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \sin \omega_d t$$

Case ii) $\xi = 1$, critically damped

$R(s) = 1$ as unit impulse input

$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

Now $L^{-1}\{e^{-at} t\} = \dfrac{1}{(s+a)^2}$

$$\therefore \quad L^{-1}[C(s)] = L^{-1}\left[\omega_n^2 \cdot \frac{1}{(s+\omega_n)^2}\right]$$

$$\therefore \quad c(t) = \omega_n^2 \cdot t\, e^{-\omega_n t}$$

Case iii) $\xi > 1$, overdamped

$R(s) = 1$ as unit impulse input

The roots are real and negative

$$\therefore \quad C(s) = \frac{\omega_n^2}{\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}$$

Finding partial fractions we get,

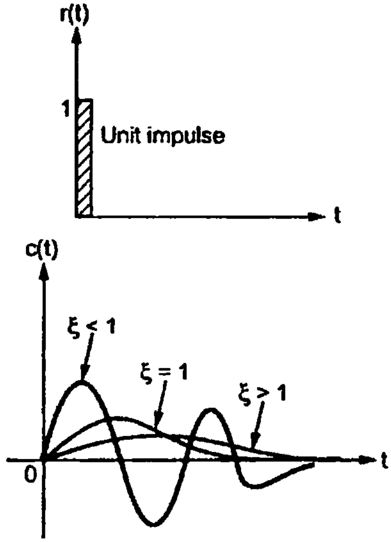
$$= \omega_n^2 \left[\frac{-\dfrac{1}{2\omega_n\sqrt{\xi^2-1}}}{\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)} + \frac{\dfrac{1}{2\omega_n\sqrt{\xi^2-1}}}{\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}\right]$$

$$\therefore \quad L^{-1}[C(s)] = c(t)$$

$$\therefore \quad c(t) = \frac{\omega_n^2}{2\omega_n\sqrt{\xi^2-1}}\left[-e^{\left[-\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right]t} + e^{\left[-\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right]t}\right]$$

$$\therefore \quad \boxed{c(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[-e^{-\omega_n t\left(\xi+\sqrt{\xi^2-1}\right)} + e^{-\omega_n t\left(\xi-\sqrt{\xi^2-1}\right)}\right]}$$

The responses are shown in the following figure.



Example 7.30 : *The open loop transfer function of a unity FBCS (Feed Back Control System) is given by* $G(s)=\frac{K}{s(sT+1)}$.

i) By what factor the amplifier gain K should be multiplied so that damping ratio is increased from 0.2 to 0.8.

ii) By what factor the time constant T should be multiplied so that the damping ratio is reduced from 0.6 to 0.3. **(M.U. : Jan.-92, May-97)**

Solution : Given $G(s) = \frac{K}{s(sT+1)}$ and $H(s) = 1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s(sT+1)}}{1+\frac{K}{s(sT+1)}}$$

$$= \frac{K}{Ts^2+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$$\omega_n^2 = \frac{K}{T} \quad \text{and} \quad 2\xi\omega_n = \frac{1}{T}$$

$$\therefore \quad \boxed{\omega_n = \sqrt{\frac{K}{T}}} \quad \text{and} \quad \boxed{\xi = \frac{1}{2\sqrt{KT}}}$$

Case i) : $\xi_1 = 0.2$, $K = K_1$ while $\xi_2 = 0.8$, $K = K_2$, T = constant

$$\therefore \quad 0.2 = \frac{1}{2\sqrt{K_1 T}} \quad \text{and} \quad 0.8 = \frac{1}{2\sqrt{K_2 T}}$$

$$\therefore \quad \frac{0.2}{0.8} = \frac{\dfrac{1}{2\sqrt{K_1 T}}}{\dfrac{1}{2\sqrt{K_2 T}}} = \frac{\sqrt{K_2}}{\sqrt{K_1}}$$

$$\therefore \quad \frac{1}{16} = \frac{K_2}{K_1} \qquad \text{... squaring both sides}$$

$$\therefore \quad K_2 = \frac{1}{16} K_1$$

So K must be multiplied by 1/16 to increase ξ from 0.2 to 0.8.

Case ii) : $\xi_1 = 0.6$, $T = T_1$ while $\xi_2 = 0.3$, $T = T_2$, K = constant

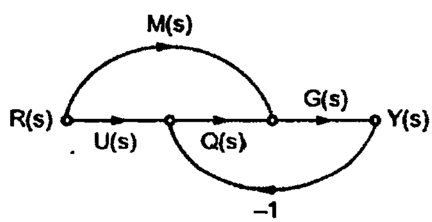
$$\therefore \quad 0.6 = \frac{1}{2\sqrt{KT_1}} \quad \text{and} \quad 0.3 = \frac{1}{2\sqrt{KT_2}}$$

$$\therefore \quad \frac{0.6}{0.3} = \frac{\sqrt{T_2}}{\sqrt{T_1}} \qquad \text{i.e. } 2 = \sqrt{\frac{T_2}{T_1}}$$

$$\therefore \quad 4 = \frac{T_2}{T_1} \qquad \text{i.e.} \qquad T_2 = 4T_1$$

So T must be multiplied by 4 to reduce ξ from 0.6 to 0.3.

➡ **Example 7.31 :** *For the system shown in the given figure, calculate the sensitivity of the closed loop system with respect to the function G(s). Does the sensitivity depend on U(s) or M(s) ?*



Solution : From given signal flow graph,

$$T_1 = U(s)Q(s)G(s), \qquad T_2 = M(s)G(s)$$

$$L_1 = -Q(s)G(s), \qquad \Delta_1 = 1, \qquad \Delta_2 = 1$$

$$\therefore \quad T(s) = \frac{Y(s)}{R(s)} = \frac{U(s)\,Q(s)\,G(s) + M(s)\,G(s)}{1 + Q(s)\,G(s)}$$

$$\therefore \quad T(s) = \frac{G(s)[U(s)Q(s) + M(s)]}{1 + Q(s)G(s)}$$

Example 7.34 : *A second order system has unity feedback and open loop transfer function.*

$$G(s)=\frac{500}{s(s+15)}$$

a) Draw the block diagram for closed loop system.

b) What is characteristic equation?

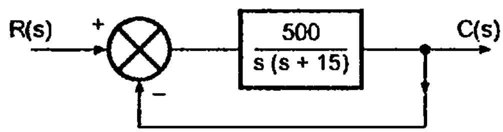
c) What is damping ratio and natural frequency values?

d) Calculate T_p (peak time), M_p (peak overshoot) and T_s (settling time) for the system output response when excited by unit step input.

e) Sketch the transient response for unit step input.

f) If input is ramp of 0.5 rad/sec, calculate steady state error. **(M.U. : Nov. - 95)**

Solution : a) Block diagram is,



b) Characteristic equation is

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1+\frac{500}{s(s+15)} = 0 \quad \text{i.e.} \quad s^2+15s+500 = 0$$

c) Comparing above with standard equation

$$s^2+2\xi\omega_n s+\omega_n^2 = 0$$

$$\therefore \quad \omega_n^2 = 500 \qquad \text{i.e.} \quad \omega_n = \mathbf{22.36068\ rad/sec}$$

$$2\xi\omega_n = 15 \qquad \text{i.e.} \quad \xi = \frac{15}{2\omega_n} = 0.3354$$

System is underdamped.

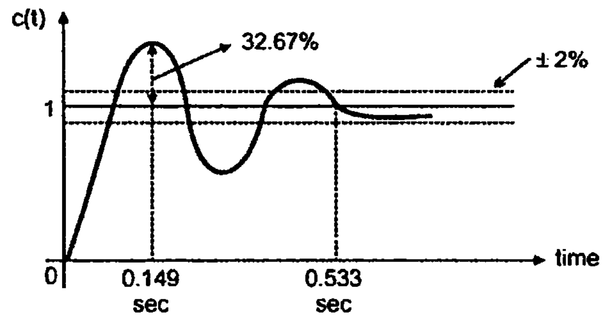
d)
$$\omega_d = \omega_n\sqrt{1-\xi^2} = 22.36\times\sqrt{1-(0.3354)^2} = \mathbf{21.0648\ rad/sec.}$$

$$T_p = \frac{\pi}{\omega_d} = \mathbf{0.1491\ sec.}$$

$$\%M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}\times 100 = 32.677\%$$

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.3354\times 22.36} = 0.5333\ \text{sec.}$$

e)



f)

$$r(t) = 0.5t$$

$$R(s) = \frac{0.5}{s^2}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{s \cdot \frac{0.5}{s^2}}{1+\frac{500}{s(s+15)}}$$

$$= \lim_{s \to 0} \frac{0.5}{s+\frac{s \times 500}{s(s+15)}} = \frac{0.5 \times 15}{500}$$

$$e_{ss} = \mathbf{0.015}$$

Example 7.35 : *For a unity feedback system $G(s) = \frac{36}{s(s+0.72)}$. Determine characteristic equation and hence calculate damping ratio, peak time, settling time, peak overshoot and number of cycles completed before output settles for a unit step input.* **(M.U. : May - 93)**

Solution : Characteristic equation is 1 + G(s)H(s) = 0

$$\therefore \quad 1+\frac{36}{s(s+0.72)} = 0$$

i.e. $s^2 + 0.72s + 36 = 0$

Comparing with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore \quad \omega_n^2 = 36, \quad \therefore \quad \omega_n = 6 \ \text{rad/sec}$$

and $2\xi\omega_n = 0.72, \quad \therefore \quad \xi = 0.06$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$\therefore \quad \omega_d = 5.9891 \ \text{rad/sec.}$$

$\therefore$ Peak time $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{5.9891} = 0.5245$ sec.

Settling time $T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.06 \times 6} = 11.11$ sec.

Peak overshoot $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.8279$

i.e. $\% M_p = 82.79\%$

Now $\omega_d = 5.9891$ rad/sec.

$= 2\pi f_d$

$\therefore$ $f_d = \frac{\omega_d}{2\pi} = 0.9531$ cycles/sec.

i.e. $T_d = \frac{1}{0.9531} = 1.0491$ sec/cycle ... Time for one cycle.

Settling time is 11.11 sec therefore output will perform,

$\frac{11.11}{1.0491} = 10.59$ ... Cycles before it settles.

Example 7.36 : *Following expression denotes the time response of a servomechanism.*

$c(t) = 1 + 0.2e^{-60t} - 1.2\,e^{-10t}$

i) Obtain the expression for the closed loop transfer function of the system.

ii) Determine the undamped natural frequency and damping ratio. Assume unit step input

(M.U. : Nov. - 96)

Solution :

i) $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$

Taking Laplace transform

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$= \frac{(s+10)(s+60) + 0.2s(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s(s+60)(s+10)}$$

$$= \frac{600}{s(s+60)(s+10)}$$

$$R(s) = \frac{1}{s}$$

$\therefore$ Closed loop transfer function $= \dfrac{C(s)}{R(s)} = \dfrac{\dfrac{600}{s(s+60)(s+10)}}{\dfrac{1}{s}} = \dfrac{600}{s^2+70s+600}$

ii) Characteristic equation is

$$s^2 + 70s + 600 = 0$$

Comparing with

$$s^2 + 2\xi\,\omega_n s + \omega_n^2 = 0$$

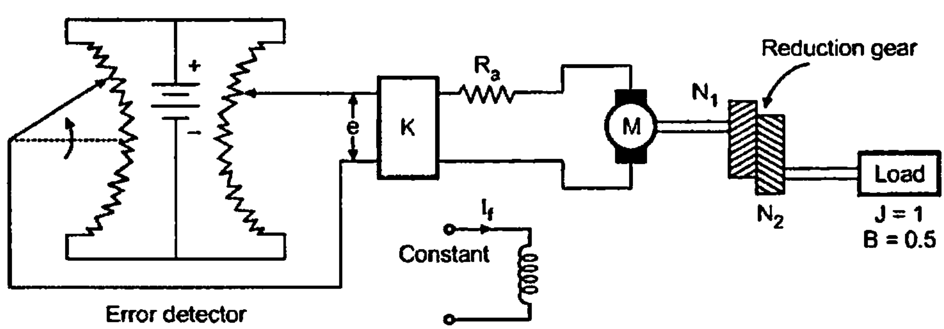
$$\omega_n^2 = 600, \quad \therefore \omega_n = \textbf{24.4948 rad/sec.}$$

and $\quad 2\xi\omega_n = 70, \quad \xi = \mathbf{1.4288}$

As $\xi > 1$, system is overdamped hence output is purely exponential without oscillations.

Example 7.37 : *A position servomechanism has as error detector generating 1 V/rad error. It is followed by an amplifier of gain 'K'. The amplifier feeds the armature of a motor which has armature resistance of 1 Ω and negligible inductance. The motor generates 0.1V/(rad/sec) back e.m.f. and has J = 0.1 kg – m² and viscous damping is negligible. The motor is coupled to the load through a reduction gear with ratio 5. The load has J = 1 kg–m² and viscous damping constant B = 0.5 Nm/(rad/sec). Determine the closed loop transfer function of the system. Also find the damping factor. Take K = 100. What is the effect of increase in K on damping factor.* **(M.U. : Nov. - 95)**

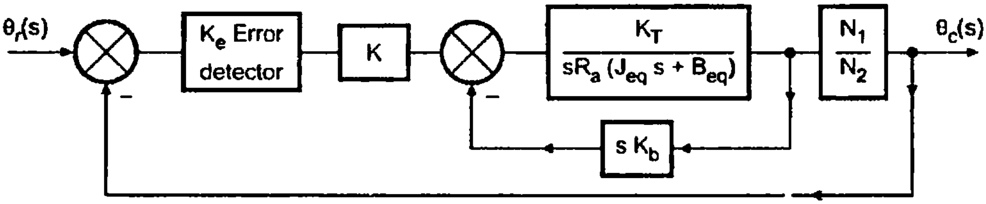
Solution : The circuit diagram is



$$J_{eq} = J_{motor} + J_{Load}\left(\frac{N_1}{N_2}\right)^2 = 0.1 + 1 \times \left(\frac{1}{5}\right)^2 = 0.14 \text{ kg-m}^2$$

$$B_{eq} = B_{motor} + B_{Load}\left(\frac{N_1}{N_2}\right)^2 = 0 + 0.5 \times \left(\frac{1}{5}\right)^2 = 0.1 \text{ Nm/(rad/sec)}$$

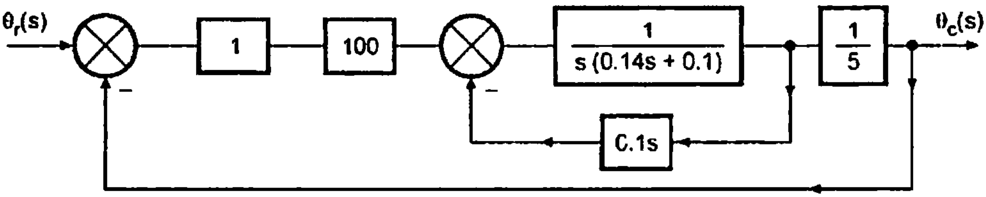
The block diagram is

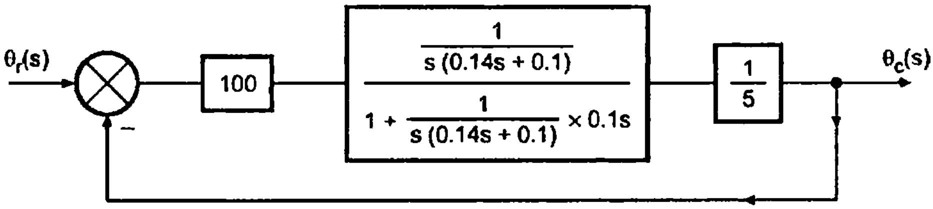


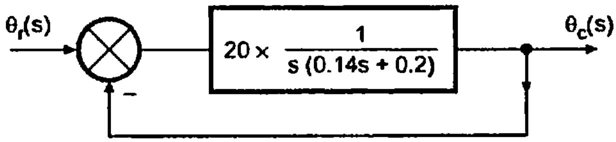
$$K_b = 0.1 \text{ V/(rad/sec)}, \quad K = 100$$

$$K_e = 1 \text{ V/rad}, \quad R_a = 1\ \Omega$$

* Assume torque constant K_T of the motor as 1.







$$\frac{\theta_c(s)}{\theta_r(s)} = \frac{\dfrac{20}{s(0.14s+0.2)}}{1+\dfrac{20}{s(0.14s+0.2)}}$$

$$\frac{\theta_c(s)}{\theta_r(s)} = \frac{20}{0.14s^2+0.2s+20} = \frac{142.857}{s^2+1.4285s+142.857}$$

The characteristic equation is

$$s^2 + 1.4285s + 142.857 = 0$$

Comparing with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = 142.857, \quad \omega_n = \mathbf{11.952\ rad/sec}$$

and $\quad 2\xi\omega_n = 1.4285, \quad \therefore \quad \xi = \mathbf{0.0597}$

Damping factor is 0.0597.

If gain K is increased, then ω_n^2 value will increase but '$2\xi\omega_n$' will remain constant. Therefore increase in K will cause decrease in damping factor.

Example 7.38 : *A system has 30% overshoot and settling time of 5 seconds, for a unit step input. Determine the transfer function. Calculate peak time and output response. Assume e_{ss} as 2%.* **(M.U. : Nov. - 95)**

Solution : $\quad M_p = 30\%$

$$\therefore \quad 30 = e^{\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$\therefore \quad 0.3 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\therefore \quad \ln(0.3) = -\frac{\pi\xi}{\sqrt{1-\xi^2}}$$

$$\therefore \quad \xi^2 = 0.128, \quad \text{i.e.} \quad \xi = 0.3578$$

$$T_s = \frac{4}{\xi\omega_n} = 5$$

$$\therefore \quad \omega_n = \frac{4}{5\xi} = 2.2358 \ \text{rad/sec}$$

$$\therefore \quad \text{T.F.} = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{4.9988}{s^2 + 1.599s + 4.9988}$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \mathbf{1.5047\ sec.}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Where $\quad \omega_d = \omega_n\sqrt{1-\xi^2} = \mathbf{2.0877\ rad/sec.}$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] \text{ radians} = 1.2048 \text{ rad}$$

$$\therefore \quad c(t) = 1 - \frac{e^{-0.8t}}{\sqrt{1-(0.3578)^2}} \sin(2.0877t + 1.2048)$$

$$\therefore \quad c(t) = \mathbf{1 - 1.07\, e^{-0.8t} \sin(2.0877t + 1.2048)}$$

Example 7.39 : *A closed loop system has two complex conjugate poles at $s_1, s_2 = -2 \pm j\,1$. Determine the form of transfer function and values of ω_n, T_p, T_r, T_s and M_p assuming standard second order system.* **(M.U. : Nov. - 95)**

Solution :
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

while $s_1, s_2 = -2 \pm j1$

$\therefore$ Denominator of T.F $= (s + 2 + j1)(s + 2 - j1) = (s+2)^2 - (j1)^2$

$$= s^2 + 4s + 5$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{5}{s^2 + 4s + 5}}$$

Comparing with standard form

$$\therefore \quad \omega_n^2 = 5, \quad \therefore \omega_n = \mathbf{2.236}$$

and $$2\xi\omega_n = 4, \quad \therefore \xi = \frac{4}{2 \times \omega_n} = \mathbf{0.8944}$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 0.9999 \text{ rad/sec} \approx \mathbf{1\ rad/sec}$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] \text{rad} = 0.4636 \text{ radians}$$

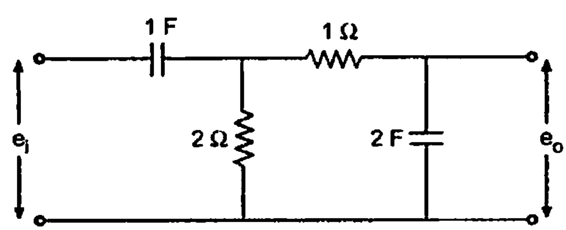
$$T_p = \frac{\pi}{\omega_d} = \mathbf{3.1415\ sec}$$

$$T_s = \frac{4}{\xi\omega_n} = \mathbf{2\ sec}$$

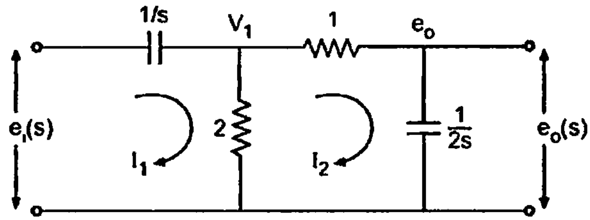
$$T_r = \frac{\pi - \theta}{\omega_d} = \mathbf{2.6779\ sec}$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{0.1867\%}$$

Example 7.40 : *Find the impulse response of electrical circuit given below.* **(M.U. : May-95)**



Solution : Taking Laplace of the network



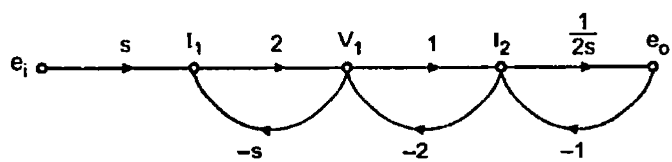
$$I_1 = \frac{e_i - v_1}{\frac{1}{s}} = s(e_i - v_1) \quad \text{..... (1)}$$

$$v_1 = 2(I_1 - I_2) \quad \text{.... (2)}$$

$$I_2 = \frac{v_1 - e_0}{1} \quad \text{..... (3)}$$

$$e_o = I_2 \times \frac{1}{2s} \quad \text{..... (4)}$$

∴ Signal flow graph is



$$T_1 = 1$$

$$L_1 = -2s \qquad L_2 = -2 \qquad L_3 = -\frac{1}{2s}$$

$$L_1 \; L_3 = \text{Non touching} = 1$$

∴ $\Delta = 1 - [L_1 + L_2 + L_3] + [L_1 \; L_3] = 1 + 2s + 2 + \frac{1}{2s} + 1$

$$= 4 + 2s + \frac{1}{2s}$$

$\Delta_1 = 1$ as all loops are touching to T_1

∴ $\frac{e_o(s)}{e_i(s)} = \frac{T_1 \Delta_1}{\Delta}$ Using Mason's gain formula

$$= \frac{1}{4 + 2s + \frac{1}{2s}} = \frac{2s}{4s^2 + 8s + 1}$$

$$\frac{e_o(s)}{e_i(s)} = \frac{0.5s}{s^2 + 2s + 0.25}$$

For impulse input $e_i(s) = 1$

$$e_o(s) = \frac{0.5s}{s^2+2s+0.25} = \frac{0.5s}{(s+0.1339)(s+1.866)}$$

$$= \frac{A}{(s+0.1339)} + \frac{B}{(s+1.866)}$$

$$A = -0.0386, \quad B = 0.5386$$

$$\therefore \quad e_o(s) = \frac{-0.0386}{(s+0.1339)} + \frac{0.5386}{(s+1.866)}$$

$$\therefore \quad e_o(t) = L^{-1}\{e_o(s)\}$$

$$e_o(t) = \mathbf{0.5386\ e^{-1.866t} - 0.0386\ e^{-0.1339t}}$$

Example 7.41 : *A system has unit step response of $c(t) = 1 - e^{-0.1t}$. Determine its impulse response and ramp response. Assume zero initial conditions.* **(M.U. : June - 94)**

Solution : For unit step input

$$R(s) = \frac{1}{s}$$

Taking Laplace of c(t),

$$C(s) = \frac{1}{s} - \frac{1}{s+0.1} = \frac{0.1}{s(s+0.1)}$$

$$\therefore \quad \text{T.F} \frac{C(s)}{R(s)} = \frac{\frac{0.1}{s(s+0.1)}}{\frac{1}{s}} = \frac{0.1}{s+0.1}$$

For impulse response, R(s) = 1

$$\therefore \quad C(s) = \frac{0.1}{s+0.1}$$

$$\therefore \quad c(t) = L^{-1}\{C(s)\} = 0.1\ e^{-0.1t}$$

For ramp, $R(s) = \frac{1}{s^2}$

$$\therefore \quad C(s) = \frac{0.1}{s^2(s+0.1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+0.1}$$

$$A(s+0.1) + Bs(s+0.1) + Cs^2 = 0.1$$

$$B + C = 0, \quad A + 0.1B = 0, \quad 0.1A = 0.1$$

$$\therefore \quad A = 1, \quad B = -10, \quad C = 10$$

$$\therefore \quad C(s) = \frac{1}{s^2} - \frac{10}{s} + \frac{10}{s+0.1}$$

$$\therefore \quad c(t) = L^{-1}\{C(s)\}$$

$$= t - 10 + 10e^{-0.1t}$$

Example 7.42 : *The open loop transfer function of unity feedback system is*

$$G(s) = \frac{K}{s(Ts+1)}$$

By what factor the gain K should be multiplied so that damping ratio is increased from 0.3 to 0.8. By what factor time constant should be multiplied so that damping ratio is reduced from 0.6 to 0.4. **(M.U. : Jan.-92, May-97)**

Solution : The characteristic equation is $1 + G(s)H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s(Ts+1)} = 0$$

$$\therefore \quad Ts^2 + s + K = 0$$

Dividing by T, to compare it with $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$$\therefore \quad s^2 + \frac{1}{T}s + \frac{K}{T} = 0$$

$$\therefore \quad \omega_n^2 = \frac{K}{T}, \quad \therefore \omega_n = \sqrt{\frac{K}{T}}$$

and $$2\xi\omega_n = \frac{1}{T}, \quad \therefore \xi = \frac{1}{2\sqrt{KT}}$$

a) ξ is increased from 0.3 to 0.8

$\xi_1 = 0.3$, let $K = K_1$ and $\xi_2 = 0.8$, $K = K_2$

$$\therefore \quad \xi_1 = \frac{1}{2\sqrt{K_1 T}} \quad \xi_2 = \frac{1}{2\sqrt{K_2 T}}$$

$$\therefore \quad \frac{\xi_1}{\xi_2} = \sqrt{\frac{K_2}{K_1}} \qquad \therefore \quad \frac{0.3}{0.8} = \sqrt{\frac{K_2}{K_1}}$$

$$\therefore \quad K_2 = 0.1406\ K_1$$

So gain should be multiplied by 0.1406.

b) ξ is reduced from 0.6 to 0.4

$\xi_1 = 0.6,$ let $T = T_1$ and $\xi_2 = 0.4,$ $T = T_2$

$$\therefore \quad \xi_1 = \frac{1}{2\sqrt{KT_1}} \qquad \xi_2 = \frac{1}{2\sqrt{KT_2}}$$

$$\therefore \quad \frac{\xi_1}{\xi_2} = \sqrt{\frac{T_2}{T_1}} \qquad \therefore \quad \frac{0.6}{0.4} = \sqrt{\frac{T_2}{T_1}}$$

$$\therefore \quad T_2 = 2.25\ T_1$$

So time constant should be multiplied by 2.25.

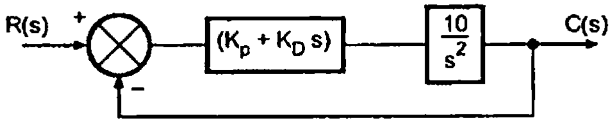
Example 7.43 : *For the control system shown*

Show

i) Trajectory on which parabolic error constant K_a is 40 sec^{-2}

ii) Trajectory on which ω_n = 40 rad/sec.

on the parameter plane $K_p - K_D$. Take K_D on x axis and K_p on y axis. (M.U. : Nov.-96)



Solution :

$$G(s)H(s) = \frac{10(K_p + K_D s)}{s^2}$$

i) $$K_a = \lim_{s \to 0} s^2\ G(s)H(s)$$

$$= \lim_{s \to 0} s^2 \cdot \frac{10(K_p + K_D s)}{s^2}$$

$$40 = 10\ K_p$$

$$K_p = 4 \text{ is line on which } K_a = 40\ sec^{-2}$$

ii) Characteristic equation is $1 + G(s)H(s) = 0$

$$1 + \frac{10(K_p + K_D s)}{s^2} = 0$$

$$s^2 + 10\,K_D s + 10K_p = 0$$

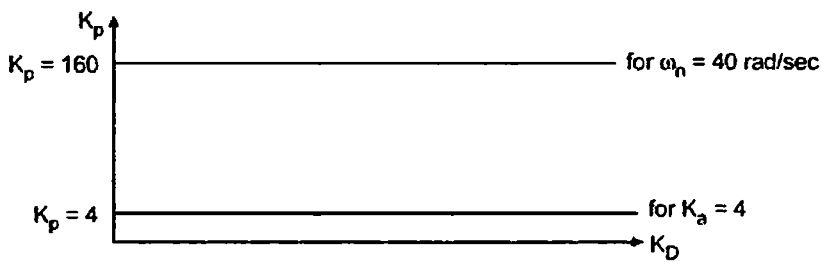
Compare with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore \quad \omega_n^2 = 10K_p \qquad \omega_n = 3.162\sqrt{K_p}$$

$$\therefore \quad 40 = 3.162\sqrt{K_p} \qquad \sqrt{K_p} = 12.649$$

$$\therefore \quad K_p = 160 \text{ is line on which } \omega_n = 40 \text{ rad/sec.}$$



Example 7.44 : *A servomechanism is used to control the angular position θ_o of a mass, using θ_i as the reference input signal. The moment of inertia of the moving parts referred to load is 200 kg $-m^2$ and the motor torque at the load is 2×10^4 Nm/rad of error. The damping torque coefficient referred to load is 3×10^3 Nm/(rad/sec). Find :*

i) The step response of the system to a step input of one radian.

ii) The natural frequency of oscillations, damped frequency of ascillations, peak time and overshoot.

iii) The steady state error which exists when a step torque of 1000 Nm is applied to load shaft.

iv) The steady state error if the reference input is a constant angular velocity of 1 r.p.m.

Solution : The open loop transfer function of the system with,

torque constant $K_T = 2 \times 10^4$ Nm/rad of error

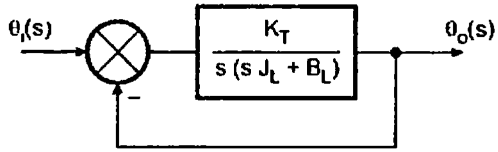
moment of inertia $J_L = 200$ kg $-$ m^2

damping torque constant $B_L = 3 \times 10^3$ Nm/(rad/sec) is

$$G(s) = \frac{K_T}{s(s\,T_L + B_L)}$$

Assume $H(s) = 1$

The system can be shown as below :



$$\therefore \quad \frac{\theta_o(s)}{\theta_i(s)} = \frac{\dfrac{K_T}{s(sJ_L + B_L)}}{1+\dfrac{K_T}{s(sJ_L + B_L)}} = \frac{K_T}{s^2 J_L + s B_L + K_T}$$

$$= \frac{\left(\dfrac{K_T}{J_L}\right)}{s^2 + s\left(\dfrac{B_L}{J_L}\right)+\left(\dfrac{K_T}{J_L}\right)}$$

Substituting the values of K_T, J_L and B_L

$$\therefore \quad \frac{\theta_o(s)}{\theta_i(s)} = \frac{100}{s^2 + 15s + 100}$$

$$\therefore \quad \omega_n^2 = 100 \quad \text{i.e.} \quad \omega_n = 10 \ \textbf{rad/sec}$$

and $$2\xi\omega_n = 15 \quad \text{i.e.} \quad \xi = \frac{15}{2\times 10} = 0.75$$

$$\therefore \quad \omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.75)^2} = \textbf{6.6143 rad/sec}$$

i) The step response for input of one radian is

$$\theta_o(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

where $$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = \tan^{-1}\left[\frac{\sqrt{1-(0.75)^2}}{0.75}\right]$$ use radian mode

$$= 0.722 \ \text{radians}$$

$$\therefore \quad \theta_o(t) = \mathbf{1 - 1.511\, e^{-7.5t} \sin(6.6143\, t + 0.722)\ radians}$$

$$\therefore \quad T_p = \frac{\pi}{\omega_d} = \mathbf{0.47\ sec}$$

and $$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = e^{-0.75\pi/\sqrt{1-(0.75)^2}} \times 100$$

$$= 2.83\%$$

iii) Load torque is 1000 Nm i.e. $T_L = 1000$ Nm

But $$K_T = \frac{T_L}{\text{rad of error}}$$

$$\therefore \quad \text{error in radians} = \frac{T_L}{K_T} = \frac{1000}{2\times 10^4} = 0.05 \text{ radians i.e. } 2.86°$$

iv) The input now is ramp input of magnitude 1 r.p.m.

i.e. $$\omega = 1 \text{ r.p.m.} = \frac{\pi}{30} \text{ rad/sec}$$

As 1 revolution i.e. 2π radians in 60 sec.

$\therefore$ Magnitude of ramp input = $A = \dfrac{\pi}{30}$

Now $$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s \cdot \frac{K_T}{s\,(s\,J_L + B_L)}$$

$$= \frac{K_T}{B_L} = \frac{2\times 10^4}{3\times 10^3} = 6.67$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{\left(\frac{\pi}{30}\right)}{6.67} = 0.0157 \text{ rad} = 0.9°$$

Example 7.45 : *A servomechanism is designed to keep a radar antenna pointed at a flying aeroplane. If the aeroplane is flying with a velocity of 600 Km/hr, at a range of 2 Km and the maximum tracking error is to be within 0.1°, determine the required velocity error coefficient.* **(Gate)**

Solution : The linear velocity of aeroplane is,

$$V = 600 \text{ km/hr}$$

The range of the aeroplane is,

$$r = 2 \text{ km}$$

Hence the angular velocity of the aeroplane is,

$$\omega = \frac{V}{r} = \frac{600}{2} = 300 \text{ rad/hr} = \frac{300}{3600} = \frac{1}{12} \text{ rad/sec}$$

This is the ramp type of input to the system.

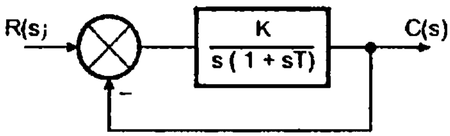
$\therefore \quad r(t) = \omega t = \frac{1}{12} t$

$\therefore \quad R(s) = \frac{1}{12s^2}$

The velocity error coefficient for system is to be calculated, for the ramp input. Hence for finite existence of velocity error coefficient, system must be of TYPE 1.

$\therefore \quad G(s)H(s) = \frac{K}{s(1+sT)}$

with H(s) = 1, the system can be shown as,



Now $\quad K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s \cdot \frac{K}{s(1+sT)} = K$

$e_{ss} = \frac{A}{K_v}$ where A = magnitude of ramp input

$\therefore \quad e_{ss} = \frac{(1/12)}{K}$

and $\quad e_{ss} = 0.1^\circ$ allowed

$\therefore \quad 0.1 = \frac{1}{12\,K}$

$\therefore \quad K = \frac{1}{1.2}$ / degree

As $K_v = K$, the require velocity error coefficient is $\frac{1}{1.2}$ / degree.

Example 7.46 : *The open loop transfer function of a unity feedback system is given by, $G(s) = e^{-2s}$. Sketch the output of the feedback system for a unit step input. Assume that the system is initially relaxed.* **(Gate)**

Solution : As system has unity feedback,

$\therefore \quad H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{e^{-2s}}{1+e^{-2s}}$$

Dividing both numerator and denominator by e^{-2s},

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{1}{1+e^{2s}}$$

The system is excited by unit step input

$$\therefore \quad r(t) = 1 \quad \text{for} \quad t \geq 0$$

$$\therefore \quad R(s) = \frac{1}{s}$$

Substituting in C(s)

$$\therefore \quad C(s) = \frac{1}{s} \cdot \frac{1}{1+e^{2s}}$$

$$\therefore \quad C(s)\left[1+e^{2s}\right] = \frac{1}{s}$$

$$\therefore \quad C(s) + C(s)\,e^{2s} = \frac{1}{s} \quad \text{... (1)}$$

Now $\quad L\{c(t+T)\} = C(s)\,e^{Ts}$

$$\therefore \quad L^{-1}\{C(s)\,e^{2s}\} = c(t+2)$$

Hence taking Laplace inverse of equation (1),

$$c(t) + c(t+2) = u(t)$$

where $\quad u(t) =$ unit step function

$$\therefore \quad c(t+2) = u(t) - c(t) \quad \text{... (2)}$$

To obtain the output waveform let us obtain the values of c(t) for various values of *t*.

Now at t = 0, c(t) has its initial value. Using initial value theorem,

$$c(0) = \lim_{t \to 0} c(t) = \lim_{s \to \infty} s\,C(s) = \lim_{s \to \infty} s \cdot \frac{1}{s\left(1+e^{2s}\right)} = 0$$

$$\therefore \quad c(0) = 0$$

Substituting $t = 0$ in equation (2),

$$\therefore \quad c(2) = u(t) - c(0) = 1 - 0$$

$$= 1 \quad \text{as} \quad u(t) = 1 \quad \text{at} \quad t = 0$$

Substituting t = 2 in equation (2),

$\therefore \qquad c(4) = u(t) - c(2) = 1 - 1 = 0$

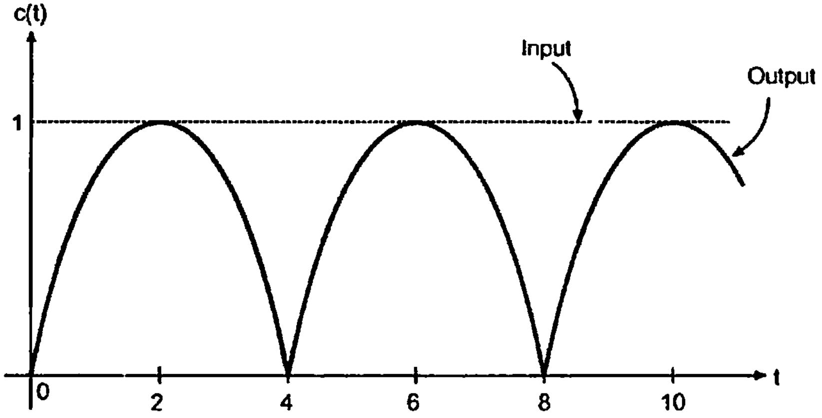
Substituting t = 4 in equation (2),

$\therefore \qquad c(6) = u(t) - c(4) = 1 - 0 = 1$

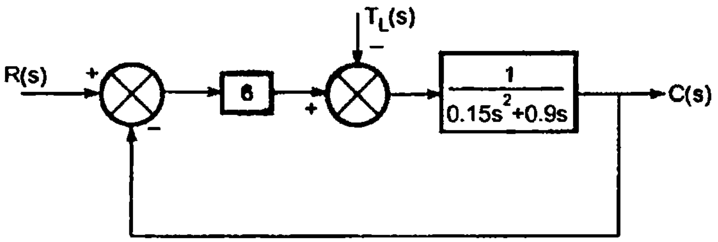
Substituting t = 6 in equation (2),

$\therefore \qquad c(8) = u(t) - c(6) = 1 - 1 = 0$ and so on.

Hence the sketch of the output response of the system for unit step input is as shown in the figure.



Example 7.47 : *For the control system shown in figure.*



I) Determine

i) M_p for a unit step input

ii) e_{ss} for unit ramp input.

II) Calculate the steady state value of the output when the input shaft is held fixed and a sudden torque T_L = 1 Nm is applied. **(M.U. : Dec. - 98)**

Solution : Assume T_L as zero for [I]

i) To calculate M_p,

$$G(s) = \frac{6}{0.15\,s^2 + 0.9s} \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{6}{s\,(0.15\,s+0.9)}}{1+\dfrac{6}{s\,(0.15s+0.9)}}$$

$$= \frac{6}{0.15s^2+0.9s+6} = \frac{40}{s^2+6s+40}$$

Comparing the characteristic equation with,

$$s^2+2\xi\omega_n s+\omega_n^{\,2} = 0$$

$$\therefore \quad \omega_n^2 = 40 \quad \text{i.e.} \quad \omega_n = \mathbf{6.3245}$$

$$2\xi\omega_n = 6$$

$$\therefore \quad \xi = \frac{6}{2\omega_n} \quad \text{i.e.}\ \xi = \mathbf{0.4743}$$

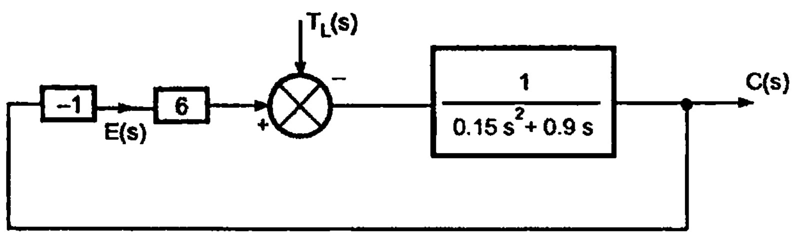
Now, $$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = e^{\frac{-\pi\times 0.4743}{\sqrt{1-(0.4743)^2}}} \times 100 = \mathbf{18.4\%}$$

ii) For unit ramp input,

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s.\frac{6}{s\,(0.15\,s+0.9)}.1 = \mathbf{6.67}$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \mathbf{0.15\ rad}$$

II) For T_L is 1 N-m, assume R(s) as zero. With R(s) zero, the system gets modified as



From the figure we can write.

$$G(s) = \frac{1}{0.15s^2+0.9s}$$

$$\therefore \quad c(t) = \frac{K}{a} - \frac{K}{a} e^{-at}$$

From graph, $\lim_{t \to \infty} c(t) = 2$

$$\therefore \quad 2 = \frac{K}{a} - \frac{K}{a} e^{-\infty} = \frac{K}{a}$$

$$\therefore \quad c(t) = 2 - 2e^{-at}$$

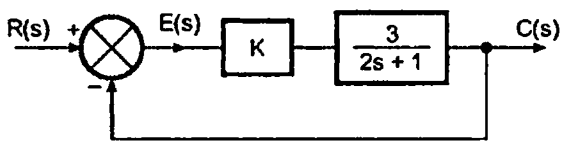
From graph, $c(t) = 0.8$ at $t = 2$

$$\therefore \quad 0.8 = 2 - 2e^{-2a}$$

Solving we get, $a = 0.2554$ and $K = 0.5108$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{0.51098}{s + 0.2554}$$

Example 7.49 : *For what values of K is the time constant of the closed loop system less than 0.2 sec.* **(M.U. : May-2003)**



Solution : The closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{\frac{3K}{2s+1}}{1+\frac{3K}{2s+1}} = \frac{3K}{2s+1+3K} = \frac{\left[\frac{3K}{1+3K}\right]}{\left[1+\frac{2}{1+3K}s\right]}$$

Comparing denominator with 1+Ts, the time constant is $\frac{2}{1+3K}$ which must be maximum 0.2.

$$\therefore \quad \frac{2}{1+3K} \le 0.2$$

$$\therefore \quad K \ge 3$$

Example 7.50 : *A pair of complex conjugate poles in the s-plane is required to meet the various specifications. For each specification sketch the region in the s-plane in which poles should be located*

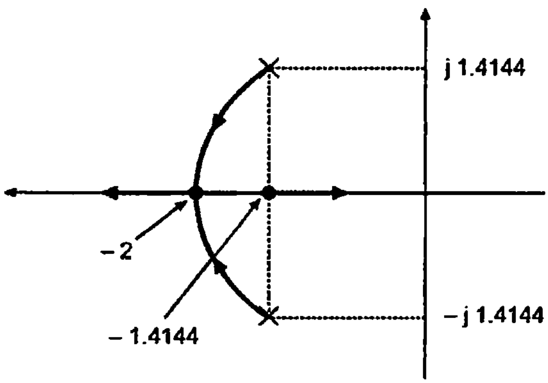
i) $\xi \ge 0.707$, $\omega_n \ge 2$ *rad/sec (positive damping)*

ii) $0 \le \xi \le 0.707$, $\omega_n \le 2$ *rad/sec (positive damping)*

iii) $\xi \le 0.5$, $1 \le \omega_n \le 5$ *rad/sec (positive damping)*

iv) $0.5 \le \xi \le 0.707$, $\omega_n \le 5$ *rad/sec (positive and negative damping)* **(M.U.: May-2003)**

Solution :



i) $\xi = 0.707 \quad \omega_n = 2$

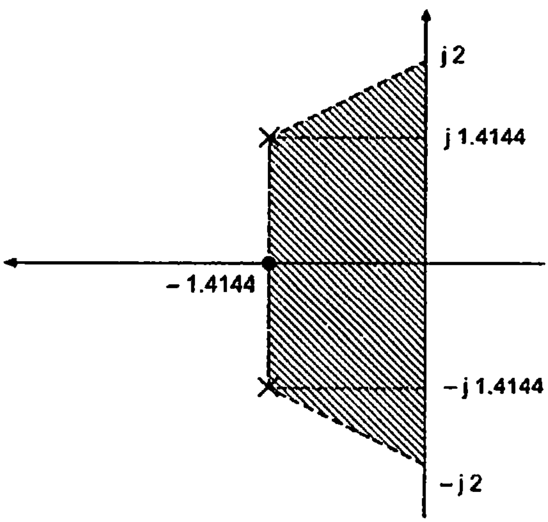
$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$= 1.4144 \text{ rad/sec}$$

$$\therefore \text{ Poles} = -\xi\omega_n \pm j\omega_d$$

$$= -1.4144 \pm j\,1.4144$$

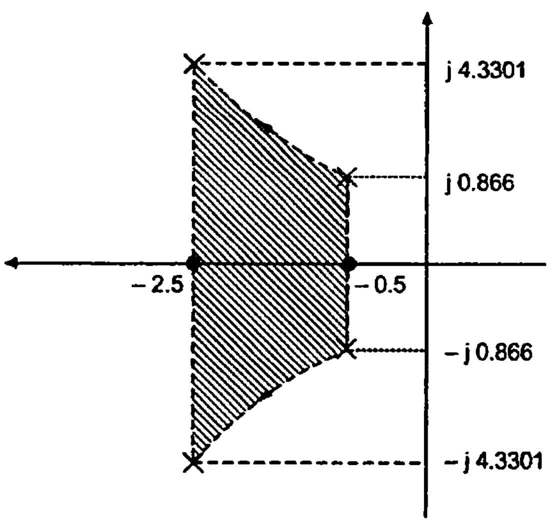
As ξ increases and becomes unity both the poles converge at – 2 and thereafter become real.



ii) $\xi = 0.707$ and $\omega_n = 2$

$\therefore$ Poles = $-1.4144 \pm j\,1.4144$

But for $0 \leq \xi \leq 0.707$ the poles remain complex and at $\xi = 0$ becomes pure imaginary. For $\omega_n \leq 2$, the imaginary part of the poles must be less than 2. So region in s-plane is as shown in adjacent figure.



iii) $\xi \leq 0.5$ and $1 \leq \omega_n \leq 5$

Let $\xi = 0.5$ and $\omega_n = 1$

then $\omega_d = 0.866$

$\therefore$ Poles are $-0.5 \pm j\,0.866$

For $\xi = 0.5$ and $\omega_n = 5$

then $\omega_d = 4.3301$.

$\therefore$ Poles are $-2.5 \pm j\,4.3301$

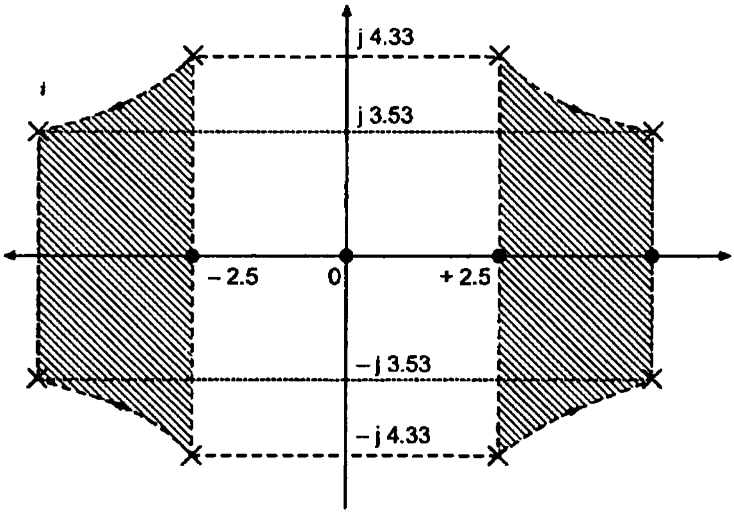
So region is as shown in the adjacent figure.

iv) $0.5 \leq \xi \leq 0.707$ and $\omega_n \leq 5$

For $\xi = 0.5$ and $\omega_n = 5$ the poles are $-2.5 \pm j\,4.3301$.

For $\xi = 0.707$ and $\omega_n = 5$ the poles are $-3.535 \pm j\,3.536$.

For positive and negative damping the region is as shown in the figure.



Example 7.51: *Consider the unity feedback control system whose open loop transfer function is* $G(s) = \dfrac{50}{s(1+0.1\,s)}$. *Determine the steady state error and its variation with time when the input is* $r(t) = 1 + t + t^2$. **(M.U.: Dec.-2003)**

Solution : For steady state error e_{ss}, $G(s)H(s) = \dfrac{50}{s(1+0.1\,s)}$.

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\times\frac{50}{s(1+0.1\,s)} = 50$$

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = 0$$

In the input $r(t) = r_1(t) + r_2(t) + r_3(t)$

$r_1(t) = 1$ i.e. $A_1 = 1$ ... step

$r_2(t) = t$ i.e. $A_2 = 1$ ... ramp

$r_3(t) = t^2 = \frac{1}{2}\times 2\,t^2$ i.e. $A_3 = 2$ ... parabolic

$$\therefore \quad e_{ss1} = \frac{A_1}{1+K_p} = 0, \quad e_{ss2} = \frac{A_2}{K_v} = \frac{1}{50}, \quad e_{ss3} = \frac{A_3}{K_a} = \infty$$

$$\therefore \quad e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \infty$$

For obtaining variation of e_{ss} with time use general error series method.

$$F_1(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{50}{s(1+0.1\,s)}} = \frac{0.1\,s^2+s}{0.1\,s^2+s+50}$$

$$e_{ss}(t) = K_0\, r(t) + K_1\, r'(t) + \frac{K_2}{2!}\, r''(t) + \ldots$$

$$K_0 = \lim_{s\to 0} F_1(s) = \lim_{s\to 0} \frac{0.1\,s^2+s}{0.1\,s^2+s+50} = 0$$

$$K_1 = \lim_{s\to 0} \frac{d\,F_1(s)}{ds} = \lim_{s\to 0} \left\{\frac{(0.1s^2+s+50)(0.2s+1)-(0.1s^2+s)(0.2s+1)}{(0.1s^2+s+50)^2}\right\}$$

$$= \lim_{s\to 0} \left\{\frac{50\,(0.2s+1)}{(0.1s^2+s+50)^2}\right\} = \frac{50}{(50)^2} = \frac{1}{50} = 0.02$$

$$K_2 = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} = \lim_{s\to 0} \frac{d}{ds}\left\{\frac{dF_1(s)}{ds}\right\} = \lim_{s\to 0} \frac{d}{ds}\left\{\frac{50\,(0.2s+1)}{(0.1s^2+s+50)^2}\right\}$$

$$= \lim_{s\to 0} \left\{\frac{(0.1\,s^2+s+50)^2\,(10)-(10\,s+50)\,(2)\,(0.1\,s^2+s+50)\,(0.2s+1)}{(0.1\,s^2+s+50)^4}\right\}$$

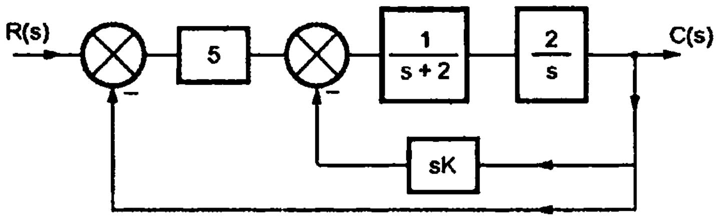
$$\therefore \quad K_2 = \frac{(50)^2\,(10)-(50)\,(2)\,(50)\,(1)}{(50)^4} = 0.0032$$

$$r(t) = 1+t+t^2, \qquad r'(t) = 1+2t, \qquad r''(t) = 2$$

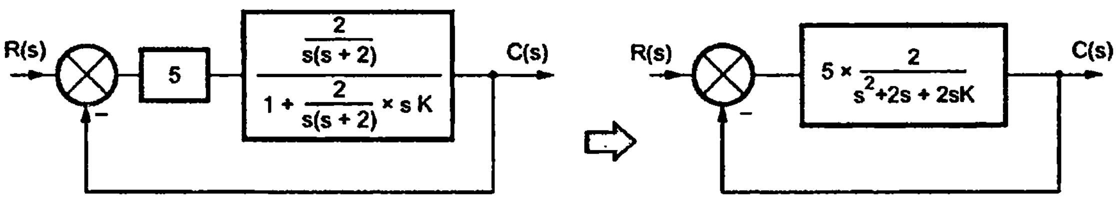
$$\therefore \quad e_{ss}(t) = 0.02\,(1+2t) + \frac{0.0032}{2}\,(2) = \mathbf{0.04t + 0.0232}$$

Example 7.52 : *In the block diagram given below, determine the output rate factor that yields a response to a step input command having a maximum overshoot of 10 %.*

(M.U. : Dec.-2003)



Solution : Solve the internal minor feedback loop,



It's a unity feedback system with $G(s) = \dfrac{10}{s^2 + 2s + 2sK}$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{10}{s^2+2s+2sK}}{1+\dfrac{10}{s^2+2s+2sK}} = \frac{10}{s^2+s\,(2+2K)+10}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$\omega_n^2 = 10$ i.e. $\omega_n = \sqrt{10} = 3.16227$ rad/sec ... (1)

$2\xi\omega_n = 2 + 2K$ i.e. $\xi = \dfrac{2+2K}{2\times\sqrt{10}} = \dfrac{1+K}{\sqrt{10}}$ ... (2)

Given, $M_p = 10\ \%$

$\therefore \quad 10 = 100\, e^{-\pi\xi/\sqrt{1-\xi^2}}$

Solving for ξ, $\xi = 0.5911$

Using (2), $0.5911 = \dfrac{1+K}{\sqrt{10}}$

$\therefore \quad K = 0.86922$ ... Output rate factor

Example 7.53 : *A feedback control system is represented by the differential equation,*

$$\frac{d^2c}{dt^2}+6.4\frac{dc}{dt} = 160\ e \text{ where } e = r - 0.4\ c.$$

The variable c denotes output. Find the value of the damping ratio and what information does this convey about the transient behaviour of the system.

(M.U. : Dec.-2003, Dec.-2006)

Solution : Substituting value of e,

$$\frac{d^2c}{dt^2}+6.4\frac{dc}{dt} = 160\ (r-0.4\ c)$$

$$\therefore \frac{d^2c}{dt^2}+6.4\frac{dc}{dt}+64\ c = 160\ r$$

Taking Laplace transform of both the sides and neglecting initial conditions,

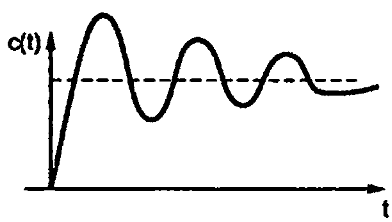
$$s^2\ C(s)+6.4\ s\ C(s)+64\ C(s) = 160\ R(s)$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{160}{s^2+6.4\ s+64} \quad \text{... Transfer function}$$

Comparing denominator with $s^2+2\xi\omega_n\ s+\omega_n^2$,

$$\omega_n^2 = 64 \quad \text{and} \quad 2\xi\omega_n = 6.4$$

i.e. $$\omega_n = \mathbf{8\ rad/sec.} \quad \text{and} \quad \xi = \frac{6.4}{2\times 8} = \mathbf{0.4.}$$



As the damping ratio ξ is less than unity, the system is **underdamped** and will produce the time response with **damped oscillatory** transients for the step input as shown in the figure.

Example 7.54 : *For a second order undamped system, show that* $c(t) = 1 - \cos \omega_n t$.

(M.U. : May-2004)

Solution : For an underdamped second order system, the output is given by,

$$c(t) = 1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) \quad \text{... (1)}$$

where $\omega_d = \omega_n \sqrt{1-\xi^2}$ and $\theta = \tan^{-1} \dfrac{\sqrt{1-\xi^2}}{\xi}$.

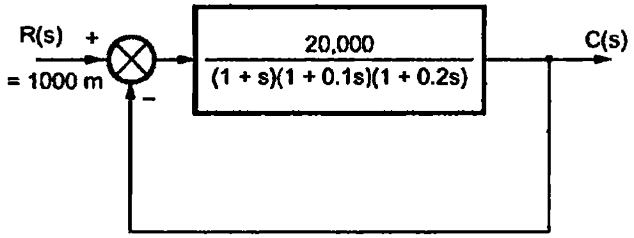
For an undamped system, $\xi = 0$.

$\therefore \quad \omega_d = \omega_n$ and $\theta = \tan^{-1} \infty = \dfrac{\pi}{2}$ rad $= 90°$.

Using in (1), $c(t) = 1 - \dfrac{e^0}{1} \sin(\omega_n t + 90°)$ but $\sin(\omega_n t + 90°) = \cos \omega_n t$

$\therefore \quad c(t) = 1 - \cos \omega_n t$ ... **Proved**

Example 7.55 : *For the block diagram shown below first obtain steady state output and then obtain steady state error.* **(M.U. : May-2004)**



Solution : From the given system, $G(s) = \dfrac{20000}{(1+s)(1+0.1\,s)(1+0.2\,s)}$, $H(s) = 1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{20000}{(1+s)(1+0.1\,s)(1+0.2\,s)}}{1+\dfrac{20000}{(1+s)(1+0.1\,s)(1+0.2\,s)}}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{20000}{(1+s)(1+0.1\,s)(1+0.2\,s)+20000} \quad \text{and} \quad R(s) = \frac{1000}{s}$$

$$\therefore C(s) = \left[\frac{20000}{(1+s)(1+0.1s)(1+0.2s)+20000}\right] \times \frac{1000}{s} = \frac{20000000}{s\,[(1+s)(1+0.1\,s)(1+0.2\,s)+20000]}$$

$$\therefore \quad C_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s \times \left\{\frac{20000000}{s\,[(1+s)(1+0.1\,s)(1+0.2\,s)+20000]}\right\}$$

$$\therefore \quad C_{ss} = \frac{20000000}{1+20000} = 999.95$$ ... Steady state output

$$\therefore \quad e_{ss} = \text{Reference input} - C_{ss} = 1000 - 999.95 = 0.05$$

Example 7.56 : *The open loop transfer function of a unity feedback control system is* $G(s) = \dfrac{100}{s(s+10)}$ *and input applied is* $r(t) = a + bt + \dfrac{ct^2}{2}$. *Obtain generalised error series.*

(M.U. : May-2004)

Solution : For generalised error series,

$$F_1(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\dfrac{100}{s(s+10)}} = \frac{s^2+10s}{s^2+10s+100}$$

$$K_0 = \lim_{s\to 0} F_1(s) = \lim_{s\to 0} \frac{s^2+10s}{s^2+10s+100} = 0$$

$$K_1 = \lim_{s\to 0} \frac{dF_1(s)}{ds} = \lim_{s\to 0} \left\{\frac{(s^2+10s+100)(2s+10)-(s^2+10s)(2s+10)}{(s^2+10s+100)^2}\right\}$$

$$= \lim_{s\to 0} \left\{\frac{100(2s+10)}{(s^2+10s+100)^2}\right\} = \lim_{s\to 0} \left\{\frac{200s+1000}{(s^2+10s+100)^2}\right\} = 0.1$$

$$K_2 = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} = \lim_{s\to 0} \frac{d}{ds}\left\{\frac{200s+1000}{(s^2+10s+100)^2}\right\}$$

$$= \lim_{s\to 0} \left\{\frac{(s^2+10s+100)^2(200)-(200s+1000)(2)(s^2+10s+100)(2s+10)}{(s^2+10s+100)^4}\right\} = 0$$

$$\therefore \quad e_{ss} = K_0\, r(t) + K_1\, r'(t) + \frac{K_2}{2!} r''(t) = 0 + 0.1[b+ct] + 0$$

$$\therefore \quad \mathbf{e_{ss} = 0.1\,[b + ct]} \qquad \text{... error series}$$

Example 7.57 : *A control design has to meet following specifications : damping ratio of 0.5, natural frequency of* $\sqrt{10}$ *rad/sec and steady state error of 10 %. For a unity feedback control system* $G(s) = \dfrac{K(s+\alpha)}{(s+\beta)^2}$, *find values of K, α and β.*

(M.U. : May-2004)

Solution : The closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K(s+\alpha)}{(s+\beta)^2}}{1+\dfrac{K(s+\alpha)}{(s+\beta)^2}} = \frac{K(s+\alpha)}{s^2+s(2\beta+K)+(k\alpha+\beta^2)}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$$\omega_n^2 = K\alpha + \beta^2 \qquad \text{i.e.} \quad \omega_n = \sqrt{K\alpha + \beta^2} \qquad \text{... (1)}$$

$$2\xi\omega_n = 2\beta + K \qquad \text{i.e.} \quad \xi = \frac{2\beta + K}{2\times\sqrt{K\alpha + \beta^2}} \qquad \text{... (2)}$$

For unit step input, A = 1.

$$K_P = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{K(s+\alpha)}{(s+\beta)^2} = \frac{K\alpha}{\beta^2}$$

$$\therefore \quad e_{ss} = \frac{A}{K_p} = \frac{1}{\frac{K\alpha}{\beta^2}} = \frac{\beta^2}{K\alpha} \qquad \text{... (3)}$$

But e_{ss} is 10% hence,

$$0.1 = \frac{\beta^2}{K\alpha} \qquad \text{i.e. } K\alpha = 10\beta^2 \qquad \text{... (4)}$$

From (1), $\sqrt{10} = \sqrt{10\beta^2 + \beta^2}$ i.e. $\beta^2 = \frac{10}{11}$ i.e. $\beta = \mathbf{0.9534}$

From (2), $0.5 = \dfrac{2\times(0.9534) + K}{2\times\sqrt{10\times(0.9534)^2 + (0.9534)^2}}$ i.e. $K = \mathbf{1.2552}$

From (4), $1.2552 \times \alpha = 10\times(0.9534)^2$ i.e. $\alpha = \mathbf{7.2416}$

Example 7.58 : *A control system is described by* $\frac{d^2y(t)}{dt^2} + \frac{8\,dy(t)}{dt} + 25\,y(t) = 50\,x(t)$. *Evaluate the output response y(t) and its peak value for a input x(t) = 2.5 u(t).*

(M.U. : May-2004, Dec.-2006)

Solution : Taking Laplace transform of both sides,

$$s^2\,Y(s) + 8\,s\,Y(s) + 25\,Y(s) = 50\,X(s)$$

$$\therefore \quad \frac{Y(s)}{X(s)} = \frac{50}{s^2 + 8s + 25} = \frac{50}{25}\left\{\frac{25}{s^2 + 8s + 25}\right\}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$$\omega_n^2 = 25 \qquad \text{i.e. } \omega_n = 5 \text{ rad/sec.}$$

$$2\xi\omega_n = 8 \qquad \text{i.e. } \xi = 0.8$$

$$\therefore \quad \omega_d = \omega_n\sqrt{1-\xi^2} = 3 \text{ rad/sec}, \qquad \theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 36.869^\circ$$

Hence the output response is,

$$y(t) = \frac{50}{25}\left\{1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)\right\}$$

$$= 2\{1-1.667\,e^{-4t}\sin(3t+36.869°)\}$$

$$\therefore \quad y(t) = 2-3.3334\,e^{-4t}\sin(3t+36.869°) \qquad \text{... Output response}$$

If the input is 2.5 u(t) i.e. step of 2.5 instead of unity then the response changes as,

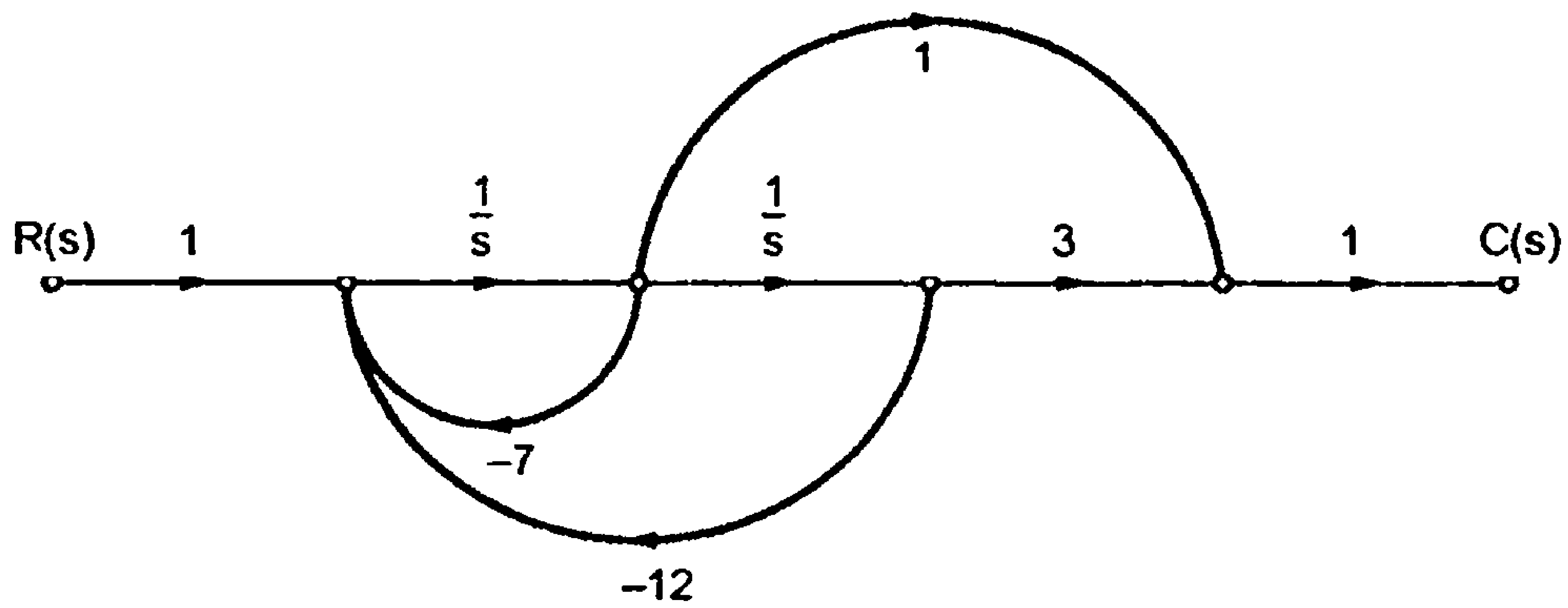
$$y(t) = 2.5\times\{2-3.334\,e^{-4t}\sin(3t+36.869°)\} = 5-8.3325\,e^{-4t}\sin(3t+36.869°)$$

The peak value is attained at $t = T_p = \frac{\pi}{\omega_d} = \frac{\pi}{3}$ sec

$$\therefore \quad y(t)|_{Peak} = 5-8.3325\,e^{-4\times\frac{\pi}{3}}\sin\left(3\times\frac{\pi}{3}+0.6434\text{ rad}\right)$$

$$= 5.0758 \qquad \text{... Use radian mode to calculate sin}$$

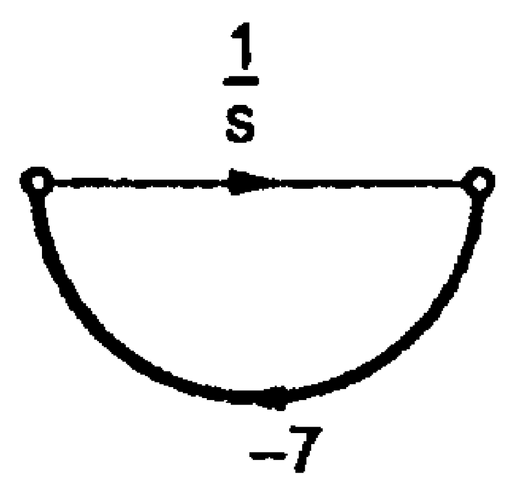
Example 7.59 : *For the signal flow graph shown below obtain the impulse response.* **(M.U. : May-2004)**



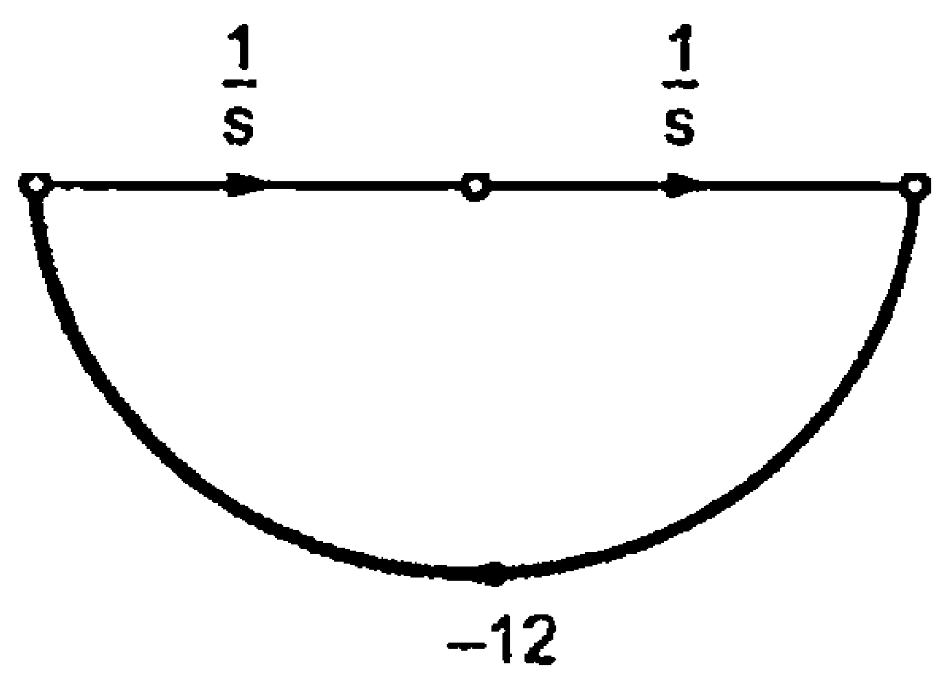
Solution : The Laplace transform of impulse response is the transfer function of the given system.

Using Mason's gain formula for the given signal flow graph,

$$T_1 = 1\times\frac{1}{s}\times\frac{1}{s}\times 3\times 1 = \frac{3}{s^2} \qquad T_2 = 1\times\frac{1}{s}\times 1\times 1 = \frac{1}{s}$$



$L_1 = -\frac{7}{s}$



$L_2 = -\frac{12}{s^2}$

Only two individual loops without any nontouching combination.

$$\therefore \quad \Delta = 1-[L_1 + L_2] = 1+\frac{7}{s}+\frac{12}{s^2} = \frac{s^2+7s+12}{s^2}$$

$$\Delta_1 = \Delta_2 = 1 \quad \text{... As both } L_1 \text{ and } L_2 \text{ touching to } T_1 \text{ and } T_2$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta} = \frac{\frac{3}{s^2}+\frac{1}{s}}{\frac{s^2+7s+12}{s^2}} = \frac{s+3}{s^2+7s+12}$$

$$= \frac{(s+3)}{(s+3)(s+4)} = \frac{1}{(s+4)}$$

For impulse response, R(s) = 1

$$\therefore \quad C(s) = \frac{1}{s+4}$$

$$\therefore \quad c(t) = L^{-1}\{C(s)\} = L^{-1}\left[\frac{1}{s+4}\right] = e^{-4t} \quad \text{... impulse response}$$

Example 7.60 : *The error response $e(t) = 2.5e^{-10t} \sin[50t+50°]$ for a unit step. Find natural frequency, damped frequency, damping ratio and comment on the type at damping.*

(M.U. : Dec.-2004, Dec.-2005)

Solution : The given response is,

$$e(t) = 2.5\,e^{-10t}\,\{\sin(50t)\cos(50°) + \cos(50\,t)\sin(50°)\}$$

$$= 2.5\,e^{-10t}\,\{0.6427\sin(50t) + 0.766\cos(50t)\}$$

$$= e^{-10t}\,\{1.60675\sin(50t) + 1.915\cos(50t)\}$$

Taking Laplace transform using $L\{e^{-at}\,f(t)\} = F(s)|_{s \to s+a}$

$$\therefore \quad E(s) = \frac{1.60675\times 50}{s^2+(50)^2} + \frac{1.915\times s}{s^2+(50)^2}\Big|_{s=s+10}$$

$$= \frac{1.915(s+10)+80.3375}{(s+10)^2+50^2} = \frac{1.915s+99.4875}{s^2+20s+2600}$$

But $$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{1}{1+G(s)H(s)} \quad \text{as } R(s) = 1 \text{ for unit step.}$$

$$\therefore \quad 1+G(s)H(s) = s^2+20s+2600 = s^2+2\xi\omega_n s+\omega_n^2$$

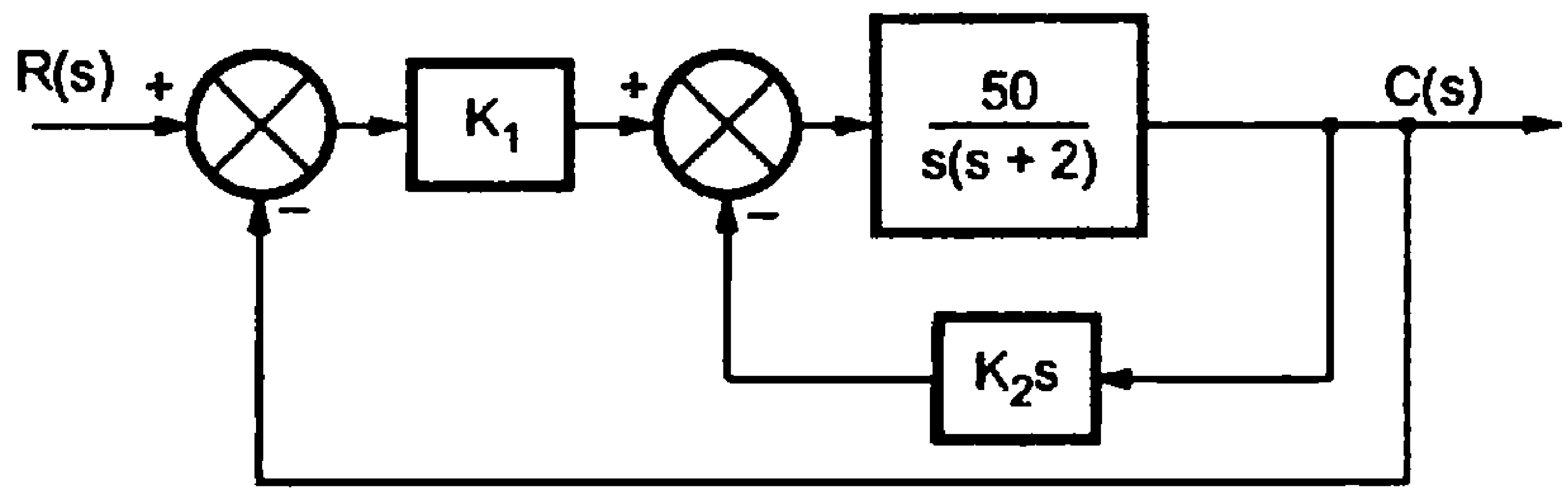
$$\therefore \quad \omega_n^2 = 2600 \quad \text{i.e. } \omega_n = 50.9901 \text{ rad/sec}$$

$$2\xi\omega_n = 20 \quad \text{i.e. } \xi = \frac{20}{2\times 50.9901} = 0.19611$$

and $\omega_d = \omega_n\sqrt{1-\xi^2} = 50$ rad/sec

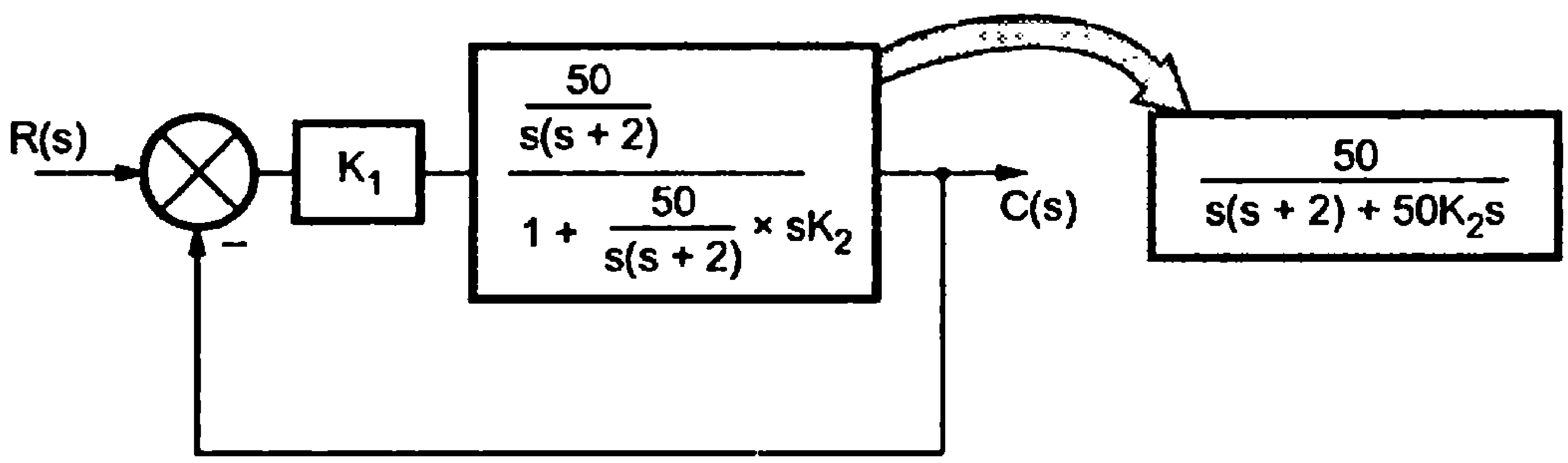
As $\xi < 1$, the system is **underdamped** in nature.

Example 7.61 : *Find value of K_1 and K_2 so as to obtain peak time = 2 sec, and settling time = 5 sec.*



(M.U. : May-2005, May-2007)

Solution : Solving inner minor feedback loop,



$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{50K_1}{s(s+2)+50\,K_2 s}}{1+\dfrac{50K_1}{s(s+2)+50\,K_2 s}} = \frac{50\,K_1}{s^2+s\,(2+50\,K_2)+50\,K_1}$$

Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$$\omega_n^2 = 50K_1 \quad \text{i.e.} \quad \omega_n = \sqrt{50K_1} \quad \text{...(1)}$$

$$2\xi\omega_n = 2+50K_2 \quad \text{i.e.} \quad \xi = \frac{2+50K_2}{2\sqrt{50K_1}} \quad \text{...(2)}$$

Given : $T_p = 2$ sec and $T_s = 5$ sec

Now, $T_p = \dfrac{\pi}{\omega_d} = 2$ and $T_s = \dfrac{4}{\xi\omega_n} = 5$ i.e. $\omega_n = \dfrac{0.8}{\xi}$

$$\therefore \quad \omega_d = \frac{\pi}{2} = \omega_n\sqrt{1-\xi^2}$$

i.e. $1.5707 = \dfrac{0.8}{\xi}\times\sqrt{1-\xi^2}$

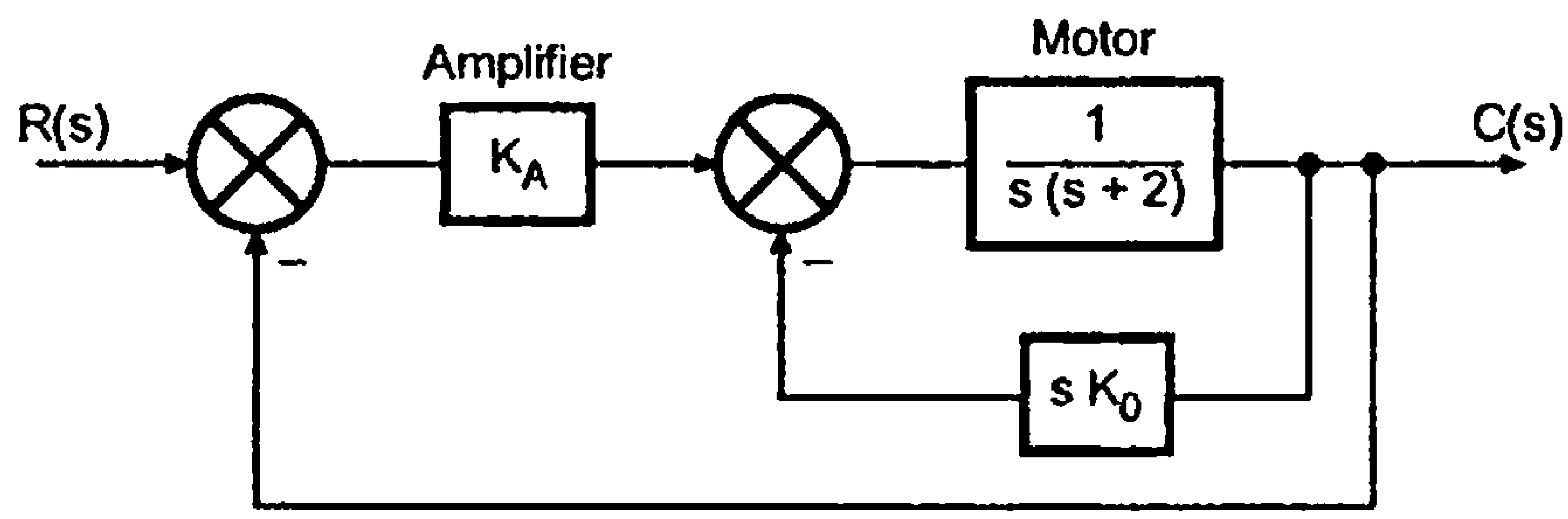
$\therefore \quad 3.8553 = \frac{1-\xi^2}{\xi^2} \quad$ i.e. $\quad 4.8553\,\xi^2 = 1$

$\therefore \quad \xi^2 = 0.20595 \quad$ i.e. $\quad \xi = 0.4538$ and $\omega_n = 1.76277$ rad/sec

Using (1), $\quad 1.7627 = \sqrt{50K_1} \quad$ i.e. $\quad K_1 = 0.06214$

Using (2), $\quad 0.4538 = \frac{2+50\,K_2}{2\times\sqrt{50\times 0.0621}} \quad$ i.e. $K_2 = -0.008$

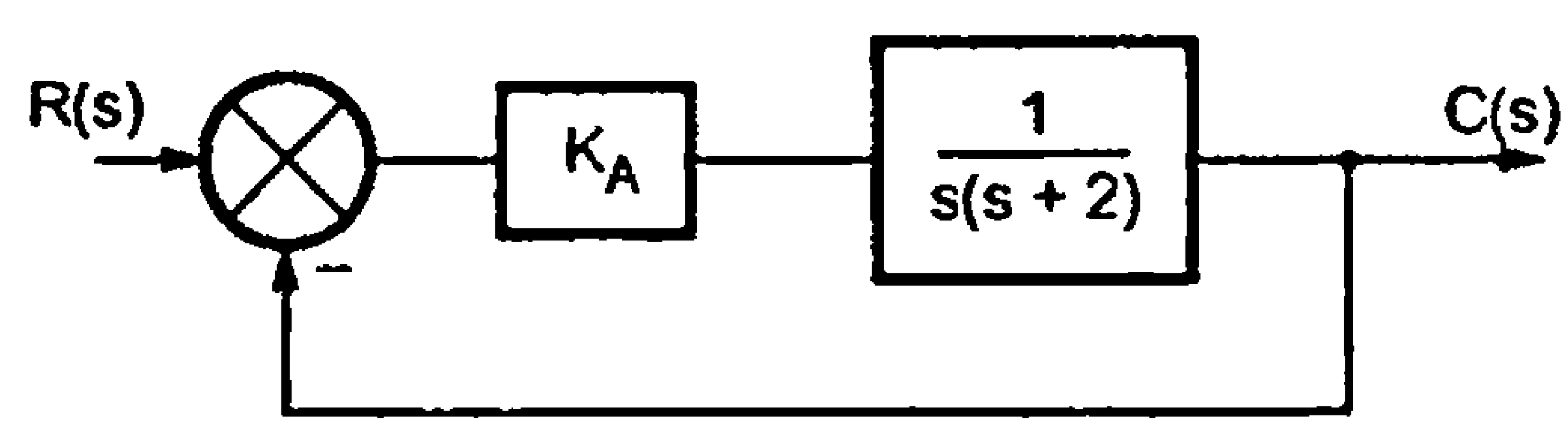
Example 7.62 : *A feedback system which uses a rate feedback controller is shown in the figure.*



a) For $K_A = 10$, in absence of derivative feedback ($K_0 = 0$) determine the damping ratio and natural frequency of oscillations. Also find the s.s error for unit ramp input.
b) Determine the constant K_0 if the damping factor required is 0.6, with $K_A = 10$. With this value of K_0, determine s.s error for unit ramp input. **(M.U. : May-2006)**

Solution : i) With $K_0 = 0$, the system is,

$$\frac{C(s)}{R(s)} = \frac{\frac{K_A}{s(s+2)}}{1+\frac{K_A}{s(s+2)}} = \frac{K_A}{s^2+2s+K_A}$$



Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$\omega_n^2 = K_A \quad$ i.e. $\omega_n = \sqrt{K_A} = \sqrt{10} = 3.16$ rad/sec

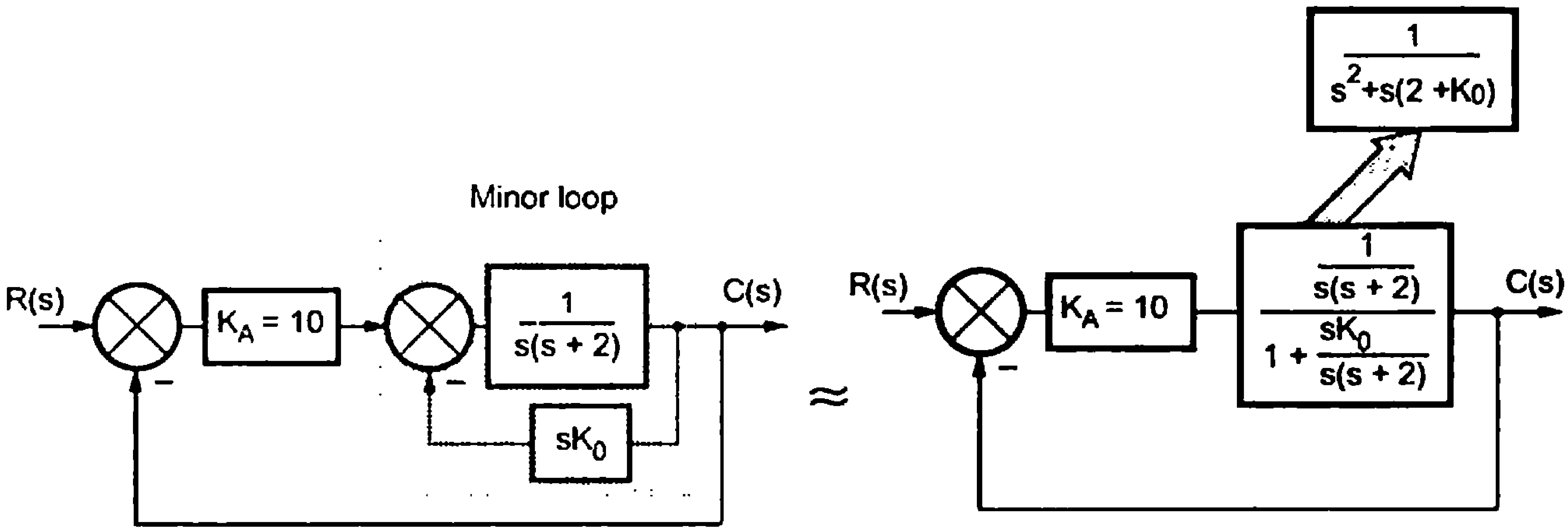
$2\xi\omega_n = 2 \quad$ i.e. $\xi = \frac{2}{2\times\sqrt{10}} = 0.3162$

$$G(s)H(s) = \frac{K_A}{s(s+2)} = \frac{10}{s(s+2)}$$

$$\therefore \quad K_v = \lim_{s\to 0} sG(s)H(s) = \frac{10}{2} = 5$$

$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{1}{5} = 0.2 \qquad$ ...For unit ramp A = 1

ii) With K_0, the system becomes,



$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\frac{10}{s^2+s(2+K_0)}}{1+\frac{10}{s^2+s(2+K_0)}} = \frac{10}{s^2+s(2+K_0)+10}$$

$$\therefore \quad \omega_n^2 = 10 \quad \text{i.e.} \quad \omega_n = \sqrt{10} \text{ rad/sec}$$

and $\quad 2\xi\omega_n = 2+K_0 \quad$ i.e. $\quad \xi = \frac{2+K_0}{2\sqrt{10}}$

But given $\quad \xi = 0.6 \quad$ hence, $\quad 0.6 = \frac{2+K_0}{2\sqrt{10}}$

$$\therefore \quad K_0 = \mathbf{1.7947}$$

$$\therefore \quad G(s)H(s) = \frac{10}{s^2+s(2+1.7949)} = \frac{10}{s(s+3.7949)}$$

$$\therefore \quad K_v = \lim_{s \to 0} sG(s)H(s) = \frac{10}{3.7949} = \mathbf{2.6351}$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{1}{2.6351} = \mathbf{0.3795}$$

Review Questions

1. *What is the difference between steady state response and transient response of a control system?*
2. *Define steady state response and steady state error.*
3. *How steady state error of a control system is determined? How it can be reduced ?*
4. *State how type of a control system is determined? How it affects the steady state error of a system?*
5. *Derive the expressions for static error coefficients? How these coefficients are useful in determining steady state error? State the limitations of static error coefficient method.*
6. *How damping ratio affects the time response of a second order system?*
7. *Show the locus of closed loop poles of a second order system as ξ is varied from 0 to ∞.*
8. *Define the following systems sketching their output waveform for a unit step input :*
   i) Underdamped system ii) Undamped system
   iii) Overdamped system iv) Critically damped system.
9. *With a neat sketch explain all the time domain specifications.*
10. *Derive the relationship between maximum overshoot and damping ratio of a second order system. Draw the graph or overshoot versus damping ratio*
11. *Derive the expressions for peak time and rise time interms of ξ and ω_n for a second order control system.*
12. *Discuss the advantages of feedback in a control system.*
13. *An unity feedback system has the loop transfer function*

$$G(s) = \frac{K}{s(s+a)}$$

If its time response is to have (i) an overshoot of less than 5% and (ii) settling time not exceeding 4 seconds, determine suitable values for K and a. **[Ans. : K = 2 ; a = 2]**

14. *A position control system is stabilised by means of acceleration feedback. If the system has a moment of inertia of 10^{-5} Kg - m^2; viscous frictional torque of 10^{-4} Nm/rad - sec^{-1} and the motor torque is given by*

$$T_m = 4 \times 10^{-3}\left(e + K\frac{d^2\theta_0}{dt^2}\right) N - m$$

where $e = \theta_i - \theta_o$;

$$K\frac{d^2\theta_o}{dt^2} = \text{Acceleration feedback}$$

i) Draw its block diagram.
ii) Write the system D.E.
iii) Determine K for critical damping and
iv) Determine the steady state error e_{ss} when the input is a constant velocity of 20 rpm.

[Ans. : K = 2.34 × 10^{-3} sec^2, 3°]

15) *Determine the step, ramp, and parabolic error constants of the following unity - feedback control systems. The open loop transfer functions are given*

a) $G(s) = \dfrac{1000}{(1+0.1s)(1+10s)}$

b) $G(s) = \dfrac{100}{s(s^2+10s+100)}$

c) $G(s) = \dfrac{100}{s(1+0.1s)(1+0.5s)}$

d) $G(s) = \dfrac{100}{s^2(s^2+10s+100)}$

e) $G(s) = \dfrac{1000}{s(s+10)(s+100)}$

f) $G(s) = \dfrac{K(1+2s)\ (1+4s)}{s^2(s^2+s+1)}$

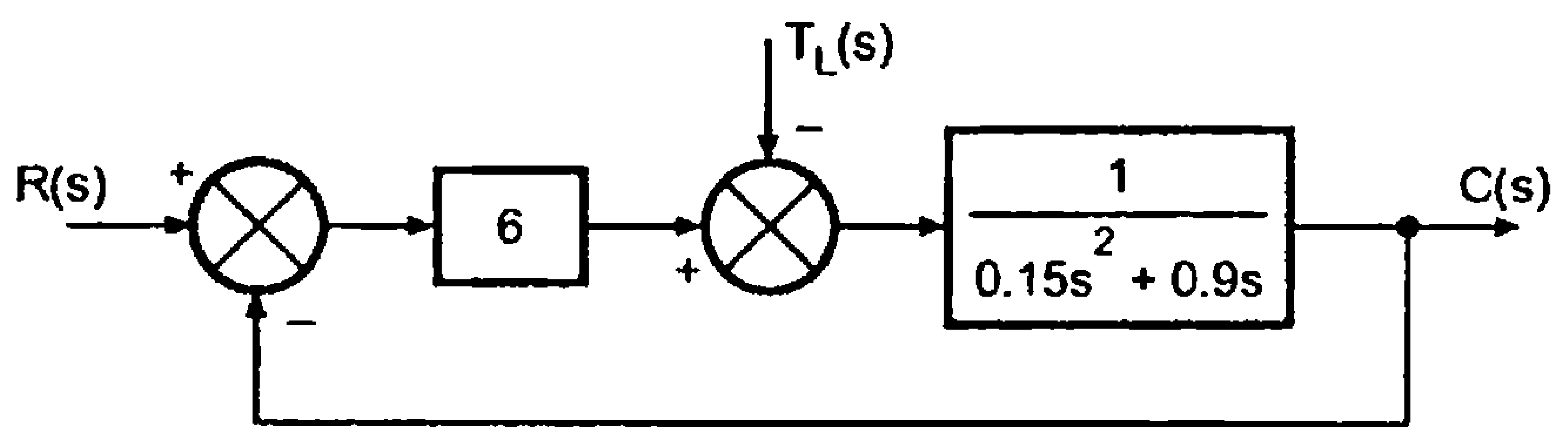
[Ans. : a) $K_p = 1000\ K_v = 0\ K_a = 0$
b) $K_p = \infty\ \ K_v = 1\ \ K_a = 0$
c) $K_p = \infty\ \ K_v = K\ \ K_a = 0$
d) $K_p = \infty\ \ K_v = \infty\ \ K_a = 1$
e) $K_p = \infty\ \ K_v = \infty\ \ K_a = 0$
f) $K_p = \infty\ \ K_v = \infty\ \ K_a = 1\,K$]

16. *A two phase a.c. servometer having a torque constant of 0.045 Nm/V controls a position load through a gear ratio of 10:1. The effective moment of inertia and co-efficient of viscous friction referred to load side are 0.25 Kg - m^2 and 1.0 Nm(rad/sec). The synchro (megslip) error detector produces an error signal of 0.1 V per degree error in misalignment. Develop the block diagram representation of the control system and there from obtain the transfer function.*

a) *Calculate the natural frequency of oscillations and damping ratio.*

b) *If damping is to be made critical using a derivative error control, determine the time constant of the phase advance circuit.* **[Ans. : $\omega_n = 3.21$ rad/sec , $\xi = 0.623$, $T_d = 0.234$ sec]**

17. *For the control system shown in figure.*



a) *Determine (i) M_p for a unit step input (ii) e_{ss} for unit ramp input*

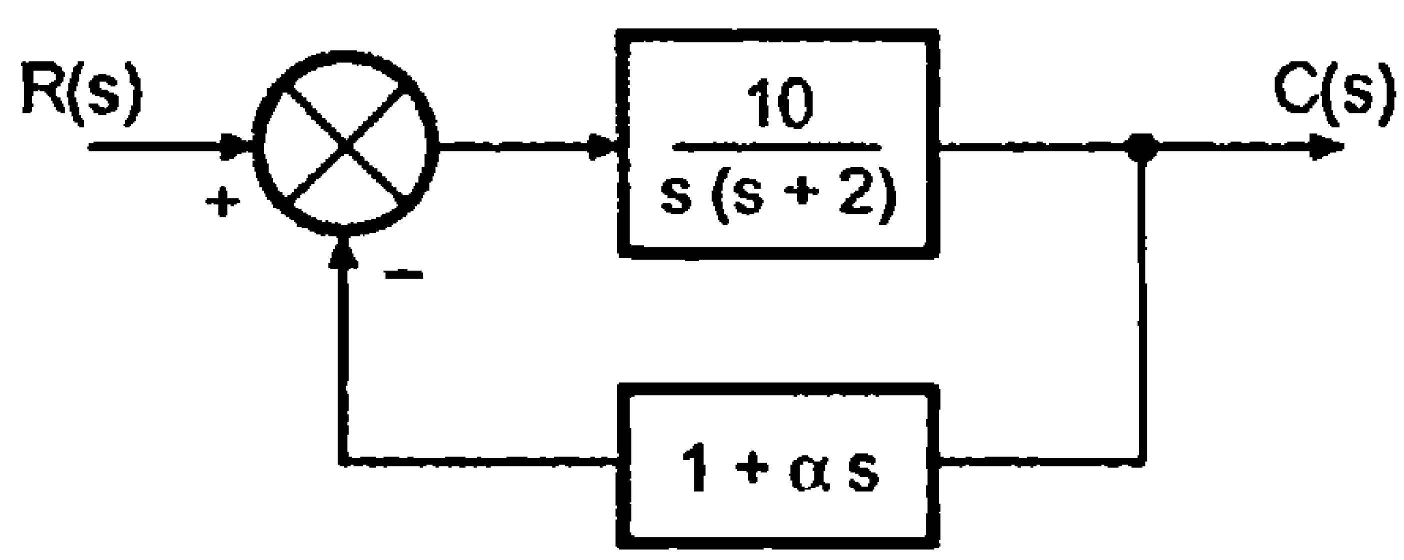
b) *Calculate the steady state value of the output when the input shaft is held fixed and sudden torque T_L = 1 Nm is applied* **[Ans. : $M_p = 18.39\%$, $e_{ss} = 0.15$ rad, $C_{ss} = -0.166$ rad]**

18. *The open loop transfer function of a unity feedback control system is given by*

$$G(s) = \frac{K}{s(1 + sT)}$$

The input to the system is 1 R.P.M. and the steady state error being 0.25. Calculate the natural frequency of oscillations if the system is critically damped. **[Ans. : 48 rad/s]**

19. *The block diagram of a position control system with velocity feedback is shown in figure. Determine the value of α so that the step response has maximum overshoot of 10 percent. What is the steady state error?*



[Ans. : $\alpha = 0.1739$, $e_{ss} = 0$]

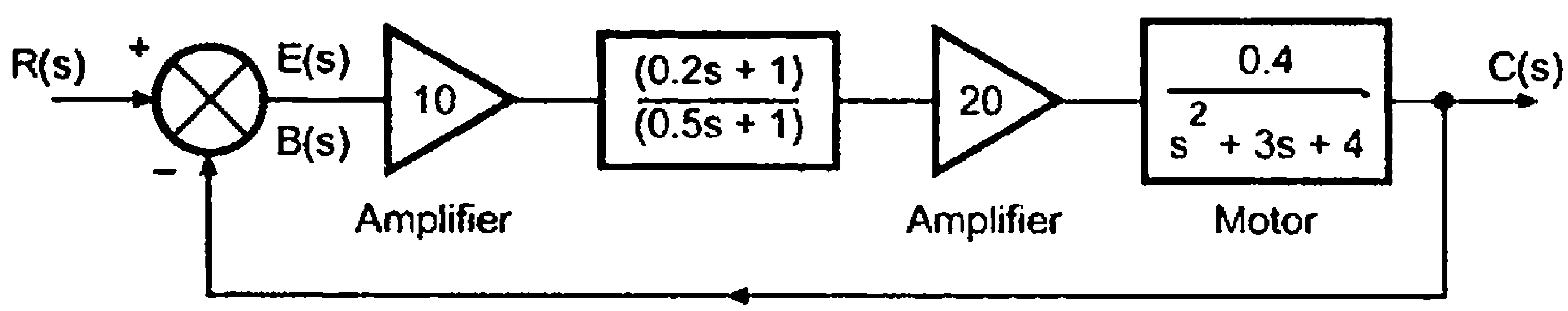
20. *The closed loop transfer function of a second order system with proportional plus error-rate feedback is given by* $\frac{C(s)}{R(s)} = \frac{K(s+z)}{s^2 + 4s + 8}$ *where parameters K and z are adjustable*

a) *If r(t) = t; determine the values of K and z such that the steady state error is zero.*

b) *For these values of K and z, determine the steady state error to a unit parabolic input; i.e.* $r(t) = -1/2t^2$.

[Ans. : (a) K = 4, z = 2 (b) 0.25]

21. *For the system shown in figure determine the steady state error for*

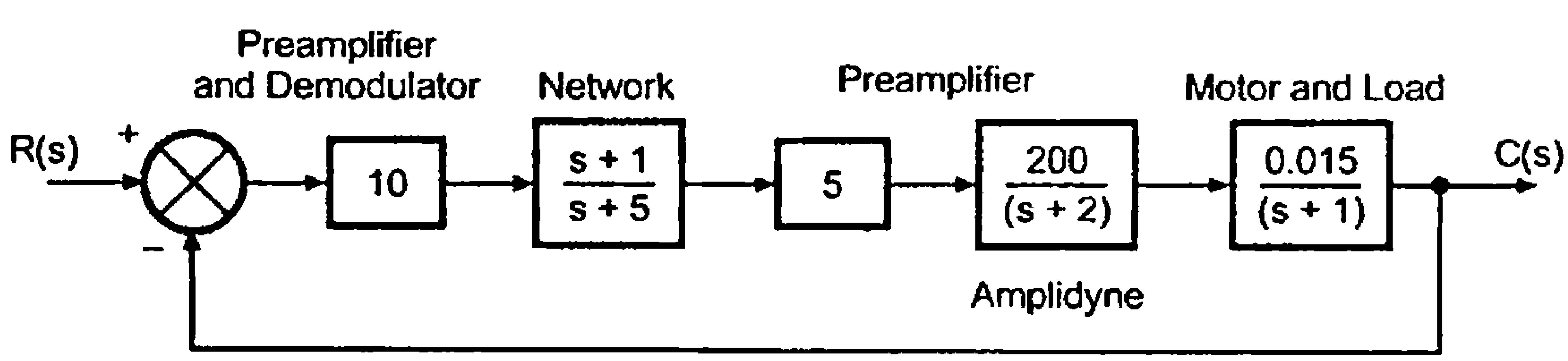


i) A unit step input

ii) A unit step velocity input

iii) A unit step acceleration input **[Ans. : (i) 0.0475 (ii) ∞ (iii) ∞]**

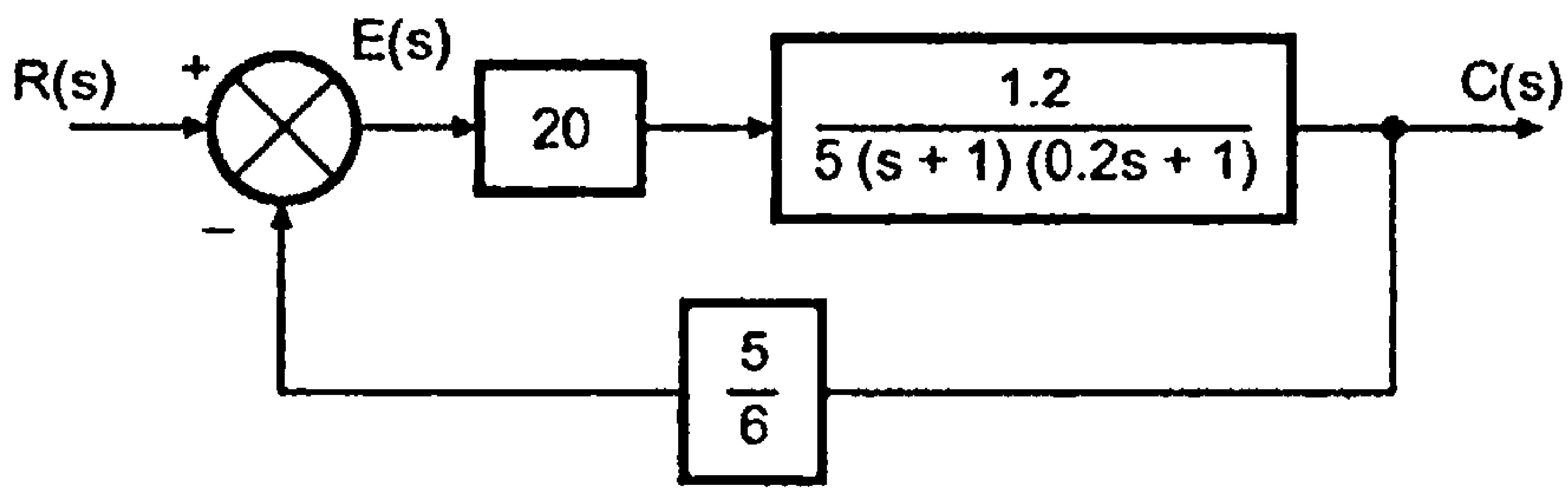
22. *The block diagram of a fire control system with unity feedback is described in figure. Using generalised error series determine the steady state error of the system when the system input is*



$$r(t) = \left(5 + 4t + \frac{3t^2}{2}\right)$$

[Ans. : $0.5 + 0.373t + 0.09375t^2$]

23. *The block diagram of a simple servo system is shown in following figure. Determine the characteristic equation of the system. Hence calculate the undamped frequency of oscillations, damping ratio, damping factor, maximum overshoot first undershoot, time intervals after which maximum and minimum occurs, settling time and the number of cycles completed before the output is settled within 2% of the final value. The input to the system is a unit step.*



[Ans. : $s^2 + 6s + 25 = 0$; $\omega_n = 5$ rad/sec; $\xi = 0.6$; $\omega_d = 4$; $M_p = 9.5\%$; 0.94%; 0.785 sec; 1.57 sec ; 1.33 sec; 0.85 cycles]

24. *The open loop transfer function of a unity feedback control system is given by*

$$G(s) = \frac{K}{s(sT+1)}$$

i) *By what factor the amplifier gain K should be multiplied so that the damping ratio is increased from 0.2 to 0.8.*

ii) *By what factor the time constant T should be multiplied so that the damping ratio is reduced from 0.6 to 0.3.*

iii) *For the system overshoot of the unit step response to reduce from 60% to 20%. Show that* $\frac{TK_1 - 1}{TK_2 - 1} = 43.22$ *where* K_1 *and* K_2 *are the values of K for 60% and 20%*

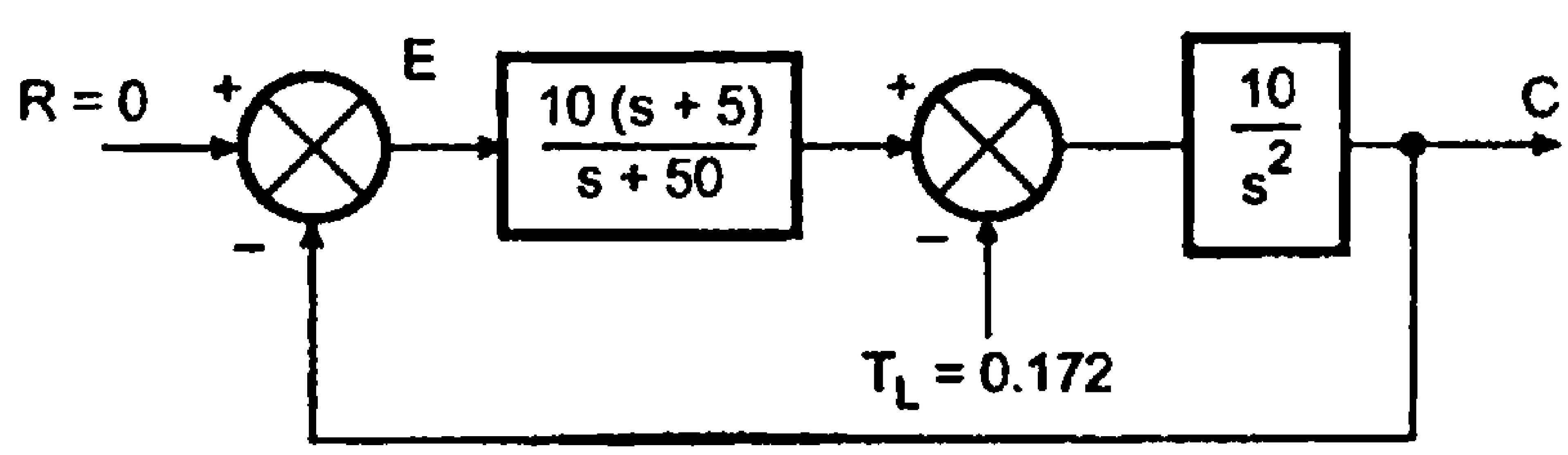
[Ans. : $\frac{1}{16}$, 4]

25. *Calculate static error coefficients for a unity feedback system with* $G(s) = \frac{12}{s(s+6)}$. *If input given is* $r(t) = 4 + 3t$ *determine steady state error. For the above system of* e_{ss} *is to be reduced to 10% of existing value, what would be the percentage change in gain?*

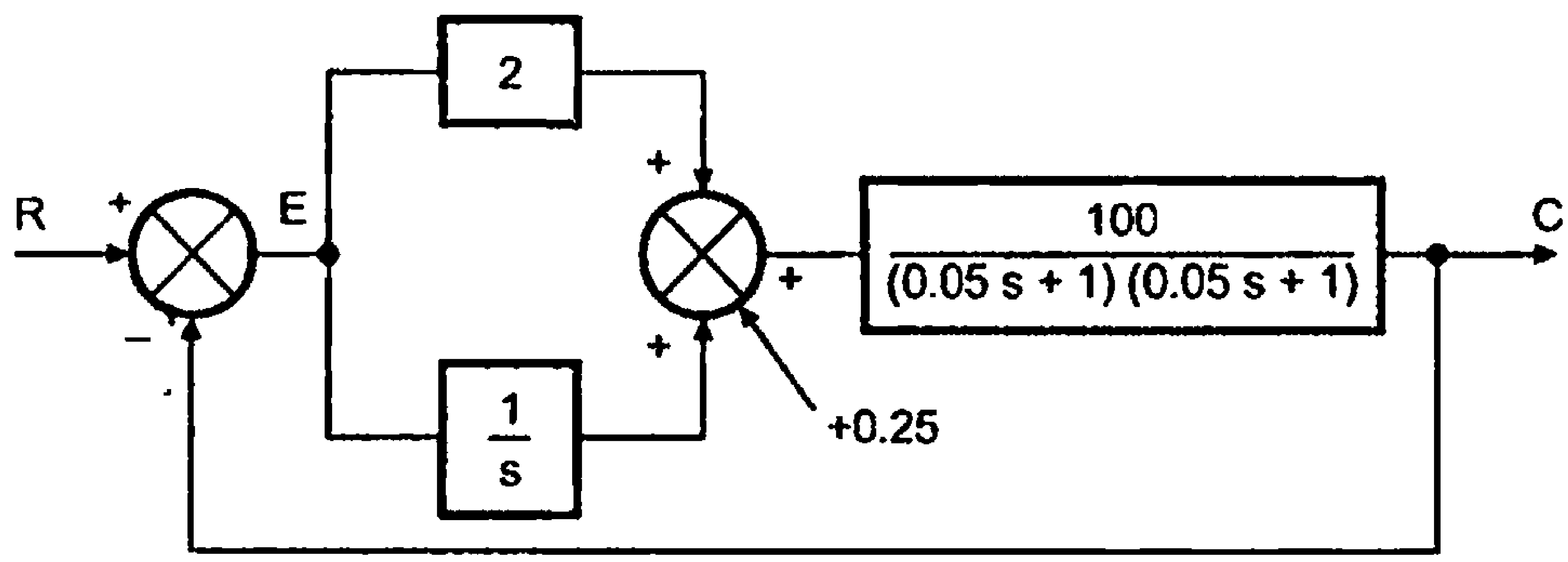
[Ans. : $K_p = \infty$, $K_v = 2$, $e_{ss} = 1.5$, 900% change is need]

26. *Given :*

What is the steady state error? **[Ans. : 0.172]**

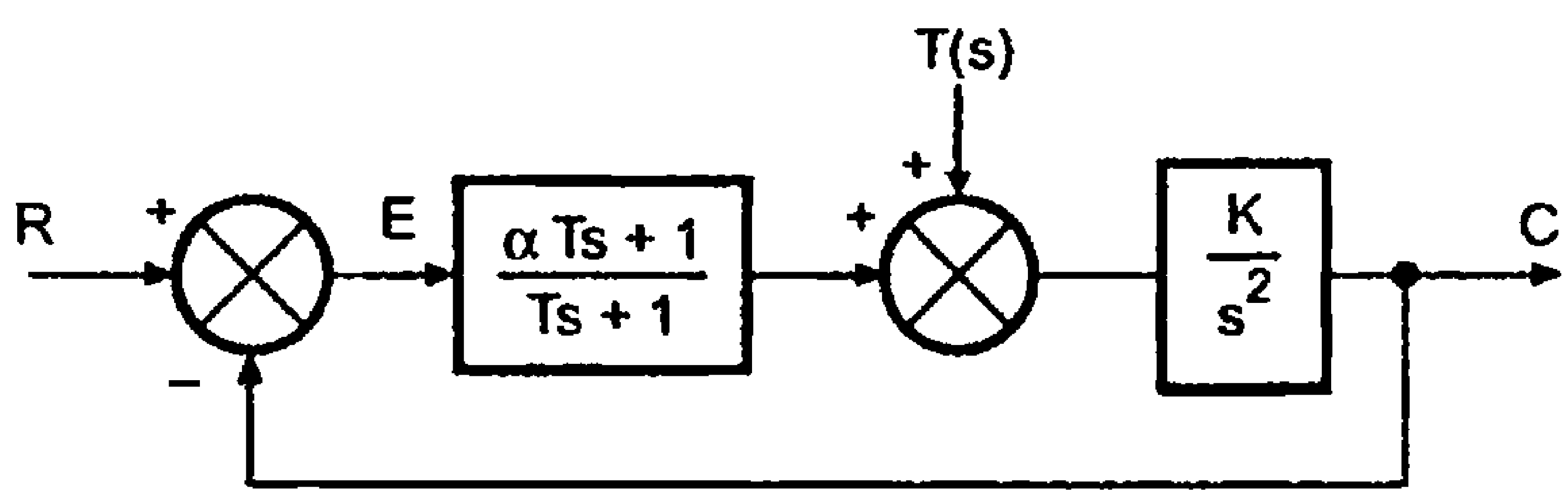


27. *Given :*



The input is a ramp, R = 1.5t . What is the steady-state error? **[Ans. : 0.015]**

28. *Given :*



If R = 0, what is the steady – state error? **[Ans. : – 1]**

29. *Find error coefficients for the given unity feedback system having*

$$G(s) = \frac{4(s^2 + 10s + 100)}{s(s+3)(s^2 + 2s + 10)}$$

(Ans. ∞, $\frac{40}{3}$, **0)**

30. *Determine error coefficients for the system having*

$$G(s)H(s) = \frac{(s+2)}{s(1 + 0.5s)(1 + 0.2s)}$$

(Ans. ∞**, 2, 0)**

31. *Find error coefficients for a system having* $G(s) = \frac{10}{s^2(1+s)}$ *and steady state error if input to the system is* $a_0 + a_1 t + a_2 t^2$. **(Ans. :** ∞, ∞, **10,** $a_2/10$**)**

32. *A system has 40% overshoot and requires a settling time of 4 secs when given a step input. The steady state error 2%. Determine the transfer function of the second order system. Also find rise time and peak time.* **(M.U. : Nov.-94)**

(Ans. : 0.5409 sec, 0.9163 sec)

33. *A standard second order system has 50% overshoot and settling time of 3 seconds. Determine* ω_n, ξ, T_p and T_r. **(M.U. : Jan.-94)**

(Ans. : 6.19, 0.2154, 0.5197, 0.2958)

34. *The open loop transfer function of a unity feedback control system is given by*

$$G(s) = \frac{K}{s(sT+1)}$$

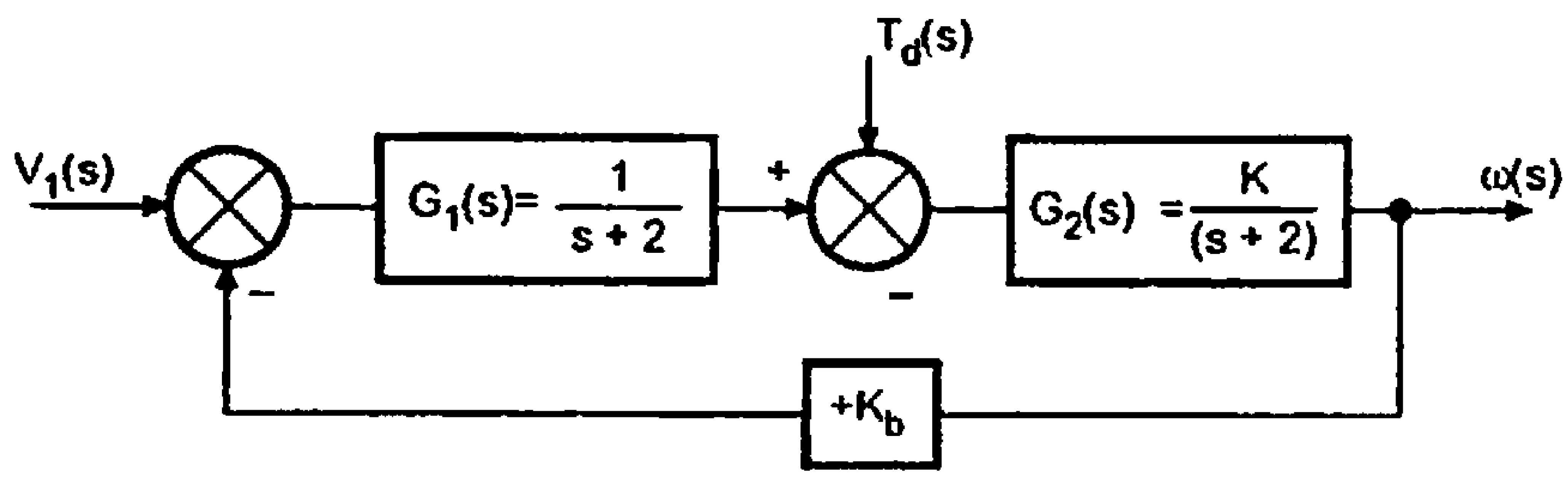
i) *By what factor the amplifier gain K should be multiplied so that the damping ratio is increased from 0.2 to 0.8.*

ii) *By what factor the time constant T should be multiplied so that damping ratio is reduced from 0.6 to 0.3.*

(M.U. : May-97)

(Ans. : 1/16, 4)

35. *The block diagram of a speed control system is shown in the figure. The disturbance is present in its forward path.*



Determine the sensitivity of the system

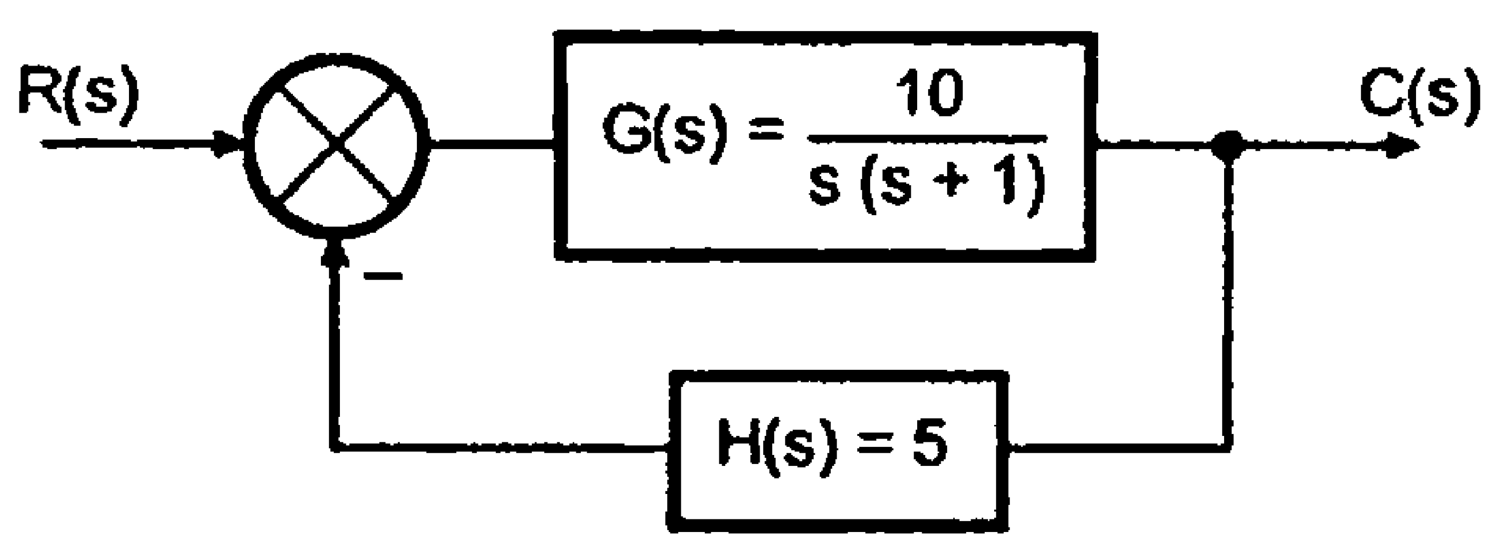
i) $S_K^{M_d}$ *ii)* $S_{K_b}^{M_d}$

where M_d (s) *is the ratio of* $\omega(s)$ and $T_d(s)$.

$$\left(\text{Ans}: \frac{s^2 + 3s + 2}{s^2 + 3s + (K K_b + 2)}, \frac{K K_b}{s^2 + 3s + (K K_b)} \right)$$

36. *The block diagram of a position control system is shown in the figure. Determine the sensitivity of closed loop transfer function* T(s) *with respect to* G(s) and H(s) *for* $\omega = 1$ rad / sec.

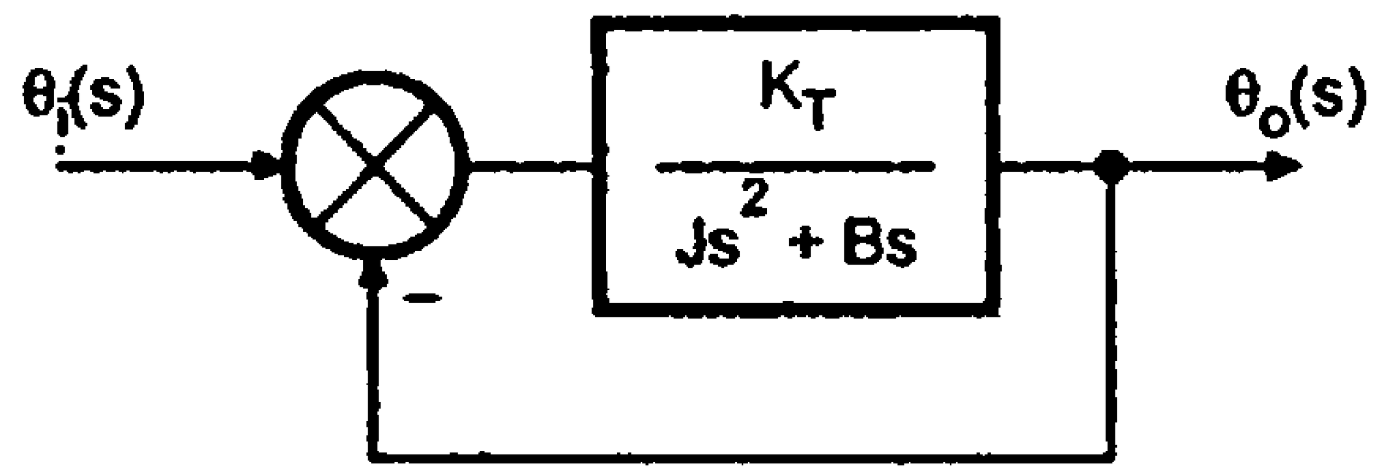
(Ans. : 0.029, 1.02)



37. *A flywheel is driven by an electric motor. It is controlled automatically by a movement of handwheel and follows its movement. The effective moment of inertial of the flywheel is 150* $\text{kg} - \text{m}^2$. *The torque developed by the motor is 2400 Nm/rad of misalignment between flywheel and handwheel. Viscous friction of the system is 600* Nm rad^{-1} sec. *If the handwheel is suddenly moved through* $\frac{\pi}{3}$ *rad from rest, determine the expression for the subsequent angular position of the flywheel.*

(Ans. : $1.05\,[1 - e^{-2t} \cos 3.5\,t + 0.6 \sin 3.5\,t]$)

38. A servo mechanism shown in the figure has moment of inertia of the moving parts referred to load shaft as 150 $kg - m^2$. The motor load torque is 4×10^4 Nm/rad of error. The damping torque coefficient referred to load shaft is 4×10^3 Nm/rad/sec.



Find :

a) The step response if input is step type of one radian. Also determine rise time, peak time and peak overshoot.

b) The S.S. error if input is constant angular velocity of 1 r.p.m.

c) The S.S. error which exists when a torque of 1200 Nm is applied to the load shaft.

(Ans. : $1 - 1.75\, e^{-13.4\,t} \sin(9.35\,t + 0.61)$, 0.6°, 1.72°)

❑❑❑

8 Stability Analysis

8.1 Background

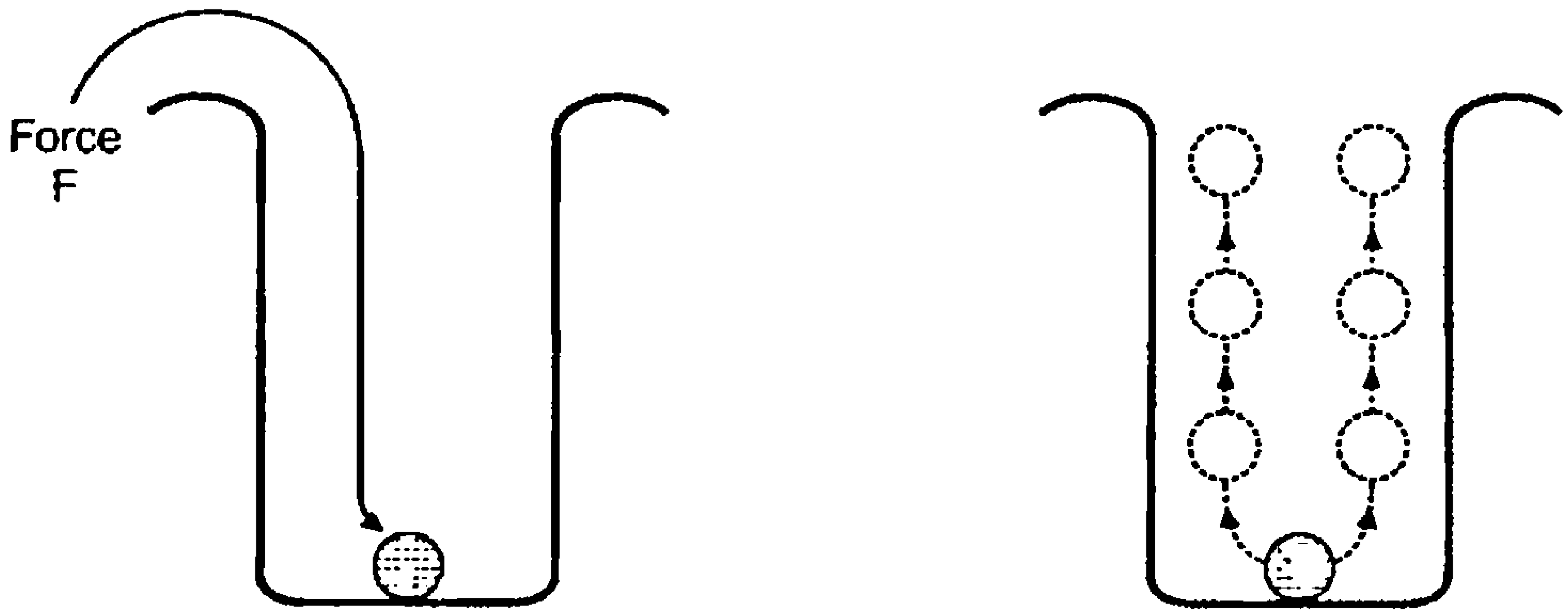
As we have seen earlier that every system, for small amount of time has to pass through a transient period. Now whether system will reach to its intended steady state after passing through transients or not? The answer to this question means to define whether system is stable or unstable. This is stability analysis.

For example, a meter is connected in a system to measure a particular parameter. Before showing the final reading, the pointer of meter will pass through the transients. The final reading is the steady state of the pointer. But during transients, it is possible that the pointer may become stationary due to certain problems in the moving system of that meter. So to achieve steady state, the system must pass through the transient period successfully.

Key Point : *The analysis of whether the given system can reach steady state; passing through the transients successfully is called* ***Stability Analysis*** *of the system.*

8.2 Concept of Stability

Consider a system i.e. a deep container with an object placed inside it as shown in the Fig. 8.1.



(a) (b)

Fig. 8.1

Now if we apply a force to take out the object, as the depth of the container is more, it will oscillate and will settle down again at its original position.

If we assume that the force required to take out the object tends to infinity i.e. always object will oscillate when force is applied and will settle down but will not come out, such a system is called ***absolutely stable*** system. No change in parameters, disturbances, changes the output. As against this, consider a container which is pointed one, on which we try to keep a circular objects shown in the Fig. 8.2.

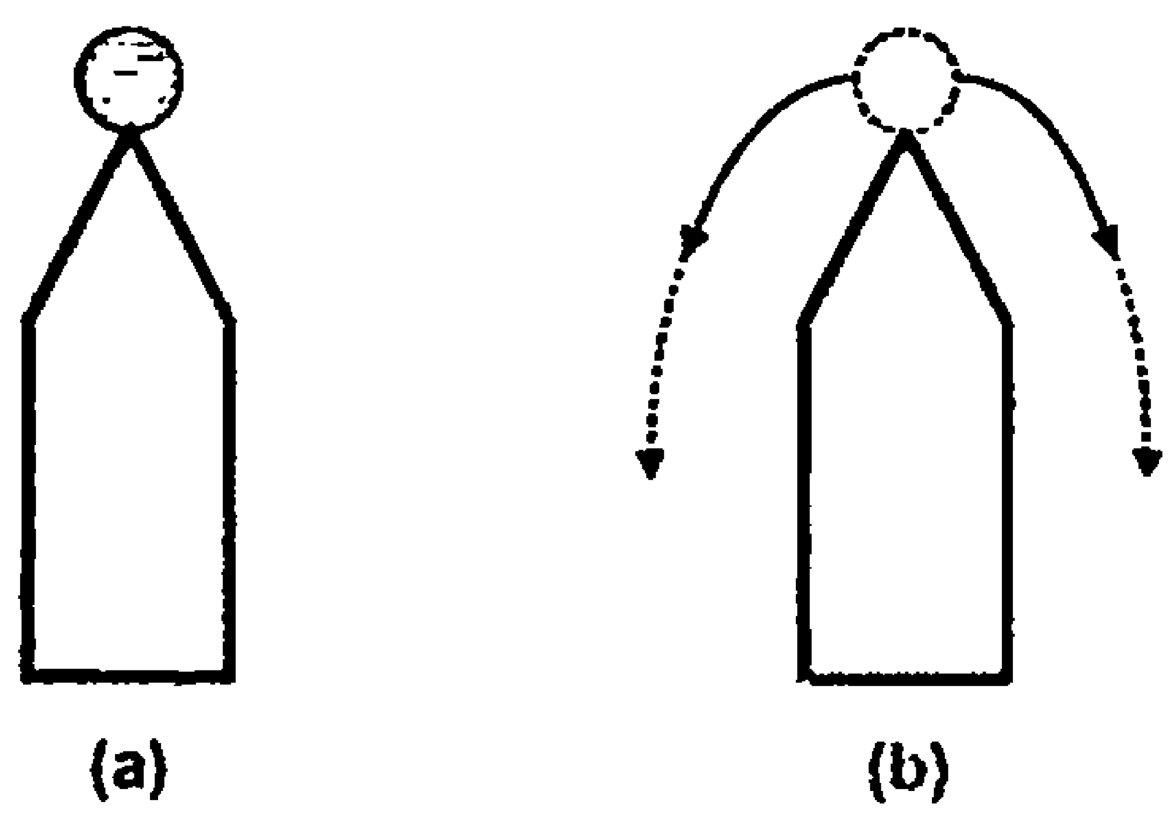


Fig. 8.2

In this case object will fall down without any external application of force. So if we try to keep the circular object, we will always fail to do so. Such system is called ***unstable system***.

While in certain cases the container is shallow then there exists a critical value of force for which object will come out of container.

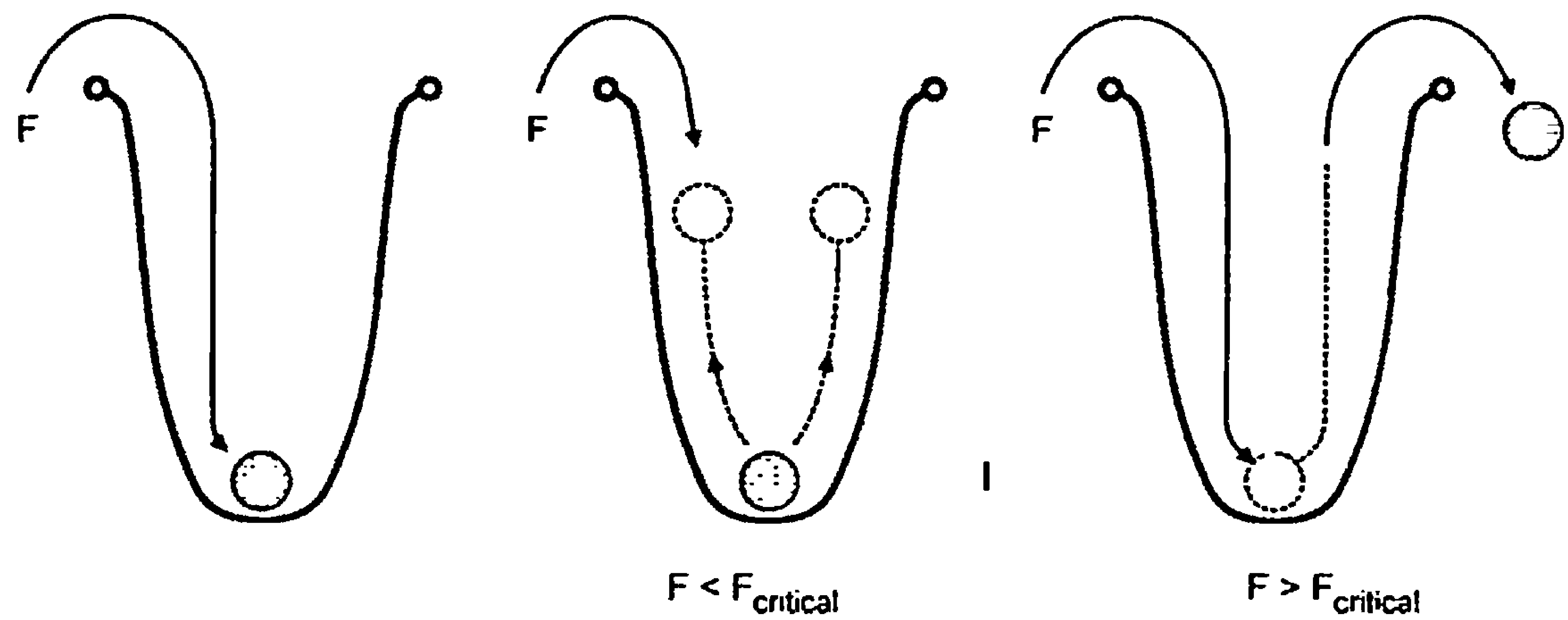


Fig. 8.3

As long as $F < F_{critical}$, object regains its original position but if $F > F_{critical}$, object will come out. Stability depends on certain conditions of the system hence system is called **conditionally stable system.**

There are few systems e.g. : Pendulum where system keeps on oscillating when certain force is applied. Such systems are neither stable nor unstable and hence called ***critically stable or marginally stable systems.***

Now let us see on which factors exactly the stability depends in a control system.

8.3 Stability of Control Systems

The stability of a linear closed loop system can be determined from the locations of closed loop poles in the s-plane.

For example : If system has closed loop T.F.

$$\frac{C(s)}{R(s)} = \frac{10}{(s+2)(s+4)}$$

Let us find out, output response for unit step input.

$$\therefore \quad R(s) = 1/s$$

$$\therefore \quad C(s) = \frac{10}{s(s+2)(s+4)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+4}$$

$$\therefore \quad C(s) = 10\left\{\frac{1/8}{s}-\frac{1/4}{s+2}+\frac{1/8}{s+4}\right\} \quad \text{... Finding the partial fractions}$$

$$\therefore \quad C(s) = \frac{1.25}{s}-\frac{2.5}{s+2}+\frac{1.25}{s+4}$$

$$c(t) = \underbrace{1.25}_{\text{Steadystate}} - \underbrace{2.5\,e^{-2t}+1.25\,e^{-4t}}_{\text{Transient}} = C_{ss} + c_t(t)$$

As closed loop poles are located in left half of s-plane, in output response there are exponential terms with negative indices i.e. e^{-2t} and e^{-4t}.

Now as $t \to \infty$ both exponential terms will approach to zero and output will be steady state output.

i.e. as $t \to \infty,\ c_t(t) = 0$

Transient output $= 0$

Such systems are called ***absolutely stable systems.***

Now transient terms are exponential terms with negative index because closed loop poles are located in left half of s-plane. For the above system under consideration, the closed loop poles are s = – 2 and s = – 4 and the negative indices of exponential terms are also – 2 and – 4.

Key Point : *Thus if closed loop poles are located in left half, exponential indices in output are negative. And if indices are negative, exponential transient terms will vanish when* $t \to \infty$.

Now let us have a system with one closed loop pole located in right half of s-plane.

$$\frac{C(s)}{R(s)} = \frac{10}{(s-2)(s+4)}$$

Find out unit step response of above system.

$$C(s) = \frac{10}{s(s-2)(s+4)} = \left\{\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+4}\right\}$$

$$C(s) = -\frac{1.25}{s} + \frac{0.833}{s-2} + \frac{0.416}{s+4}$$

$$\therefore \quad c(t) = -1.25 + 0.833\, e^{+2t} + 0.416\, e^{-4t}$$

Now due to pole located in right half, there is one exponential term with positive index in transient output.

while $\quad c_{ss}(t) = -1.25$

t	c(t)
0	0
1	+ 4.91
2	+ 44.23
4	+ 2481.88
∞	∞

As it is clear from the table that instead of approaching to steady state value as $t \to \infty$, due to exponential term with positive index, transients go on increasing in amplitude. So such system is said to be unstable.

Key Point : *So it is clear that if any of the closed loop poles lie in right half of s-plane, then it gives the exponential term of positive index and due to that, transient response of increasing amplitude, making system unstable.*

In such systems output is uncontrollable and unbounded one. Output response of such systems is as shown in the Fig. 8.4.

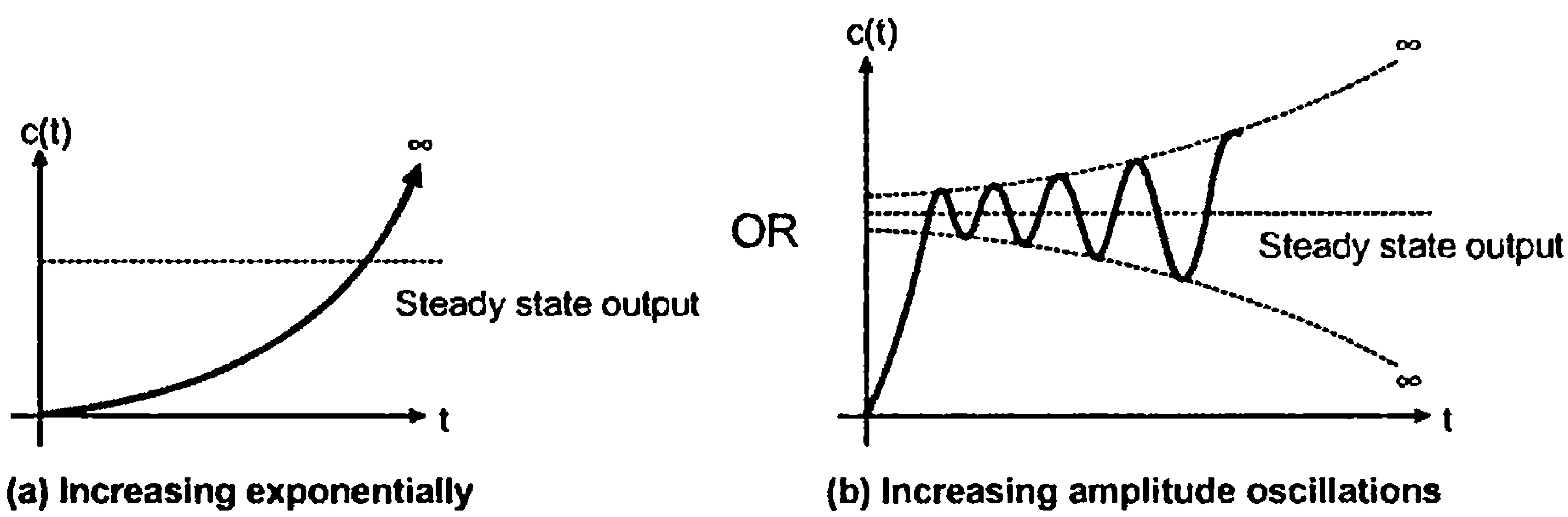


Fig. 8.4 Uncontrollable response

For such unstable systems, if input is removed output may not return to zero. And as soon as input power is turned on, output tends to ∞. If no saturation takes place in system and no mechanical stop is provided then system may get damaged and failed.

Remember that the stability depends on locations of closed loop poles. And the closed loop poles are the roots of the characteristic equation of the system.

So, | Closed loop poles = Roots of the characteristic equation |

If all the closed loop poles or roots of characteristic equation lies in left half of s-plane then in the output response there will be exponential terms with negative indices along with steady state terms. Such transient terms approach to zero as time advances. Eventually output reaches to equilibrium and attains steady state value. So transient terms in such systems may give oscillations but the amplitudes of such oscillations will be decreasing w.r.t. time and finally will vanish. So output response of such system can be shown as in the Fig. 8.5 (a) and (b).

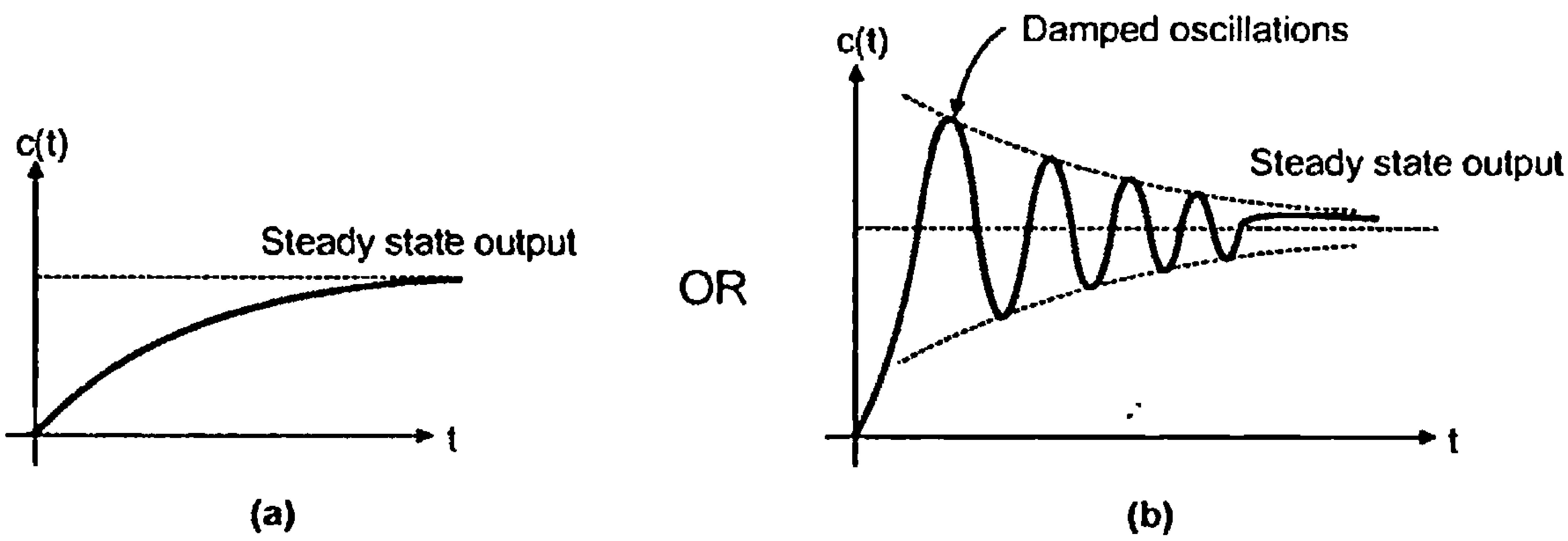


Fig. 8.5 Stable response

Definition of BIBO Stability : *This is Bounded Input Bounded Output stability (BIBO).*

A linear time invariant system is said to be stable if following conditions are satisfied :

i) When the system is excited by a bounded input, output is also bounded and controllable.

ii) In the absence of the input, output must tend to zero irrespective of the initial conditions.

Unstable System : *A linear time invariant system is said to be unstable if,*

i) For a bounded input it produces unbounded output.

ii) In absence of the input, output may not return to zero. It shows certain output without input.

Besides these two cases, if one or more pairs of simple **nonrepeated roots** of characteristic equation are located on the imaginary axis of the s-plane, but there are no roots in the right half of s-plane, the output response will be undamped sinusoidal oscillations of constant frequency and amplitude. Such systems are said to be critically or marginally stable systems.

Critically or Marginally Stable System : *A linear time invariant system is said to be critically or marginally stable if for a bounded input its output oscillates with constant frequency and amplitude. Such oscillations of output are called* ***undamped oscillations*** *or* ***sustained oscillations.***

For such systems, one or more pairs of **nonrepeated** roots are located on imaginary axis as shown in the Fig. 8.6 (b).

Output response of such systems is as shown in the Fig. 8.6 (a).

Key Point : *The stability or instability is a property of the system itself i.e. closed loop poles of the system and does not depend on input or driving function. The poles of input do not affect stability of system, they affect only steady state output.*

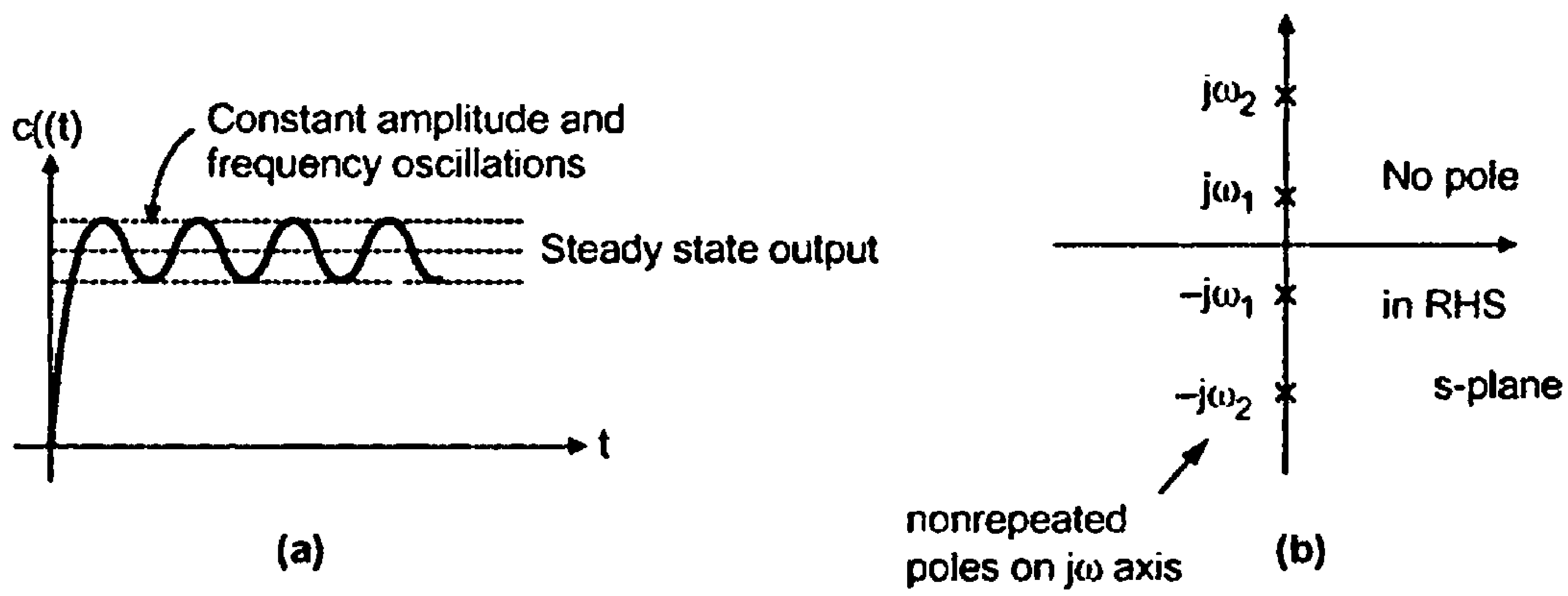
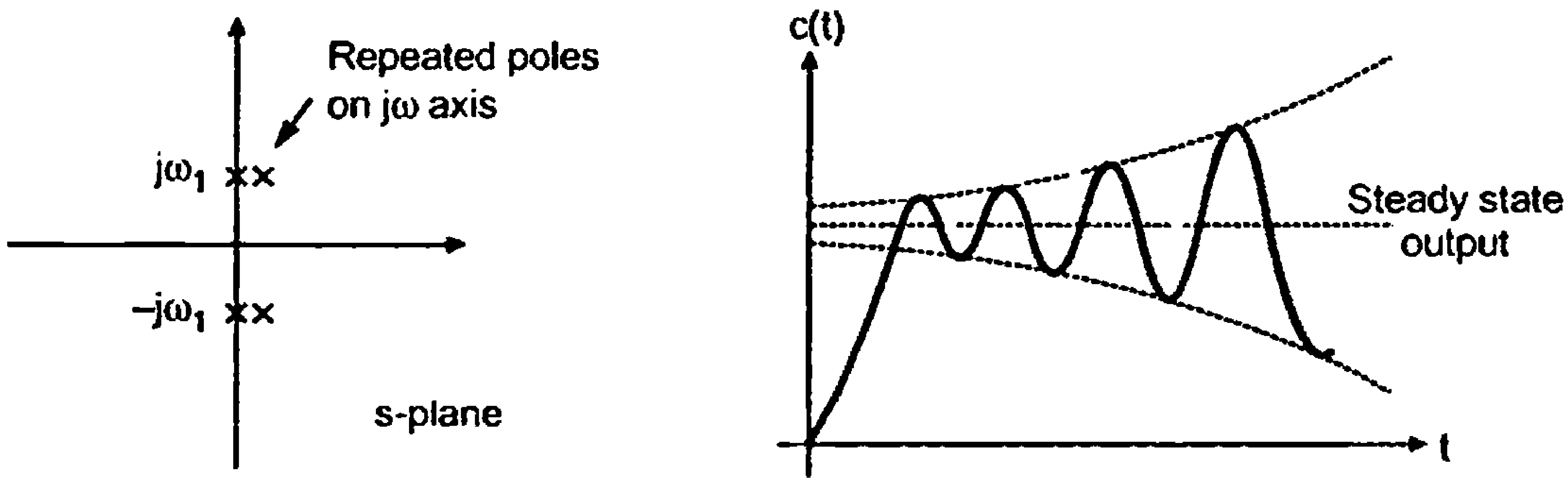


Fig. 8.6 Critically or marginally stable

Special Case : If there are **repeated roots** located purely on imaginary axis, system is said to be unstable.



(a) Location of poles **(b) Unstable system**

Fig. 8.7

Conditionally Stable System : A linear time invariant system is said to be conditionally stable if for a certain condition of a particular parameter of the system, its output is bounded one. Otherwise if that condition is violated output becomes unbounded and system becomes unstable i.e. stability of system depends on condition of parameter of the system. Such system is called **conditionally stable system.**

So s-plane can be divided into three distinct zones from stability point of view as shown in the Fig. 8.8.

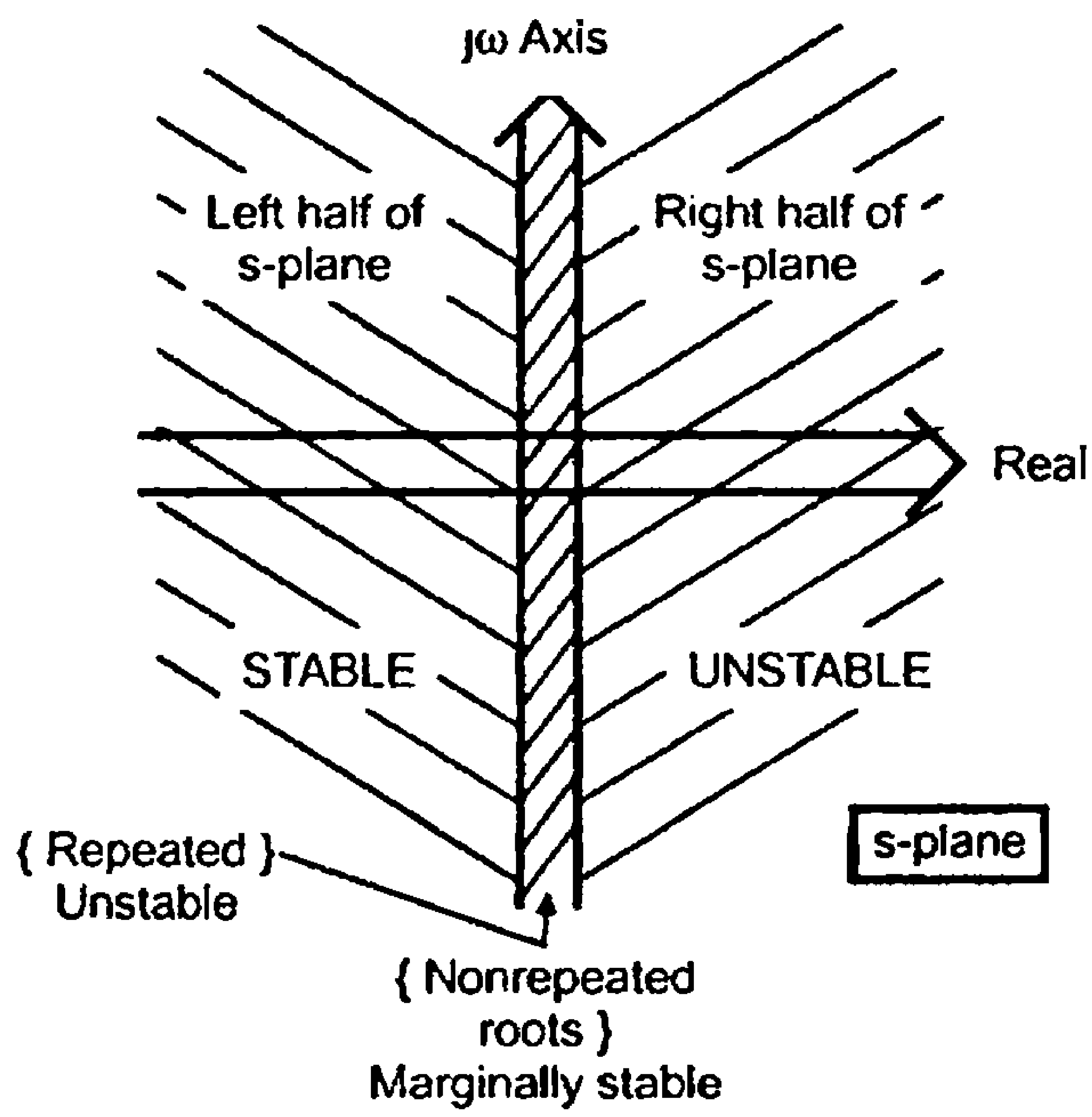


Fig. 8.8 Division of s-plane from stability point of view

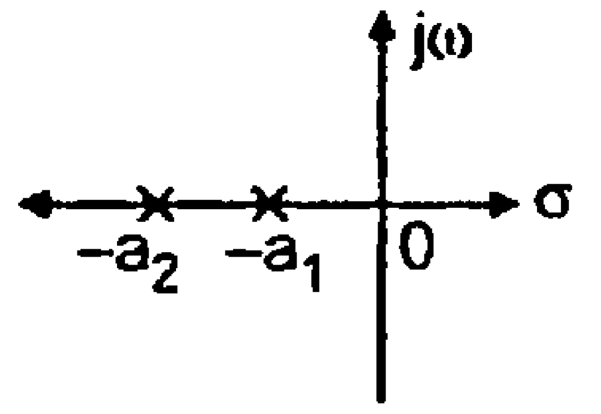
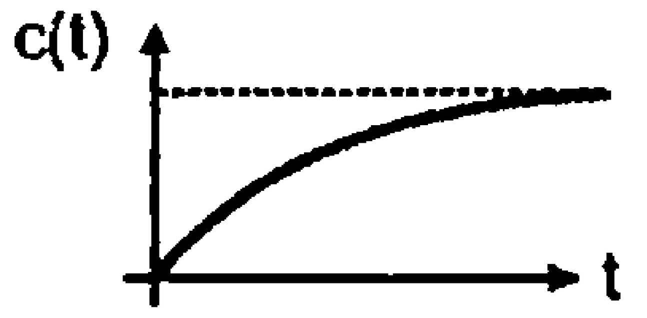
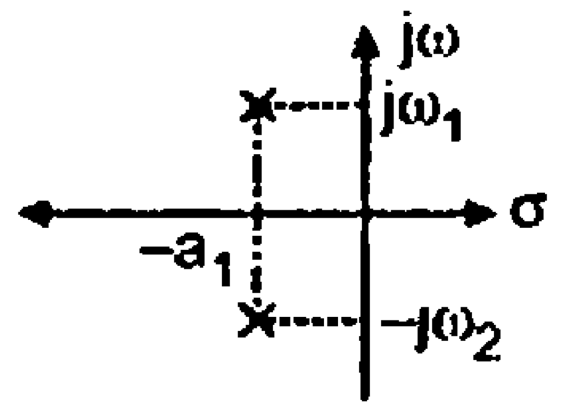
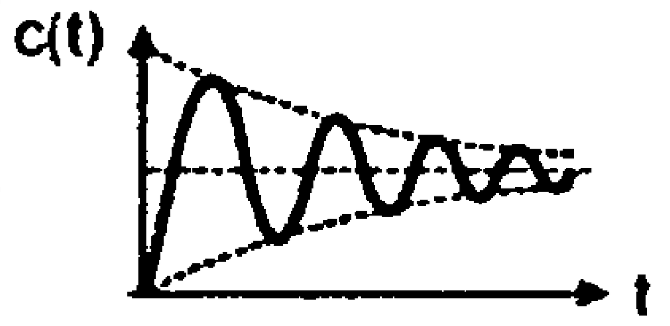
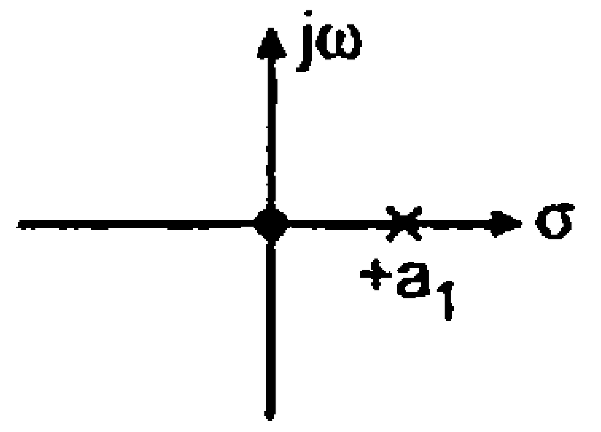
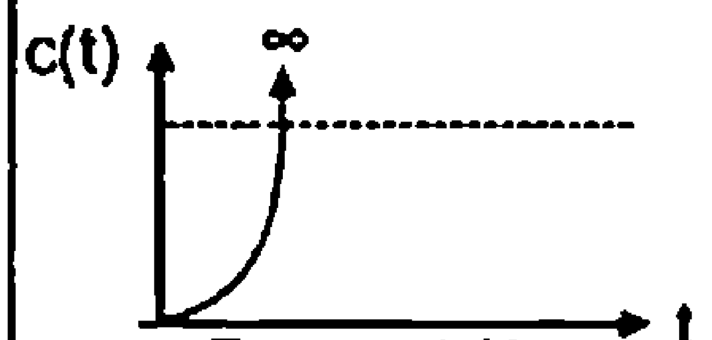
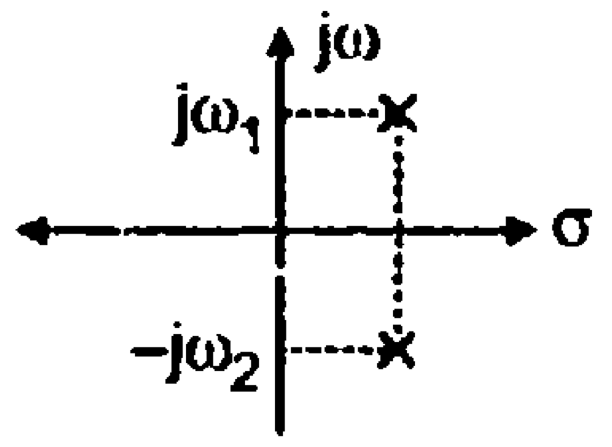
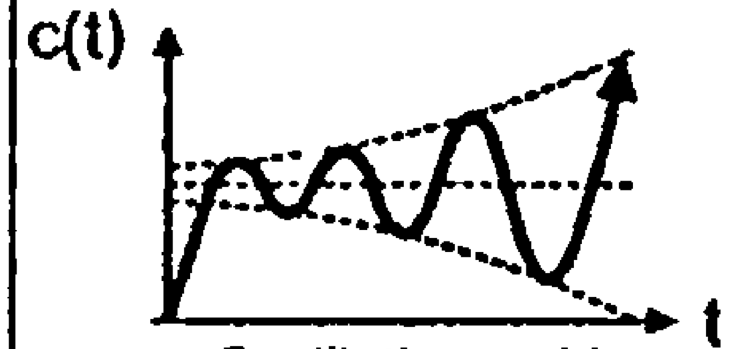
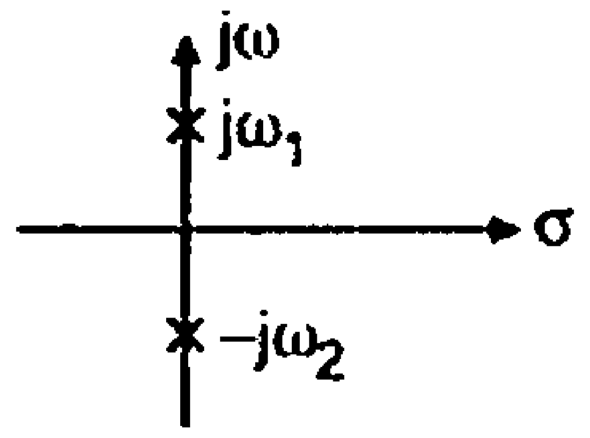
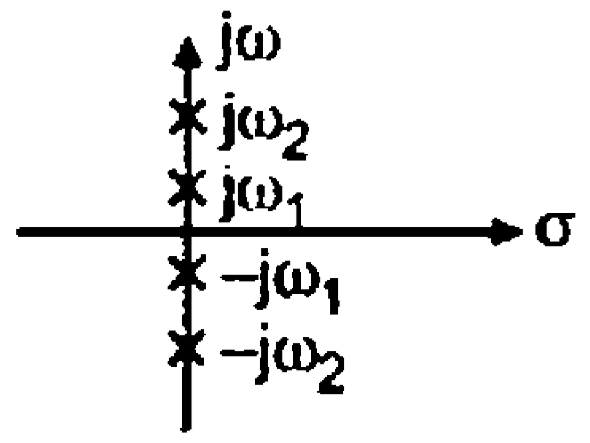
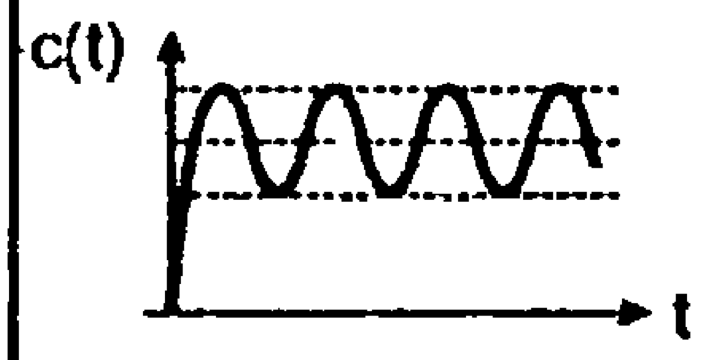
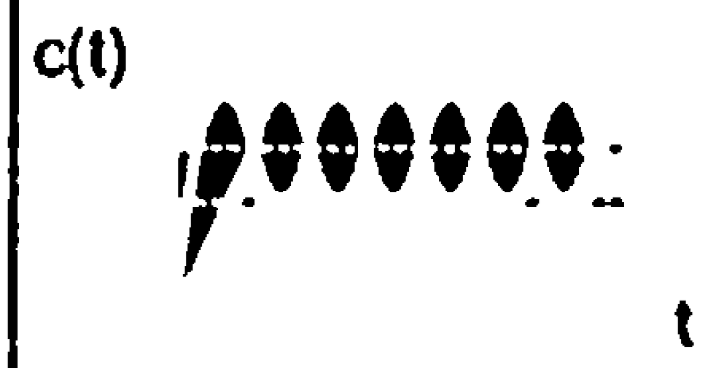
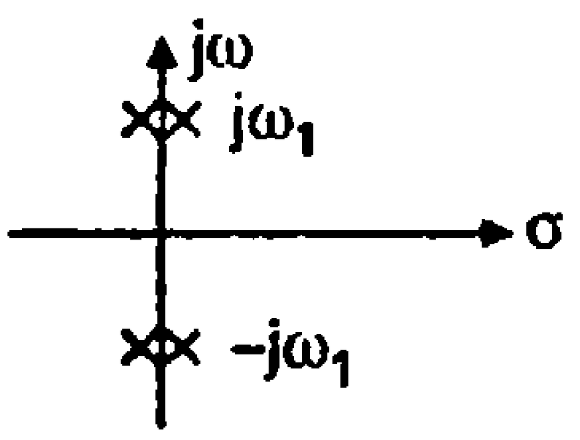
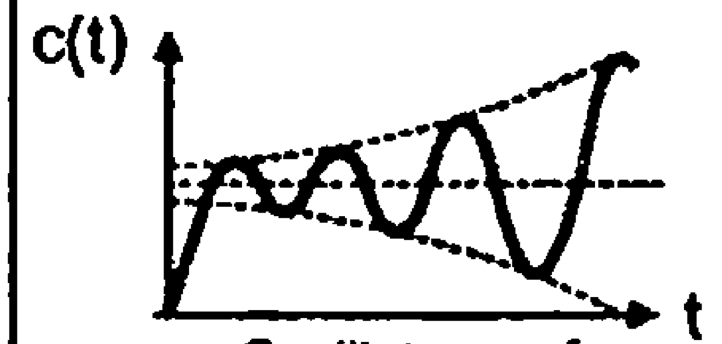
Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane.		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane.		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane).		 Exponential but increasing towards ∞	Unstable
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane.		 Oscillations with increasing amplitude	Unstable
5.	Nonrepeated pair on imaginary axis without any pole in R.H.S. of s-plane	 or  Two nonrepeated pairs on imaginary axis.	 Frequency of oscillations = ω_1  Sustained oscillations with two frequency components ω_1 and ω_2.	Marginally or critically stable Marginally or critically stable
6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane.		 Oscillations of increasing amplitude	Unstable

Table 8.1 Closed loop poles and stability

8.4 Zero Input and Asymptotic Stability

Some systems in practice may get driven by the initial conditions, without any input applied to it. For example a series RC circuit with capacitor initially charged to some voltage. This initial voltage is enough to start the operation of the system. This initial voltage, without any external input, drives the current till capacitor gets fully discharged. The stability related to such a system which is under zero input condition but operated under initial condition is called **zero input stability** of the system. The current through RC circuit reduces to zero as capacitor gets fully discharged. The current in such a case is called **zero input response** of the system, which is only due to the initial conditions. From this, zero input stability can be defined as :

If the zero input response of the system subjected to the finite initial conditions, reaches to zero as time t approaches infinity, the system is said to be **zero input stable** otherwise it is called **zero input unstable.**

Mathematically if c(t) is the zero input response of the system then for zero input stability there exists a positive number M, which depends on set of finite initial conditions such that,

$$\therefore \quad |c(t)| \leq M < \infty \text{ for all } t \geq t_0$$

$$\text{and} \quad \lim_{t \to \infty} |c(t)| = 0$$

As magnitude of zero input response reaches zero as Lim $t \to \infty$, the zero input stability is also called the **asymptotic stability.**

8.4.1 Remarks about Asymptotic Stability

Following are the important remarks about zero input or asymptotic stability,

1. The zero input or asymptotic stability depends on the roots of the characteristic equation i.e. closed loop poles of the systems.
2. All the requirements about the locations of roots of the characteristic equation related to BIBO stability are applicable to zero input or asymptotic stability. For zero input or asymptotic stability also, all the roots of the characteristic equation must be located in left half of s-plane.
3. If a system is BIBO stable, then it must be zero input or asymptotically stable.

Thus hereafter the system is said to be just stable, unstable or marginally stable, for all practical purposes.

Note that for nonrepeated pair of roots of the characteristic equation on jω axis, system is marginally stable. But an integrator having transfer function 1/s i.e. root located at origin is treated to be stable for all practical purposes as an exception.

8.5 Relative Stability

The system is said to be relatively more stable or unstable on the basis of settling time. System is said to be relatively more stable if settling time for that system is less than that of the other system.

The settling time of the root or pair of complex conjugate roots is inversely proportional to the real part of the roots.

So for the roots located near the jω axis, settling time will be large. As roots or pair of complex conjugate roots moves away from jω – axis i.e. towards left half of s-plane, settling time becomes lesser or smaller and system becomes more and more stable.

So relative stability of the system improves, as the closed loop poles move away from the imaginary axis in left half of s-plane.

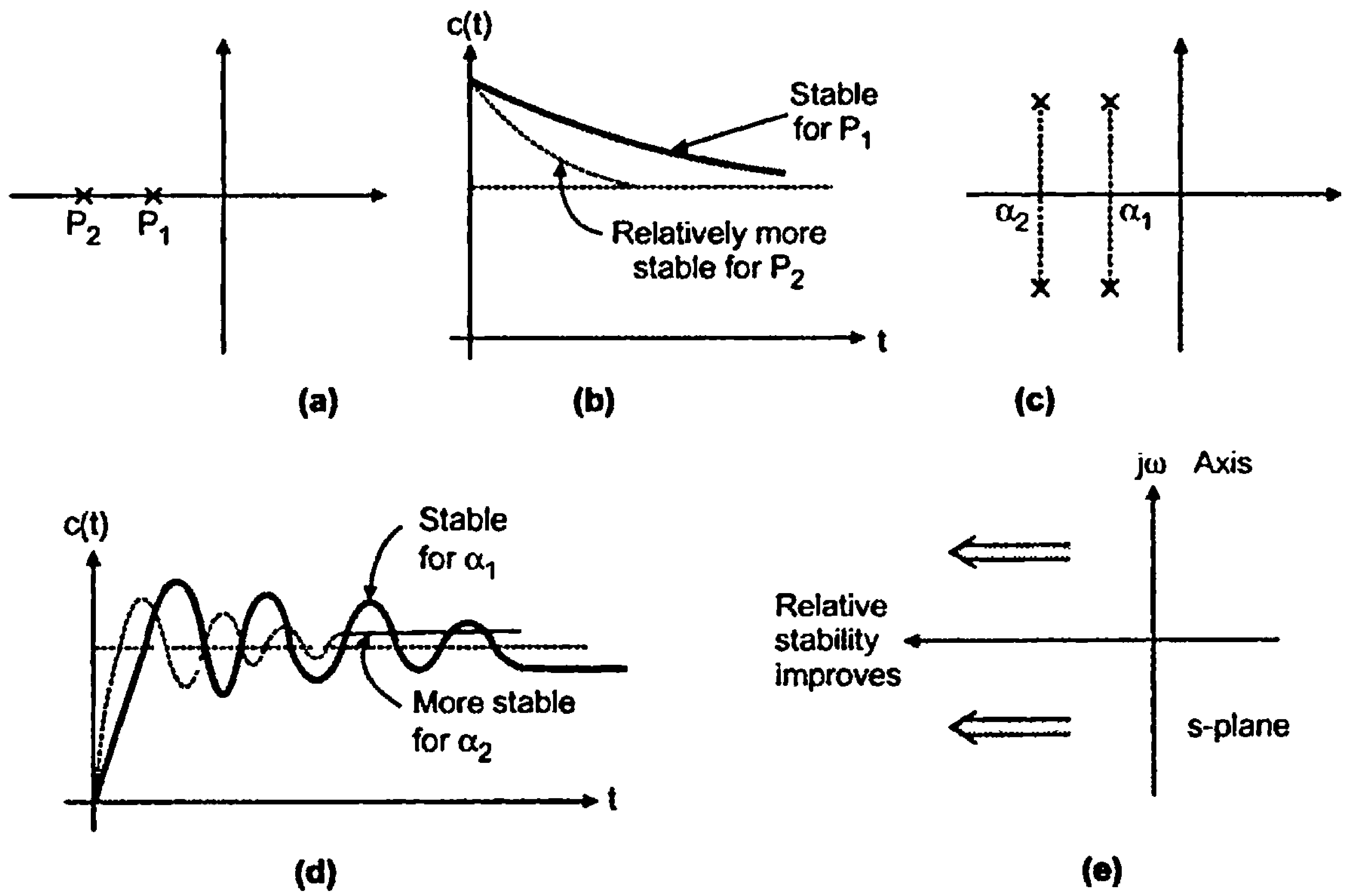


Fig. 8.9 Concept of relative stability

8.6 Routh-Hurwitz Criterion

This represents a method of determining the location of poles of a characteristic equation with respect to the left half and right half of the s-plane without actually solving the equation.

The T.F. of any linear closed loop system can be represented as,

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{B(s)}{F(s)}$$

where 'a' and 'b' are constants.

To find closed loop poles we equate F(s) = 0. This equation is called **characteristic**

equation of the system.

i.e. $F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

8.6.1 Necessary Conditions

In order that the above characteristic equation has no root in right of s-plane, it is **necessary but not sufficient** that,

1) All the coefficients of the polynomial have the same sign.
2) None of the coefficient vanishes i.e. all powers of `s' must be present in descending order from `n' to zero.

These conditions are not sufficient.

8.6.2 Hurwitz's Criterion

The sufficient condition for having all the roots of characteristic equation in left half of s-plane is given by Hurwitz. It is referred as Hurwitz criterion. It states that :

The necessary and sufficient condition to have all roots of characteristic equation in left half of s-plane is that the sub-determinants D_K , K = 1, 2, ...n obtained from Hurwitz's determinant `H' must all be positive.

Method of forming Hurwitz determinant :

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots\ldots & a_{2n-1} \\ a_0 & a_2 & a_4 & \ldots\ldots & a_{2n-2} \\ 0 & a_1 & a_3 & \ldots\ldots & a_{2n-3} \\ 0 & a_0 & a_2 & \ldots\ldots & a_{2n-4} \\ 0 & 0 & a_1 & \ldots\ldots & a_{2n-5} \\ . & . & . & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ 0 & . & . & \ldots\ldots & a_n \end{vmatrix}$$

The order is n × n where n = order of characteristic equation. In Hurwitz determinant all coefficients with suffices greater than `n' or negative suffices must all be replaced by zeros. From Hurwitz determinant subdeterminants D_K, K= 1, 2, ... n must be formed as follows :

$$D_1 = |a_1| \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_K = |H|$$

For the system to be stable, all the above determinants must be positive.

... Hurwitz's ...

... $= 4$, $n = 3$

$$\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

As D_2 and D_3 are negative, given system is unstable.

8.6.3 Disadvantages of Hurwitz's Method

i) For higher order systems, to solve the determinants of higher order is very complicated and time consuming.

...ber of roots located in right half of s-plane for unstable system cannot be ... this method.

...t marginal stability of the system.

... new method is suggested by the scientist Routh called ...th-Hurwitz method.

...witz's method.

...f characteristic equation in a ...h's array.

...ne = Number of sign changes = 2.

...y from characteristic equation.

From these two rows next and

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

This process is to be continued till the coefficient for s

From this array stability of system can be predicted.

8.7.1 Routh's Criterion

The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in the first column of Routh's array."

If there are any sign changes existing then,

a) System is unstable.

b) The number of sign changes equals the number of roots lying in the right half of the s-plane.

Examine the stability of given equations using Routh's method :

Example 8.2 : $s^3 + 6s^2 + 11s + 6 = 0$

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

s^3	1	11
s^2	6	6
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0
s^0	6	

As there is no sign change in first column, **system is stable.**

Example 8.3 : $s^3 + 4s^2 + s + 16 = 0$

Solution : $a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$

s^3	1	1
s^2	+ 4	16
s^1	$\frac{4 - 16}{4} = -3$	0
s^0	+ 16	

As there are two sign changes, **system is unstabl**

Number of roots located in the right half of s-pl

8.8 Special Cases of Routh's Criterion

8.8.1 Special Case 1

First element of any of the rows of Routh's array is zero and the same remaining row contains at least one non-zero element.

Effect : *The terms in the new row become infinite and Routh's test fails.*

e.g. : $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

s^5	1	3	2	
s^4	2	6	1	
s^3	$\boxed{0}$	1.5	0	Special case 1
s^2	∞	...	...	Routh' array failed

Following two methods are used to remove above said difficulty.

First method : Substitute a small positive number 'ε' in place of a zero occurred as a first element in a row. Complete the array with this number 'ε'. Then examine the sign change by taking $\lim\limits_{\varepsilon \to 0}$. Consider above example.

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon - 3}{\varepsilon}$	1	0
s^1	$\frac{\frac{1.5(6\varepsilon - 3)}{\varepsilon} - \varepsilon}{\frac{(6\varepsilon - 3)}{\varepsilon}}$	0	
s^0	1		

To examine sign change,

$$\lim_{\varepsilon \to 0}\left(\frac{6\varepsilon - 3}{\varepsilon}\right) = 6 - \lim_{\varepsilon \to 0}\frac{3}{\varepsilon}$$

$$= 6 - \infty$$

$$= -\infty \text{ sign is negative.}$$

$$\lim_{\varepsilon \to 0} \frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \to 0} \frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$$= +1.5 \text{ sign is positive.}$$

Routh's array is,

s^5	1	3	2
s^4	2	6	1
s^3	$+\varepsilon$ ↓	1.5	0
s^2	$-\infty$ ↓	1	0
s^1	+ 1.5	0	0
s^0	1	0	0

As there are two sign changes, **system is unstable.**

Second method : To solve the above difficulty one more method can be used. In this, replace 's' by '1/z' in original equation. Taking L.C.M., rearrange characteristic equation in descending powers of 'z'. Then complete the Routh's array with this new equation in 'z' and examine the stability with this array.

Consider $\quad F(s) = s^5+2s^4+3s^3+6s^2+2s+1 = 0$

Put $\quad s = 1/z$

$$\therefore \quad \frac{1}{z^5}+\frac{2}{z^4}+\frac{3}{z^3}+\frac{6}{z^2}+\frac{2}{z}+1 = 0$$

$$z^5+2z^4+6z^3+3z^2+2z+1 = 0$$

z^5	1	6	2
z^4	2	3	1
z^3	4.5	1.5	0
z^2	2.33	1	0
z^1	− 0.429	0	
z^0	1		

As there are two sign changes, **system is unstable**. The result is same.

In this book, the method 1 is used to solve the special case 1 of the Routh's array. In method 1, it is not always necessary to find limits of the terms consisting of ε in the first column of Routh's array. Just by mere inspection of such terms, remembering $\varepsilon \to 0$, the signs of such terms can be predicted.

8.8.2 Special Case 2

All the elements of a row in a Routh's array are zero.

Effect : The terms of the next row cannot be determined and Routh's test fails

s^5	a	b	c	
s^4	d	e	f	
s^3	0	0	0	← Row of zeros, Special case 2

This indicates non-availablility of coefficient in that row.

Example 8.1 : *Determine the stability of the given characteristic equation by Hurwitz's method.*

$F(s) = s^3 + s^2 + s^1 + 4 = 0$ is characteristic equation.

Solution : $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ n = 3$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

As D_2 and D_3 are negative, given system is unstable.

8.6.3 Disadvantages of Hurwitz's Method

i) For higher order systems, to solve the determinants of higher order is very complicated and time consuming.

ii) Number of roots located in right half of s-plane for unstable system cannot be judged by this method.

iii) Difficult to predict marginal stability of the system.

Due to these limitations, a new method is suggested by the scientist Routh called Routh's method. It is also called Routh-Hurwitz method.

8.7 Routh's Stability Criterion

It is also called **Routh's array method or Routh-Hurwitz's method.**

Routh suggested a method of tabulating the coefficients of characteristic equation in a particular way. Tabulation of coefficients gives an array called **Routh's array.**

Consider the general characteristic equation as,

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0$$

Method of forming an array :

s^n	a_0	a_2	a_4	a_6	....
s^{n-1}	a_1	a_3	a_5	a_7	
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
:	:	:	:		
s^0	a_n				

Coefficients for first two rows are written directly from characteristic equation.

From these two rows next rows can be obtained as follows.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

From 2nd and 3rd row, 4th row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n. From this array stability of system can be predicted.

8.7.1 Routh's Criterion

The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in the first column of Routh's array."

If there are any sign changes existing then,

a) System is unstable.

b) The number of sign changes equals the number of roots lying in the right half of the s-plane.

Examine the stability of given equations using Routh's method :

Example 8.2 : $s^3 + 6s^2 + 11s + 6 = 0$

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

s^3	1	11
s^2	6	6
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0
s^0	6	

As there is no sign change in first column, **system is stable.**

Example 8.3 : $s^3 + 4s^2 + s + 16 = 0$

Solution : $a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$

s^3	1	1
s^2	+ 4	16
s^1	$\frac{4 - 16}{4} = -3$	0
s^0	+ 16	

As there are two sign changes, **system is unstable.**

Number of roots located in the right half of s-plane = Number of sign changes = 2.

8.8.2.1 Procedure to Eliminate this Difficulty

i) Form an equation by using the coefficients of a row which is just above the row of zeros. Such an equation is called an **Auxiliary Equation** denoted as A(s). For above case such an equation is,

$$A(s) = ds^4 + es^2 + f$$

Key Point : *The coefficients of any row are corresponding to alternate powers of 's' starting from the power indicated against it.*

So 'd' is coefficient corresponding to s^4 so first term is ds^4 of A(s).

Next coefficient 'e' is corresponding to alternate power of 's' from 4 i.e. s^2 hence the term es^2 and so on.

ii) Take the derivative of an auxiliary equation with respect to 's'.

i.e. $$\frac{dA(s)}{ds} = \underline{4d}\, s^3 + \underline{2e}\, s$$

iii) Replace row of zeros by the coefficients of $\frac{dA(s)}{ds}$.

s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0

iv) Complete the array interms of these new coefficients.

8.8.2.2 Importance of an Auxiliary Equation

Auxiliary equation is always the part of the original characteristic equation. This means the roots of the auxiliary equation are some of the roots of original characteristic equation. Not only this but the **roots of auxiliary equation are the most dominant roots of the original characteristic equation, from the stability point of view.**

Key Point : *The stability can be predicted from the roots of A(s) = 0 rather than the roots of characteristic equation as the roots of A(s) = 0 are the most dominant from the stability point of view. The remaining roots of the characteristic equation are always in the left half and they do not play any significant role in the stability analysis.*

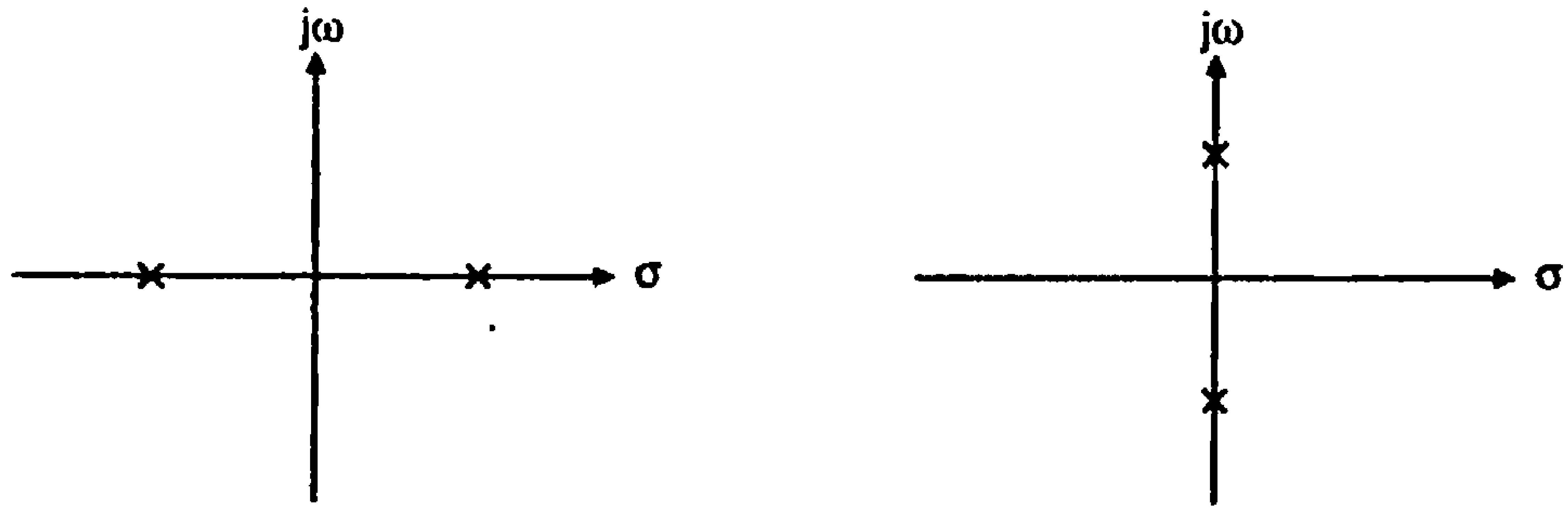
e.g. Let F(s) = 0 is the original characteristic equation of say order n = 5.

Let A(s) = 0 be the auxiliary equation for the system due to occurrence of special case 2 of the order m = 2.

Then out of 5 roots of F(s) = 0, the 2 roots which are most dominant (Dominant means very close to imaginary axis or on the imaginary axis or in the right half of s-plane) from the stability point of view are the 2 roots of A(s) = 0. The remaining 5 – 2 = 3 roots are not significant from stability point of view as they will be far away from the imaginary axis in the left half of s-plane.

The roots of auxiliary equation may be,

i) A pair of real roots of opposite sign i.e. as shown in the Fig. 8.10 (a).

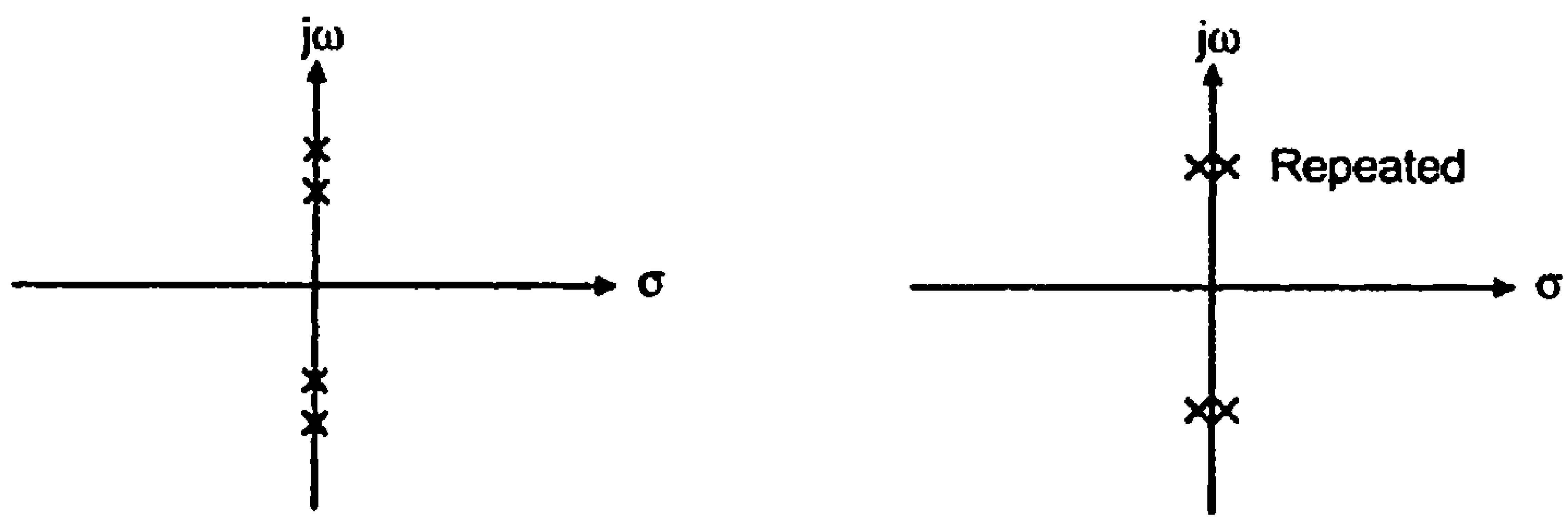


(a) (b)

Fig. 8.10

ii) A pair of roots located on the imaginary axis as shown in the Fig. 8.10 (b).

iii) The nonrepeated pairs of roots located on the imaginary axis as shown in the Fig. 8.10 (c).



(c) (d)

Fig. 8.10

iv) The repeated pairs of roots located on the imaginary axis as shown in the Fig. 8.10 (d).

Hence total stability can be determined from the roots of A(s) = 0, which can be out of four types shown above.

8.8.2.3 Change in Criterion of Stability in Special Case 2

After replacing a row of zeros by the coefficients of $\frac{dA(s)}{ds}$, complete the Routh's array.

But now, the criterion that, no sign change in 1^{st} column of array for stability, **no longer remains sufficient but becomes a necessary**. This is because though A(s) is a part of orginial characteristic equation, $\frac{dA(s)}{ds}$ is not, which is infact used to complete the array.

So if sign change occurs in first column, system is unstable with number of sign changes equal to number of roots of characteristic equation located in right half of s-plane.

Key Point : *But if there is no sign change, system cannot be predicted as stable. And in such case stability is to be determined by actually solving A(s) = 0 for its roots. And from the locations of roots of A(s) = 0 in the s-plane the system stability must be determined, because roots A(s) = 0 are always dominant roots of characteristic equation.*

8.9 Applications of Routh's Criterion

8.9.1 Relative Stability Analysis

If it is required to find relative stability of system about a line $s = -\sigma$. i.e. how many roots are located in right half of this line $s = -\sigma$, the Routh's method can be used effectively.

To determine this from Routh's array, shift the axis of s-plane and then apply Routh array i.e. substitute $s = s' - \sigma$, (σ = constant) in characteristic equation. Write polynomial in terms of s'. Complete array from this new equation. The number of sign changes in first column is equal to number of roots those are located to right of the vertical line $s = -\sigma$.

Key Point : *Instead of variable s', any other variable may be used. Thus if z is new variable then s must be replaced by z − σ in the equation.*

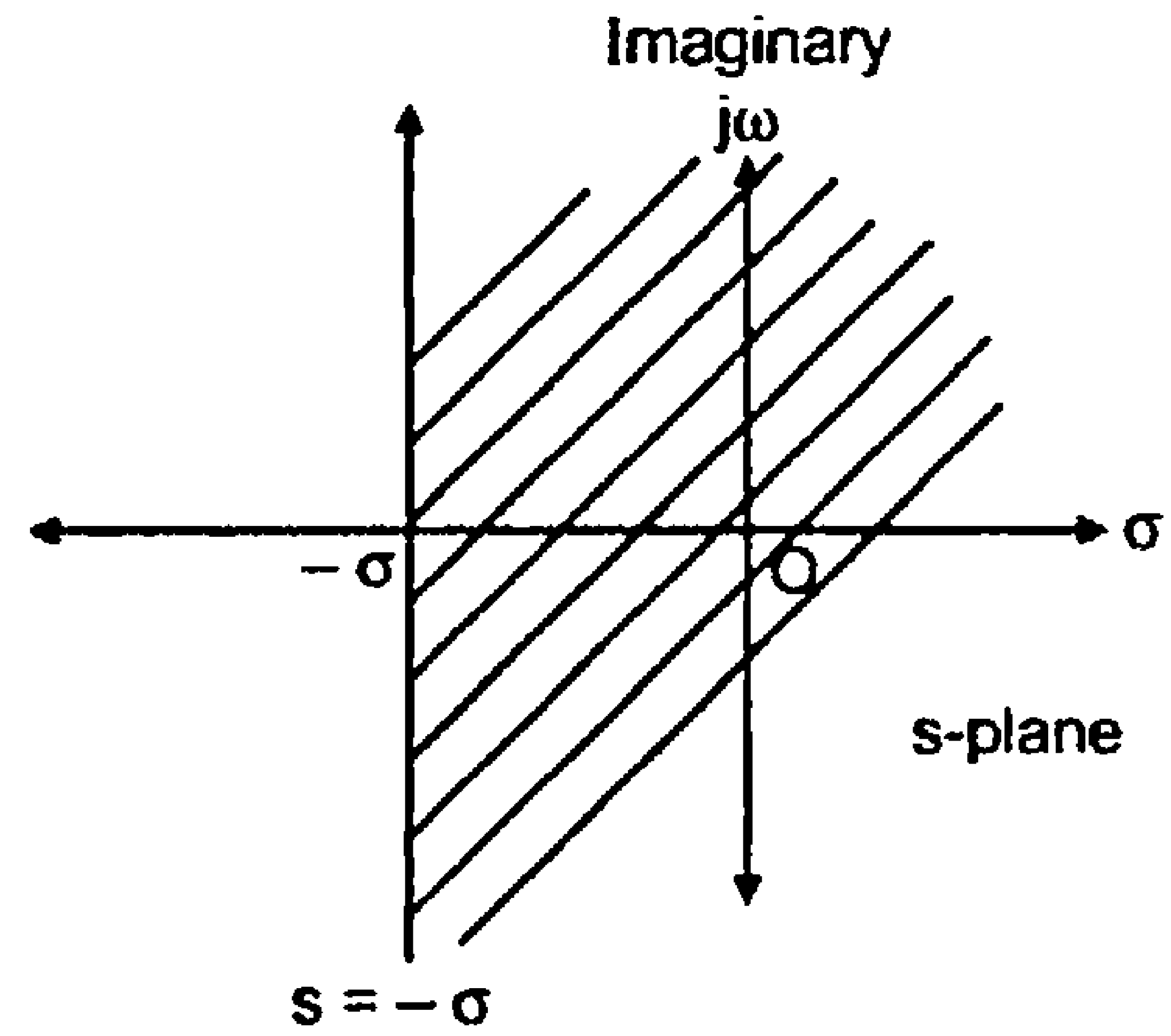


Fig. 8.11

8.9.2 Determining Range of Values of K

In practical system, an amplifier of variable gain K is introduced as shown in the Fig. 8.12.

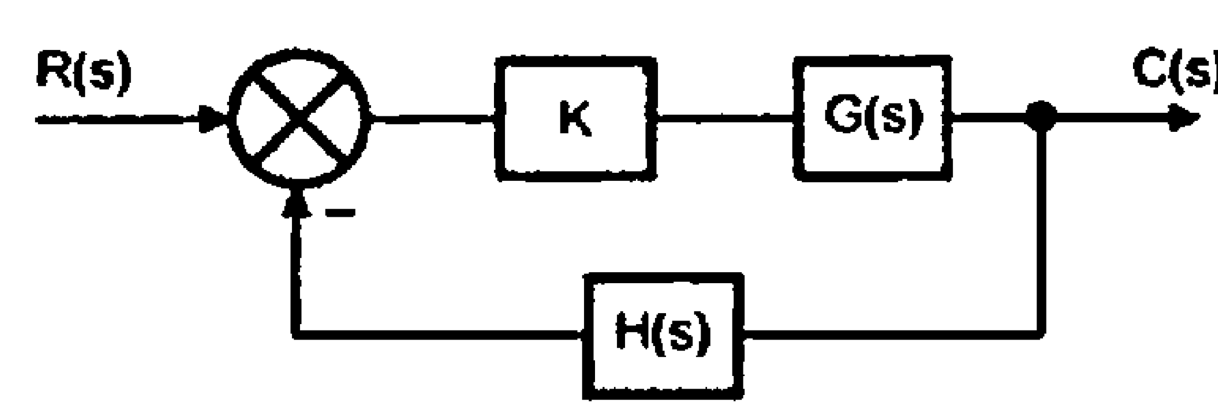


Fig. 8.12

The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Hence the characteristic equation is

$$F(s) = 1 + KG(s)H(s) = 0$$

So the locations of roots of the above equation are dependent on the proper selection of value of 'K'.

So unknown 'K' appears in the characteristic equation. In such case Routh's array is to be constructed in terms of K and then the range of values of K can be obtained in such a way that it will not produce any sign change in the first column of the Routh's array. Hence it is possible to obtain the range of values of K for absolute stability of the system using Routh's criterion. Such a system where stability depends on the condition of parameter K, is called conditionally stable system.

8.10 Advantages of Routh's Criterion

Advantages of Routh's array method are :

i) Stability of the system can be judged without actually solving the characteristic equation.

ii) No evaluation of determinants, which saves calculation time.

iii) For unstable system it gives number of roots of characteristic equation having positive real part.

iv) Relative stability of the system can be easily judged.

v) By using this criterion, critical value of system gain can be determined hence frequency of sustained oscillations can be determined.

vi) It helps in finding out range of values of K for system stability.

vii) It helps in finding out intersection points of root locus with imaginary axis.

8.11 Limitations of Routh's Criterion

i) It is valid only for real coefficients of the characteristic equation.

ii) It does not provide exact locations of the closed loop poles in left or right half of s-plane.

iii) It does not suggest methods of stabilising an unstable system.

iv) Applicable only to linear systems.

Example 8.4 : $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$

Solution :

s^5	1	2	3
s^4	1	2	15
s^3	0	– 12	0

Replace 0 by small positive number ε.

s^5	1	2	3
s^4	1	2	15
s^3	ε	–12	0
s^2	$\dfrac{2\varepsilon+12}{\varepsilon}$	15	0
s^1	$\dfrac{\left(\dfrac{2\varepsilon+12}{\varepsilon}\right)(-12)-15\varepsilon}{\dfrac{2\varepsilon+12}{\varepsilon}}$	0	0
s^0	15		

$$\lim_{\varepsilon \to 0} = \frac{2\varepsilon+12}{\varepsilon} = 2+\frac{12}{\varepsilon} = +\infty$$

$$\lim_{\varepsilon \to 0} \frac{\frac{(2\varepsilon+12)}{\varepsilon}(-12)-15\varepsilon}{\frac{(2\varepsilon+12)}{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{-24\varepsilon-144-15\varepsilon^2}{2\varepsilon+12}$$

$$= \frac{0-144-0}{0+12} = -12$$

s^5	1	2	3
s^4	1	2	15
s^3	ε	– 12	0
s^2	+ ∞	15	0
s^1	–12	0	
s^0	15		

There are two sign changes, so system is unstable.

Example 8.5 : *$s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$. Find the number of roots of this equation with positive real part, zero real part and negative real part.*

Solution :

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	0	0	0	

$$A(s) = 3s^4 - 48 = 0 \qquad \frac{dA}{ds} = 12s^3$$

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	12	0	0	0
s^2	[ε] 0	– 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	– 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

$\therefore$ One sign change and system is unstable . Thus there is one root in R.H.S. of s-plane i.e. with positive real part. Now solve A(s) = 0 for the dominant roots.

$$A(s) = 3s^4 - 48 = 0$$

Put $s^2 = y$

$\therefore$ $3y^2 = 48$ $\quad\therefore$ $y^2 = 16,$ $\quad\therefore$ $y = \pm\sqrt{16} = \pm 4$

$\therefore$ $s^2 = +4$ $\qquad s^2 = -4$

$s = \pm 2$ $\qquad s = \pm 2j$

So $s = \pm 2j$ are the two roots on imaginary axis i.e. with zero real part. Root in R.H.S. indicated by a sign change is $s = +2$ as obtained by solving A(s) = 0. Total there are 6 roots as n = 6.

Roots with positive real part = 1

Roots with zero real part = 2

Roots with negative real part = 6 – 2 – 1 = 3.

8.12 Marginal K and Frequency of Sustained Oscillations

Marginal value of 'K' is that value of 'K' for which system becomes marginally stable. For a marginally stable system there must be a row of zeros occurring in Routh's array. **So value of 'K' which makes any row of Routh array as row of zeros is called marginal value of K.** Now K = 0 makes row of s^0 as row of zeros but K = 0 can not be marginal value, because for K = 0, constant term in the characteristic equation becomes zero i.e. one coefficient for s^0 vanishes which makes system unstable instead of marginally stable.

Key Point : *Hence marginal value of 'K' is a value which makes any row other than* s^0 *as row of zeros.*

To obtain the frequency of oscillations, solve the auxiliary equation A(s) = 0 for $K = K_{mar}$. The magnitude of imaginary roots of A(s) = 0 obtained for marginal value of K (K_{mar}) indicates the frequency of sustained oscillations, which system is going to produce.

Example 8.6 : *For unity feedback system,* $G(s) = \dfrac{K}{s(1+0.4s)(1+0.25s)}$ *, find range of values of K, marginal value of K and frequency of sustained oscillations.*

Solution : Characteristic equation, 1 + G(s)H(s) = 0 and H(s) = 1

$$\therefore \quad 1+\frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s[1 + 0.65s + 0.1s^2] + K = 0$$

$$\therefore \quad 0.1s^3 + 0.65s^2 + s + K = 0$$

s^3	0.1	1	From s^0, $K > 0$
s^2	0.65	K	From s^1,
s^1	$\frac{0.65-0.1K}{0.65}$	0	$0.65 - 0.1K > 0$ $\therefore$ $0.65 > 0.1\,K$
s^0	K		$\therefore$ $6.5 > K$

$\therefore$ Range of values of K, $0 < K < 6.5$.

The marginal value of `K' is a value which makes any row other than s^0 as row of zeros.

$$\therefore \quad 0.65 - 0.1\,K_{mar} = 0$$

$$\therefore \quad \boxed{K_{mar} = 6.5}$$

To find frequency, find out roots of auxiliary equation at marginal value of `K'.

$$A(s) = 0.65s^2 + K = 0\,;$$

$$\therefore \quad 0.65s^2 + 6.5 = 0 \qquad \because \; K_{mar} = 6.5$$

$$s^2 = -10$$

$$s = \pm\, j\, \mathbf{3.162}$$

Comparing with $s = \pm\, j\omega$

ω = Frequency of oscillations

= **3.162 rad/sec.**

Examples with Solutions

Example 8.7 : *For system $s^4 + 22s^3 + 10s^2 + s + K = 0$, find K_{mar} and ω at K_{mar}.*

Solution :

s^4	1	10	K
s^3	22	1	0
s^2	9.95	K	0
s^1	$\frac{9.95-22K}{9.95}$	0	
s^0	K		

Marginal value of `K' which makes row of s^1 as row of zeros.

$$9.95 - 22\,K_{mar} = 0$$

$$\therefore \quad K_{mar} = \mathbf{0.4524}$$

Hence $A(s) = 9.95s^2 + K = 0$

$$9.95s^2 + 0.4524 = 0$$

$$s^2 = -0.04546$$

$$s = \pm j\,0.2132$$

Hence frequency of oscillations = **0.2132 rad/sec.**

Example 8.8 : *For a system with characteristic equation* $F(s) = s^6 + 3s^5 + 4s^4 + 6s^3 + 5s^2 + 3s + 2 = 0$, *examine stability.*

Solution :

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	0	0	0	0

Row of zeros

$$A(s) = 2s^4 + 4s^2 + 2 = 0 \quad \text{i.e.} \quad s^4 + 2s^2 + 1 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 + 4s$$

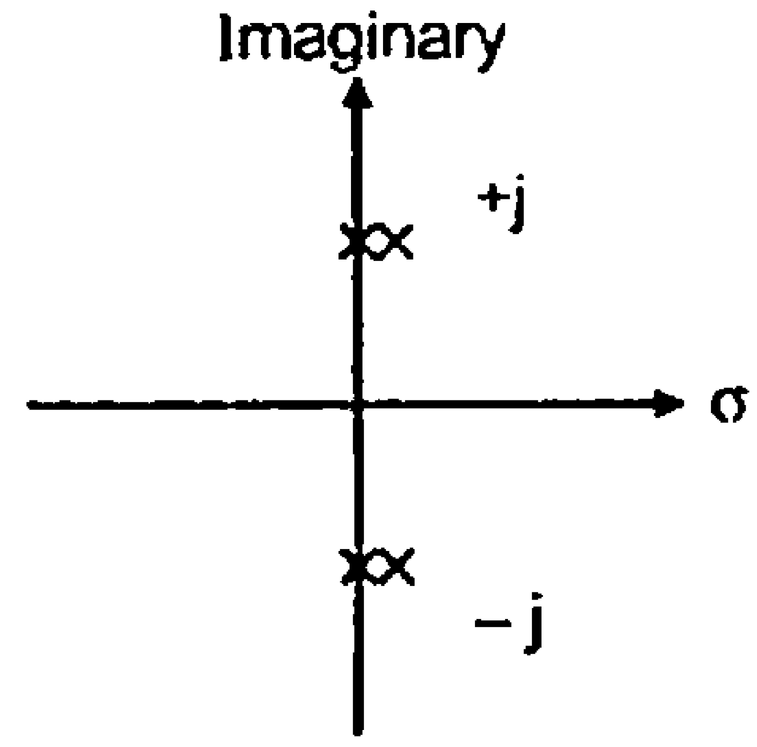
s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	0	0	0	0

Row of zeros again

$$\therefore \quad A'(s) = 2s^2 + 2 = 0$$

$$\frac{dA'(s)}{ds} = 4s = 0$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	4	0	0	0
s^0	2	0	0	0



No sign change, hence no root is located in R.H.S. of s-plane. As row of zeros occur, system may be marginally stable or unstable. To examine that find the roots of first auxiliary equation.

$A(s) = s^4 + 2s^2 + 1 = 0 \qquad s^2 = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

$s^2 = -1, \qquad s^2 = -1, \qquad s_{1,2} = \pm j, \qquad s_{3,4} = \pm j$

The roots of A '(s) = 0 are the roots of A(s) = 0. So do not solve second auxiliary equation. Predict the stability from the nature of roots of first auxiliary equation.

As there are repeated roots on imaginary axis, system is unstable.

Example 8.9 : *$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$. Check the stability of given characteristic equation using Routh's method.* **(M.U. : May-1996)**

Solution :

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0 ⇦ Special case 2

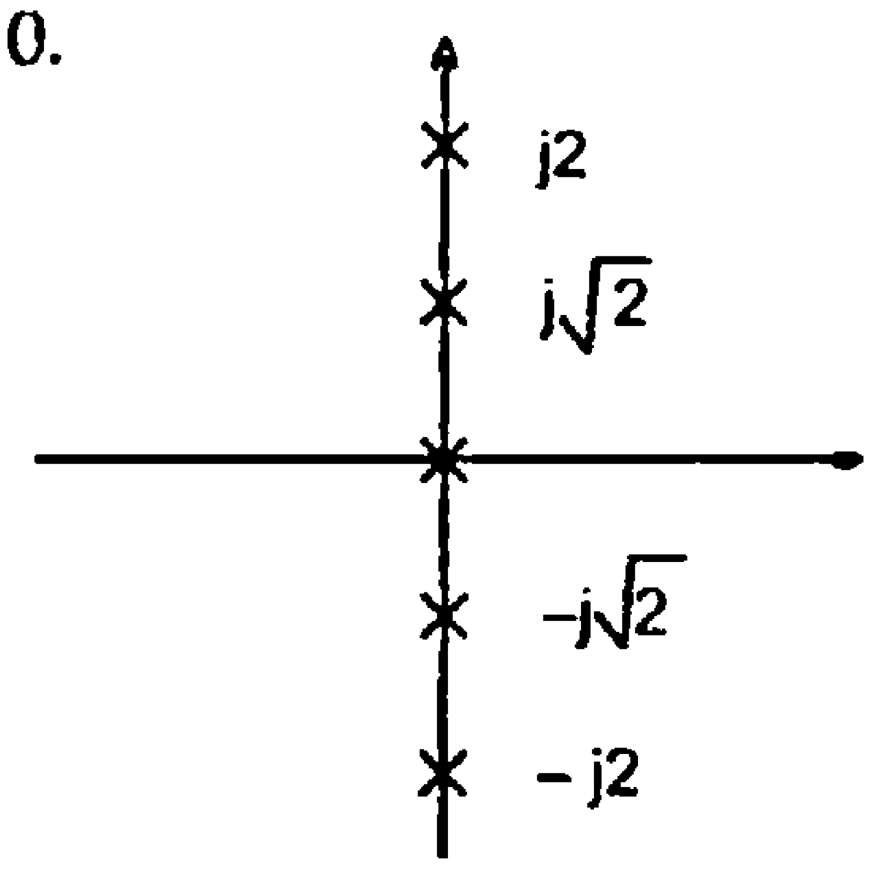
Row of zeros

$$A(s) = 2s^4 + 12s^2 + 16 = 0$$

$$\frac{dA}{ds} = 8s^3 + 24s = 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	
s^1	2.67	0		
s^0	16			

No sign change, so system may be stable. But as there is row of zero, system will be (i) marginally stable or (ii) unstable. To examine this solve A(s) = 0.



$$2s^4 + 12s^2 + 16 = 0$$

$$s^4 + 6s^2 + 8 = 0$$

Put $s^2 = y$

$\therefore \quad y^2 + 6y + 8 = 0$

$$y = -6 \pm \frac{\sqrt{36-32}}{2}$$

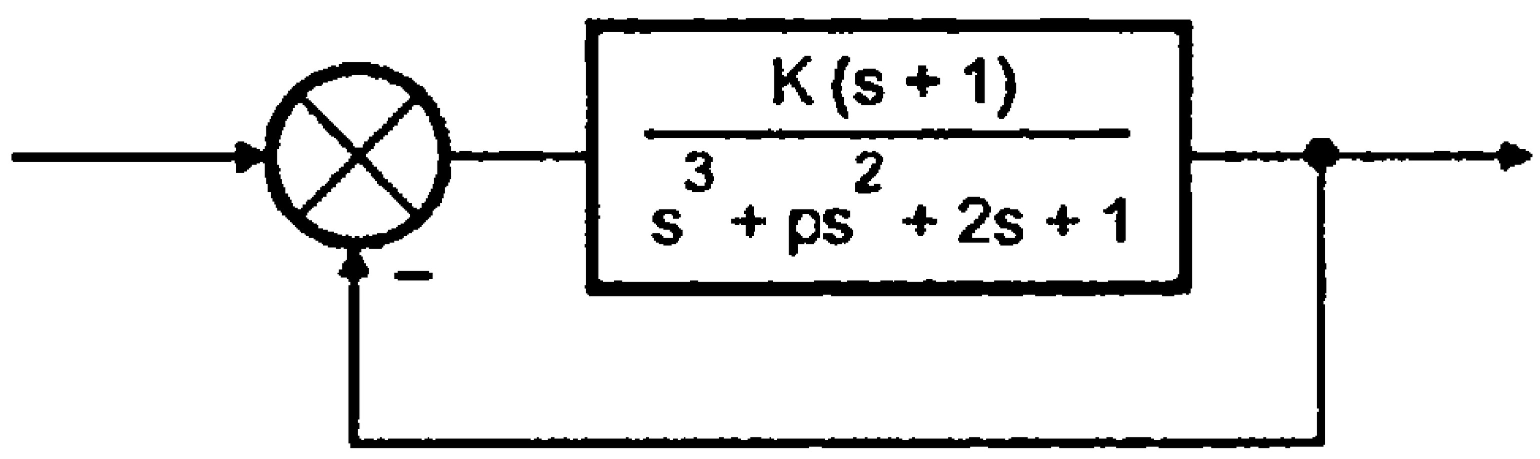
$$= -3 \pm 1 = -2, -4$$

$\therefore \quad s^2 = -2 \quad$ and $\quad s^2 = -4$

$\therefore \quad s = \pm j\sqrt{2} \quad$ and $\quad s = \pm j2$

Nonrepeated roots on imaginary axis. Hence system is marginally stable.

Example 8.10 : *A given system oscillates with frequency 2 rad/sec. Find values of `$K_{mar}$' and `p'. No poles are in R.H.S.* **(M.U. : May-1996)**



Solution : As system oscillates, it is marginally stable and value of `K' at this situation is marginal value of `K'. As system is marginally stable there must be row of zeros occurring in Routh's array.

Characteristic equation is given by,

$$1 + \frac{K(s+1)}{s^3 + ps^2 + 2s + 1} = 0$$

$$\therefore \quad s^3 + ps^2 + (2+K)s + (1+K) = 0$$

s^3	1	$2 + K$
s^2	p	$1 + K$
s^1	$\frac{(2+K)p - (1+K)}{p}$	0
s^0	$1 + K$	

At marginal value of `K'

$$(2 + K)\, p - (1 + K) = 0$$

$$\therefore \quad (2 + K)\, p = (K + 1)$$

$$\therefore \quad p = \frac{K+1}{K+2} \qquad \dots (1)$$

Now at this value,

$$A(s) = ps^2 + K + 1 = 0$$

$$\therefore \quad s^2 = -\frac{(K+1)}{p}, \qquad s = \pm j\sqrt{\frac{K+1}{p}}$$

Compare with $s = \pm j\omega$ and frequency ω is given as 2.

$$\therefore \quad \sqrt{\frac{K+1}{p}} = 2$$

$$\frac{K+1}{p} = 4$$

$$\therefore \quad p = \frac{K+1}{4} \quad \text{... (2)}$$

$$\therefore \quad \frac{K+1}{K+2} = \frac{K+1}{4}$$

$$\therefore \quad 4 = K + 2 \qquad \therefore K_{mar} = 2$$

$$\therefore \quad p = \frac{K_{mar}+1}{4} = \frac{3}{4} = 0.75$$

Example 8.11 : *The open loop transfer function of a feedback system is*

$$G(s)H(s) = \frac{K(s+5)}{s(1+Ts)(1+2s)}$$

Parameters K and T are represented on a plane with K on x axis and T on y axis. Determine region in which a closed loop system is stable. **(M.U. : Nov.-1996)**

Solution : The characteristic equation of the system is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+5)}{s(1+Ts)(1+2s)} = 0$$

$$\therefore \quad s(1 + Ts)(1 + 2s) + K(s + 5) = 0$$

$$\therefore \quad s(2Ts^2 + (T+2)s + 1) + K(s + 5) = 0$$

$$\therefore \quad 2Ts^3 + (T+2)s^2 + s(K + 1) + 5K = 0$$

∴ Routh's array is

s^3	$2T$	$K + 1$
s^2	$(T + 2)$	$5K$
s^1	$\frac{(T+2)(K+1) - 2T \times 5K}{(T+2)}$	0
s^0	$5K$	

From last row $5K > 0$

K must be positive.

From row of s^1

$$(T + 2)(K + 1) - 10KT > 0$$

$$KT + 2K + T + 2 - 10KT > 0$$

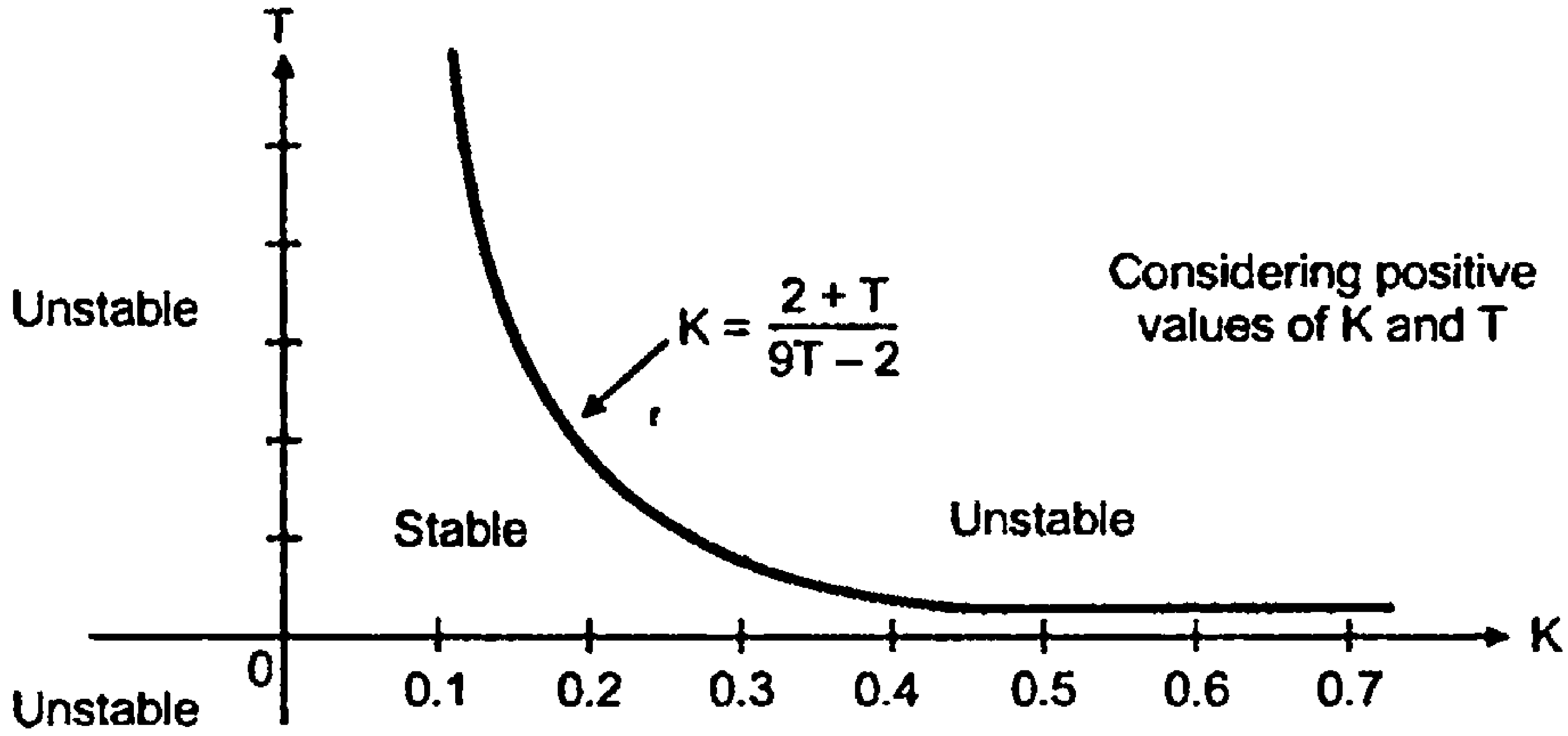
$$\therefore \quad 2K + T + 2 - 9KT > 0$$

Limiting value is

$$2K + T + 2 - 9KT > 0$$

i.e. $$K < \frac{2+T}{9T-2}$$

$\therefore$ Region in which a closed loop system is stable.



Example 8.12 : *The open loop transfer function of a unity feedback system is given by,*

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

Derive an expression for gain K interms of T_1 and T_2 for the stability of the system.

Solution : The characteristic equation is,

$$1 + G(s)H(s) = 0$$

$$\therefore \quad 1+\frac{K}{s(1+sT_1)(1+sT_2)} = 0$$

$$\therefore \quad T_1T_2s^3 + s^2(T_1+T_2) + s + K = 0$$

The Routh's array is

s^3	T_1T_2	1
s^2	T_1+T_2	K
s^1	$\frac{(T_1+T_2)-KT_1T_2}{(T_1+T_2)}$	0
s^0	K	

From s^0, $$K > 0$$

From s^1, $$(T_1+T_2)-KT_1T_2 > 0$$

$\therefore \quad T_1 + T_2 > KT_1T_2$

$\therefore \quad K < \frac{T_1+T_2}{T_1T_2} < \frac{1}{T_2}+\frac{1}{T_1}$

So $0 < K < \left(\frac{1}{T_1}+\frac{1}{T_2}\right)$ is the range of K for stability.

Example 8.13 : *Determine the ranges of K such that the characteristic equation.*

$s^3 + 3(K+1)s^2 + (7K+5)s + (4K+7) = 0$ has roots more negative than $s = -1$.

Solution : Put $s = x - 1$

x is dummy variable to shift origin at s = –1. The Routh's array interms of x indicates how many roots exist to the right or left of s = –1 in the s-plane. Instead of x, any other variable such as s′ may be used.

$\therefore \quad (x-1)^3 + 3(K+1)(x-1)^2 + (7K+5)(x-1) + (4K+7) = 0$

$\therefore \quad x^3 - 3x^2 + 3x - 1 + 3Kx^2 - 6Kx + 3K$
$+ 3x^2 - 6x + 3 + 7Kx - 7K + 5x - 5 + 4K + 7 = 0$

$\therefore \quad x^3 + x^2[-3+3K+3] + x[3-6K-6+7K+5]$
$+ [-1+3K+3-7K-5+4K+7] = 0$

$\therefore \quad x^3 + 3Kx^2 + (K+2)x + 4 = 0$

Routh's array is,

x^3	1	K + 2
x^2	3K	4
x^1	$\frac{3K(K+2)-4}{3K}$	0
x^0	4	

From x^2, $\quad 3K > 0$

$\therefore \quad K > 0$

$3K(K+2) - 4 > 0$

$\therefore \quad 3K(K+2) > 4$

$\therefore \quad 3K^2 + 6K - 4 > 0$

$K^2 + 2K - 1.33 > 0$

$\therefore (K - 0.5275)(K + 2.5275) > 0$

$\therefore \quad K > 0.5275$ and $K > -2.5275$

Ultimate range of K is 0.5275 < K < ∞, to have roots more negative than s = −1 i.e. located to the left of s = −1.

Example 8.14 : *$s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$. Check the stability of given characteristic equation using Routh's method.*

Solution :

s^6	1	3	− 16	− 48	
s^5	4	0	− 64	0	
s^4	3	0	− 48	0	
s^3	0	0	0		⇐ Special case 2

$$A(s) = 3s^4 - 48 = 0$$

$$\frac{dA}{ds} = 12s^3$$

s^6	1	3	− 16	− 48	
s^5	4	0	− 64	0	
s^4	3	0	− 48	0	
s^3	12	0	0	0	
s^2	[ε] 0	− 48	0	0	⇐ Special case 1
s^1	$\frac{576}{\varepsilon}$	0	0		
s^0	− 48				

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

∴ One sign change and system is unstable .

$$A(s) = 3s^4 - 48 = 0$$

Put $s^2 = y$

∴ $3y^2 = 48$ ∴ $y^2 = 16$, ∴ $y = \pm\sqrt{16}$

∴ $s^2 = +\sqrt{16}$ $s^2 = -\sqrt{16}$

$s^2 = 4$ $s^2 = -4$

$s = \pm 2$ $s = \pm 2j$

Roots with positive real part	→	One
Roots with negative real part	→	Three
Roots with zero real part	→	Two

Example 8.15 : *For a system* $G(s)H(s) = \frac{K(1+s)^2}{s^3}$, *find range of `K' for system to be stable.*

Solution : Characteristic equation :

$$1 + G(s)H(s) = 0$$

i.e.
$$1 + 1 + \frac{K(1+s)^2}{s^3} = 0$$

∴
$$s^3 + K[s^2 + 2s + 1] = 0$$

$$s^3 + Ks^2 + 2Ks + K = 0$$

s^3	1	2K	from s^0 and s^2, $K > 0$
s^2	K	K	$\frac{2K^2 - K}{K} > 0$
s^1	$\frac{2K^2 - K}{K}$	0	$2K - 1 > 0$
s^0	K		$2K > 1 \quad \therefore K > \frac{1}{2}$

Range of values of K is **K > 0.5 and < ∞**

Example 8.16 : *For unity feedback system, system is marginally stable and oscillates with frequency 4 rad/sec. Find* K_{mar} *and `q'.* **(M.U. : Dec.-2007)**

$$G(s) = \frac{4}{(s^2 + qs + 2K)\, s}$$

Solution : Characteristic equation

$$1 + G(s) = 0$$

$$1 + \frac{4}{s(s^2 + qs + 2K)} = 0$$

$$s^3 + qs^2 + 2Ks + 4 = 0$$

s^3	1	2 K
s^2	q	4
s^1	$\frac{2Kq - 4}{q}$	0
s^0	4	

For marginal value of K,

$$2\,Kq - 4 = 0$$

$$\therefore \quad 2\,Kq = 4$$

$$\therefore \quad q = \frac{4}{2K} \qquad \text{... (1)}$$

Now for this value of K,

$$A(s) = qs^2 + 4 = 0$$

$$s^2 = -\frac{4}{q}$$

$$s = \pm j\,\frac{2}{\sqrt{q}}$$

$$\text{Frequency} = 4$$

$$\therefore \quad \frac{2}{\sqrt{q}} = 4 \qquad \therefore \quad \frac{4}{q} = 16$$

$$\therefore \quad q = \frac{1}{4} = 0.25$$

$$\therefore \quad 2K = \frac{4}{q}$$

$$\therefore \quad K = \frac{4}{0.25 \times 2} = 8$$

$$\therefore \quad K_{marginal} = 8, \quad q = 0.25$$

Example 8.17 : *Determine the stability of the system having characteristic equation,*
$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$ **(M.U. : May-1996)**

Solution : Routh's Array

s^5	1	2	3	
s^4	1	2	5	
s^3	0	−2	0	⇐ Special case 1

Occurrence of special case 1, replacing '0' by '+ ε'

s^5	1	2	3	
s^4	1	2	5	
s^3	$+\varepsilon$	−2	0	⇐ Special case 1
s^2	$\frac{2\varepsilon+2}{\varepsilon}$	5		
s^1	$\frac{\left(\frac{2\varepsilon+2}{\varepsilon}\right)(-2)-5\varepsilon}{\left(\frac{2\varepsilon+2}{\varepsilon}\right)}$	0		
s^0	5			

As 'ε' is positive

$$\therefore \quad \frac{2\varepsilon+2}{\varepsilon} \rightarrow +ve$$

$$\text{and} \frac{\left(\frac{2\varepsilon+2}{\varepsilon}\right)(-2)-5\varepsilon}{\left(\frac{2\varepsilon+2}{\varepsilon}\right)} \rightarrow \frac{(-ve)+(-ve)}{(+ve)} \rightarrow -ve$$

∴ There are two sign changes in the first column sc system is **unstable** with two roots in the right half of s-plane.

Example 8.18 : *The characteristic equation of a system is,*
$s^3 + 3Ks^2 + (K+2)s + 4 = 0$

Determine range of K for stability. **(M.U. : Nov.-1994)**

Solution : $s^3 + 3Ks^2 + (K+2)s + 4 = 0$

Routh's Array

s^3	1	K + 2
s^2	3K	4
s^1	$\frac{3K^2+6K-4}{3K}$	0
s^0	4	

For stability

$$3K^2 + 6K - 4 > 0$$

$$\therefore \quad (K - 0.5275)(K + 2.5275) > 0$$

$$\therefore \quad K > 0.5275, \quad K > -2.5275$$

From row of s^2, $3K > 0$

$$\therefore \quad K > 0$$

∴ Range of K is $\mathbf{0.5275 < K < \infty}$.

Example 8.19 : *The output c(t) of a control system is related to its input r(t) by* $[s^4 + 2s^3 + 2s^2 + (3+K)s + K]\,C(s) = K(s+1)\,R(s)$

where K is positive gain of an amplifier.

i) With K = 6, will the output response by stable

ii) Determine limiting positive values of K for stability

(M.U. : May-1996, Dec.-1998)

Solution : The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K(s+1)}{(s^4 + 2s^3 + 2s^2 + (3+K)s + K)}$$

∴ The characteristic equation is

$$s^4 + 2s^3 + 2s^2 + (3+K)s + K = 0$$

i) For K = 6 it is , $s^4 + 2s^3 + 2s^2 + 9s + 6 = 0$

Routh's array

s^4	1	2	6
s^3	2	9	0
s^2	– 2.5	6	
s^1	13.8	0	
s^0	6		

There are two sign changes in the first column, so for K = 6 **output will not remain stable.**

ii) Routh's array in terms of `K' is

s^4	1	2	K
s^3	2	3 + K	0
s^2	$\frac{1-K}{2}$	K	
s^1	$\dfrac{\frac{(1-K)}{2}(3+K)-K}{\left(\frac{1-K}{2}\right)}$		
s^0	K		

From last row, $K > 0$

From row of s^2, $1 - K > 0$ ∴ $K < 1$

From row of s^1, $\frac{1-K}{2}(3 + K) - 2K > 0$

∴ $(1 - K)(3 + K) - 4K > 0$

∴ $3 - 2K - K^2 - 4K > 0$

$\therefore \quad -K^2 - 6K + 3 > 0 \quad$ i.e. $\quad K^2 + 6K - 3 < 0$

$\therefore \quad (K - 0.464)(K + 6.464) < 0$

$\therefore \quad K < 0.464, \quad K < -6.464$

Considering the three conditions range of positive values of K is **0 < K < 0.464.**

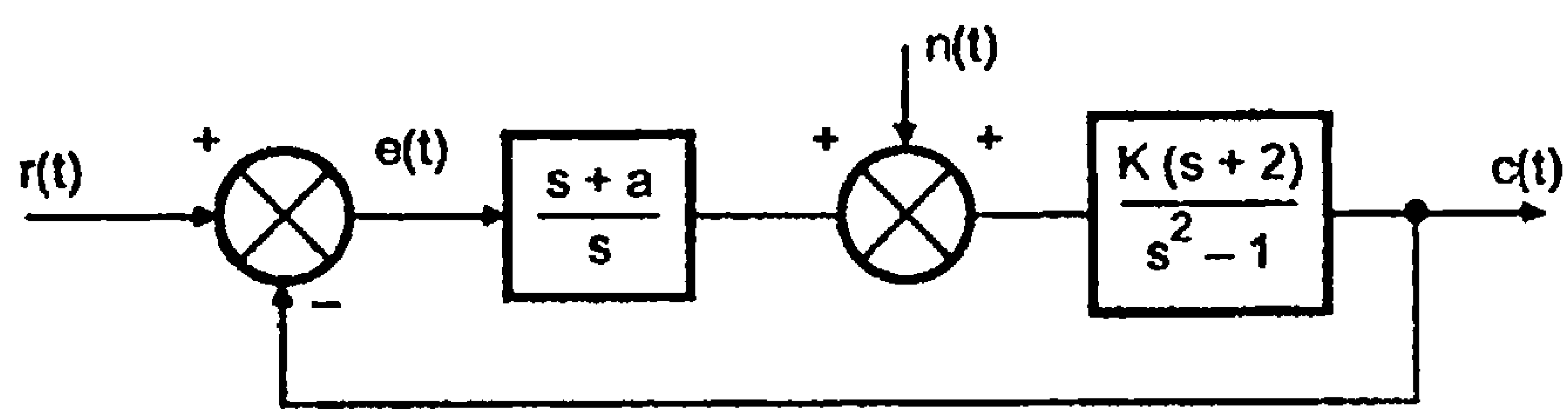
Example 8.20 : *For the system shown*

i) Find C_{ss} when r(t) = 0 and n(t) is unit step.

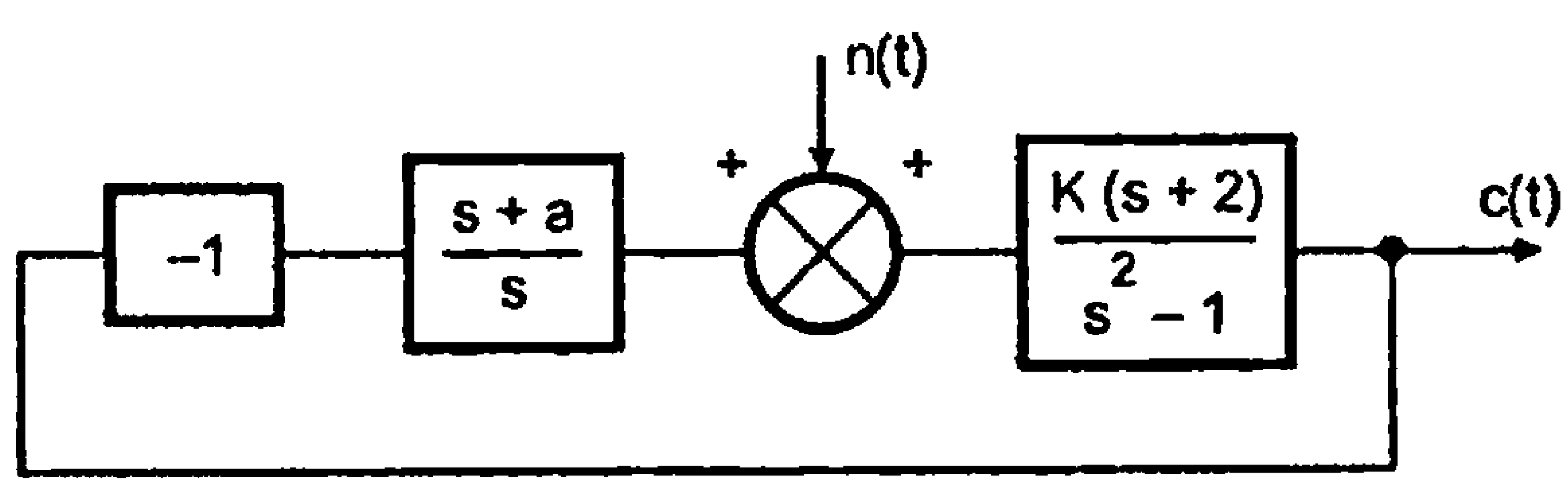
ii) Find the requirement of 'K' and 'a' so that system is stable.

iii) Show stable region on a plane with K on y axis and 'a' on x axis.

(M.U. : May-1996)



Solution : When r(t) = 0, – c(t) = e(t) and block diagram becomes



$$\therefore \quad \frac{C(s)}{N(s)} = \frac{\dfrac{K(s+2)}{s^2-1}}{1-\left[\dfrac{K(s+2)}{(s^2-1)}\right]\left[\dfrac{-(s+a)}{s}\right]} \quad \left(\text{using } \frac{G}{1-GH}\right)$$

$$\therefore \quad \frac{C(s)}{N(s)} = \frac{K(s-2)s}{s(s^2+1)+K(s+2)(s+a)}$$

$$= \frac{sK(s+2)}{s^3+Ks^2+(2K+aK-1)s+2aK}$$

i) For $\quad N(s) = \frac{1}{s}$ as unit step

$$\therefore \quad C(s) = \frac{1}{s}\cdot\frac{s(s+2)K}{s^3+Ks^2+(2K+aK-1)s+2aK}$$

$$\therefore \qquad C_{ss} = \lim_{s \to 0} sC(s) = 0$$

But for unit ramp input $C_{ss} = \frac{1}{a}$

ii) The characteristic equation is,

$$s^3 + Ks^2 + (2K + aK - 1)s + 2aK = 0$$

$\therefore$ Routh's array is

s^3	1	2K + aK − 1
s^2	K	2aK
s^1	$\frac{(K)(2K + aK - 1) - 2aK}{K}$	0
s^0	2aK	

From the last row, $2aK > 0$

From the row of s^1,

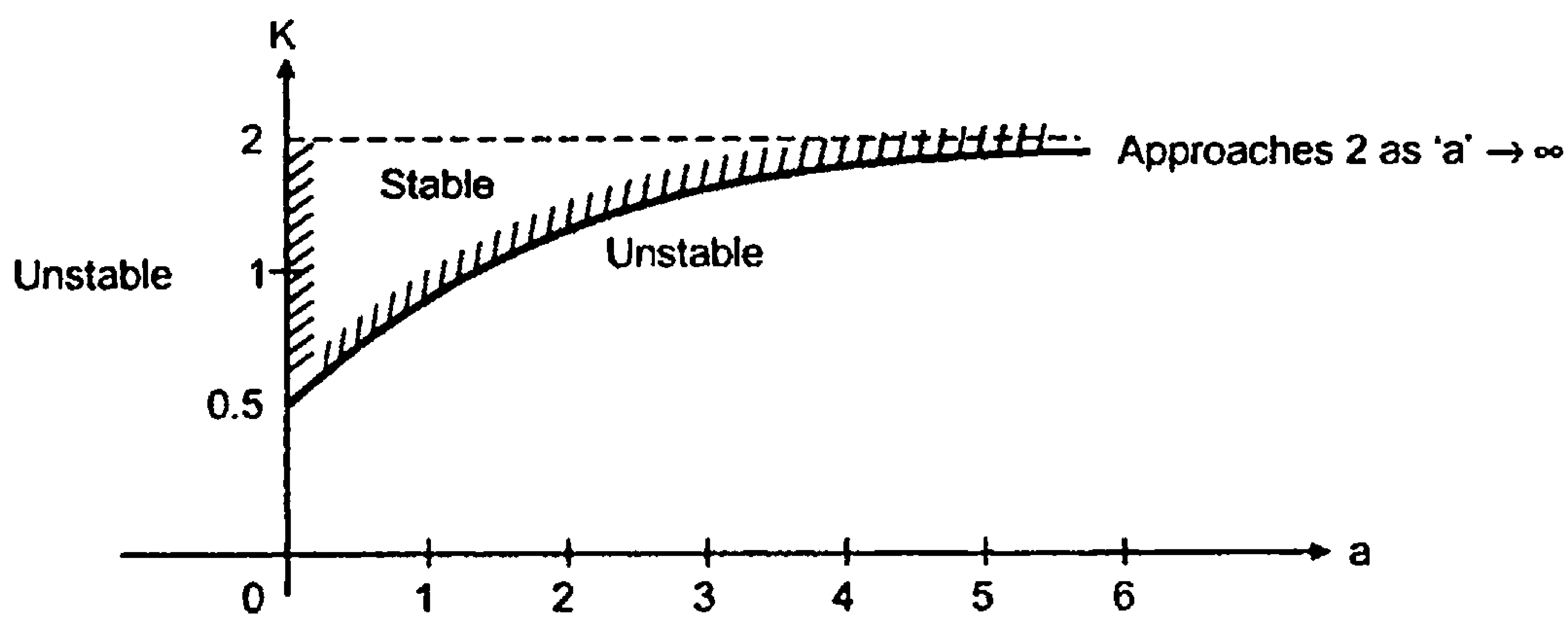
$$2K^2 + aK^2 - K - 2aK > 0$$

i.e. $2K + aK - 1 - 2a > 0$

i.e.

$$\boxed{K > \frac{1+2a}{a+2}}$$

iii)



Example 8.21 : *Determine the number of roots on the imaginary axis for the characteristic equation given below.,*

$$s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24 = 0$$

(M.U. : May-1998, Dec.-1996)

Solution : Routh's array

s^5	1	15	44
s^4	6	30	24
s^3	10	40	0
s^2	6	24	0
s^1	0	0	0 ⇐ Special case 2
s^0			

Since we have a row of all zeros we take the row above this take it as auxiliary equation and differentiate.

$$6s^2 + 24 = 0 \quad \text{Auxiliary equation.}$$

differentiating $$12s = 0$$

Now replace row of zeros with coefficient of the above equation.

s^5	1	15	44
s^4	6	30	24
s^3	10	40	0
s^2	6	24	0
s^1	12	0	0
s^0	24		

There is no sign change i.e. no root is in right half.

Now roots can be found out taking roots of auxiliary equation.

$$6s^2 + 24 = 0$$

$$\therefore \quad s^2 + 4 = 0$$

$$s^2 = -4$$

$$\therefore \quad s = \pm 2j$$

Two roots on imaginary axis at s = +2 j and –2 j

➠ **Example 8.22 :** *The characteristic equation of a system is* $s^3 + 10s^2 + 50s + 500 = 0$

Determine absolute stability using Routh's array **(M.U. : Nov.-1994)**

Solution :

s^3	1	50
s^2	10	500
s^1	0	0 ⇐ Special case 2

We have row of zeros therefore we can't find out stability.

∴ Take auxiliary equation of the row just above row of zeros.

$$10s^2 + 500 = 0 \quad \text{..... (1)}$$

Take differential w.r.t. s i.e. 20s ..... (2)

Replace row of zeros with this equation (2) and solve further

s^3	1	50
s^2	10	500
s^1	20	0
s^0	500	

No sign change in first column, system may be stable. As sufficient condition does not remain sufficient due to occurrence of special case 2. So stability is to be predicted by the nature of the roots of A(s) = 0 i.e. roots of

$$10s^2 + 500 = 0$$

$$s^2 = -50$$

$$\therefore \quad s = \pm j\sqrt{50} \quad \text{Roots}$$

$$= \pm j\, 7.071$$

$$\therefore \quad s = +j\, 7.071 \text{ and } -j\, 7.071$$

As these roots are purely imaginary, **system is marginally stable.**

Example 8.23 : *The open loop transfer function of a unity feedback control system is given by -*

$$G(s) = \frac{K(s+5)(s+40)}{s^3(s+200)(s+1000)}$$

Discuss the stability of the closed loop system as a function of K. Determine the value of K that will cause sustained oscillation in the closed loop system. What is the frequency oscillation? **(M.U. : Nov.-1993)**

Solution : Take equation 1 + G(s) H(s)

$$H(s) = 1 \quad \text{as it is unity feedback system}$$

$$1 + G(s)H(s) = 1 + \left(\frac{K(s+5)(s+40)}{s^3(s+200)(s+1000)}\right)1$$

$$= \frac{s^3(s+200)(s+1000) + K(s+5)(s+40)}{s^3(s+200)(s+1000)}$$

Now equate $1 + G(s)H(s) = 0$

$$\frac{s^3(s+200)(s+1000) + K(s+5)(s+40)}{s^3(s+200)(s+1000)} = 0$$

$$s^3 (s+200)(s+1000) + K(s+5)(s+40) = 0$$

$$s^3 (s^2 + 1200s + 200000) + K(s^2 + 45s + 200) = 0$$

$$s^5 + 1200s^4 + 200000s^3 + Ks^2 + 45Ks + 200K = 0$$

Routh's Array is

s^5	1	200000	45K
s^4	1200	K	200K
s^3	$\dfrac{2.4\times 10^8 - K}{1200}$	44.833K	0
s^2	$\dfrac{\left[\dfrac{2.4\times 10^8 - K}{1200}\right][K] - 5.38\times 10^4 K}{\left[\dfrac{2.4\times 10^8 - K}{1200}\right]}$	200K	After simplification $K - 1200\dfrac{(54000K-200K)}{(24\times 10^7 - K)}$
s^1	$\dfrac{-54000K^3 + 9.534\times 10^{12} K^2 - 11.52\times 10^{18} K}{1200(1.7544\times 10^8 - K)}$		
s^0	200K		

For stability

From s^3 row, $K < 2.4 \times 10^8$

From s^2 row, $1.7544\times 10^8 K - K^2 > 0$

i.e. $K > 0, \; K < 1.7544 \times 10^8$

From s^1 row, $-54000K^3 + 9.534\times 10^{12} K^2 - 11.52\times 10^{18} K > 0$

i.e. $54000K^2 - 9.534\times 10^{12} K + 11.52\times 10^{18} < 0$

i.e. $(K - 0.01214 \times 10^{18})(K - 1.7535\times 10^{18}) < 0$

Thus range of K is

$K >$ **1.214×10^6** and $K >$ **$1.7535 \times 10^{10^8}$**

For $K = 1.214\ 10^6$, $\omega =$ **16.55 rad/sec**

$K = 1.7535 \times 10^8$, $\omega =$ **379 rad/sec.**

Example 8.24 : *Using Routh criterion, investigate the stability of a unity feedback system whose open loop transfer function is*

$$G(s) = \frac{e^{-sT}}{s(s+1)}$$ **(M.U. : Dec.-1998)**

Solution : The characteristic equation is

$$1 + G(s)H(s) = 0$$

$$\therefore \quad 1 + \frac{e^{-sT}}{s(s+1)} = 0$$

$$\therefore \quad s^2 + s + e^{-sT} = 0$$

Now e^{-sT} can be expressed in the series from as

$$e^{-sT} = \left(1 - sT + \frac{s^2T^2}{2!} + \text{..........}\right)$$

Trancating the series and considering only first two terms we get

$$e^{sT} \approx 1 - sT$$

$$\therefore \quad s^2 + s + 1 - sT = 0$$

$$\therefore \quad s^2 + s(1 - T) + 1 = 0$$

So Routh's Array is

s^2	1	1
s	$1 - T$	0
s^0	1	

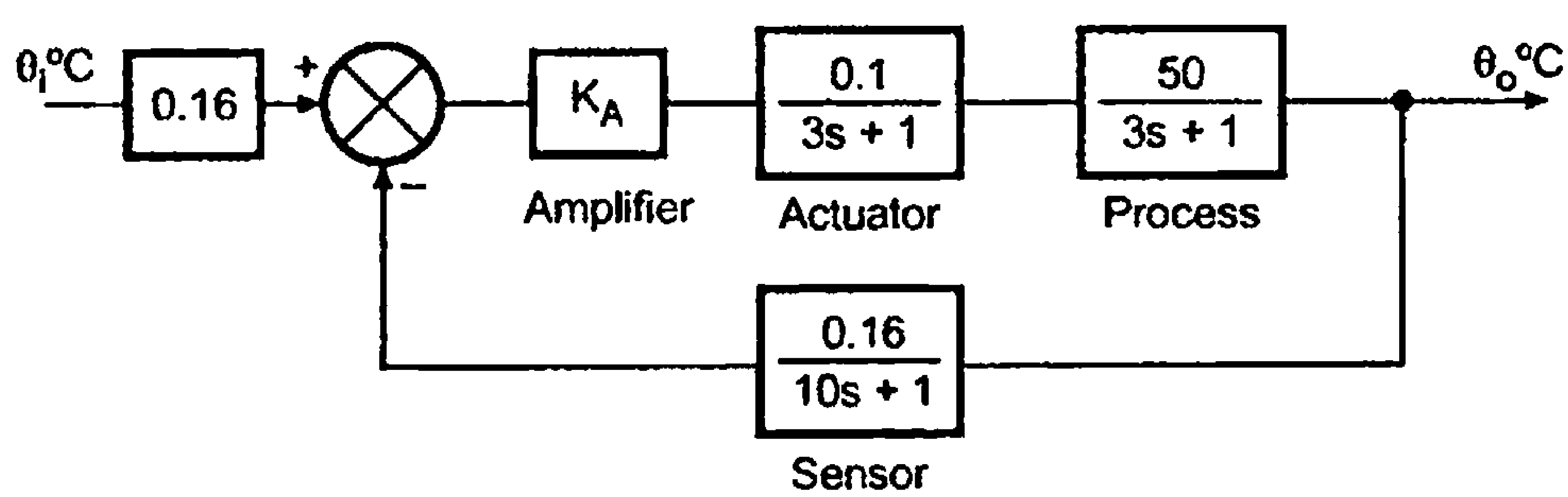
$$\therefore \quad 1 - T > 0 \quad \text{for stability}$$

$$\therefore \quad T < 1$$

This is the required condition for stability of the system.

Example 8.25 : *Consider the heat exchanger temperature control loop shown in figure.*
i) Find the range of values of $K_A > 0$ for which system is stable.

ii) The acceptable value of steady state error is 1°C for the step input of 10°C. Find the value of K_A that meets this specification on static accuracy. **(M.U. : May-2003)**



Solution : i) Find the transfer function of the system given.

$$\frac{\theta_o(s)}{\theta_i(s)} = 0.16\left\{\frac{\frac{K_A \times 0.1 \times 50}{(3s+1)^2}}{1+\frac{K_A \times 0.1 \times 50}{(3s+1)^2} \times \frac{0.16}{(10s+1)}}\right\} = \frac{0.8K_A(10s+1)}{(3s+1)^2(10s+1)+0.8K_A}$$

$$= \frac{0.8K_A(10s+1)}{90s^3+69s^2+16s+1+0.8K_A}$$

From the characteristics equation Routh's array is,

s^3	90	16
s^2	69	$1 + 0.8\,K_A$
s^1	$\frac{1104-90-72K_A}{69}$	0
s^0	$1 + 0.8\,K_A$	

For $K_a > 0$, $1104 - 90 - 72\,K_A > 0$

$\therefore \quad 1014 - 72\,K_A > 0$

$\therefore \quad 1014 > 72\,K_A$

$\therefore \quad K_A < 14.0833$

So range of values of K_A for system to be stable is $0 < K_A < 14.0833$

ii) For steady state error find G(s)H(s).

$$G(s)H(s) = \frac{0.1 \times K_A \times 50}{(3s+1)^2} \times \frac{0.16}{(10s+1)}$$

$$\therefore \quad K_p = \lim_{s \to 0} G(s)H(s) = 0.8\,K_A$$

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} \text{ where A = Magnitude of step input}$$

But here step of 10 °C is modified by 0.16 while applying to the system.

$$\therefore \quad A = 10 \times 0.16 = 1.6$$

$$\therefore \quad e_{ss} = \frac{1.6}{1+0.8\,K_A} \text{ and } e_{ss} = 1$$

$$\therefore \quad 1 = \frac{1.6}{1+0.8K_A}$$

$$\therefore \quad K_A = 0.75$$

Example 8.26 : *Determine whether the largest time constant of the roots of the characteristics equation given below is greater than, less than or equal to 1.0 sec.*

$$s^3 + 4s^2 + 6s + 4 = 0.$$ **(M.U. : May-2003)**

Solution : The time constant is the reciprocal of the actual values of the roots of characteristic equation.

Hence check whether roots are located to the right or left of s = – 1, for the given characteristic equation.

Replace $s = s' - 1$

$\therefore (s'-1)^3 + 4(s'-1)^2 + 6(s'-1) + 4 = 0$

$\therefore (s')^3 + (s')^2 + (s') + 1 = 0$

s'^3	1	1
s'^2	1	1
s'	0	0 ⇐ Row of zeros
s'^0		

$$A(s) = (s')^2 + 1 = 0$$

$$\therefore \quad \frac{dA(s)}{ds} = 2s'$$

Replacing row of zeros by dA(s)/ds, there is no sign change in the first column. Roots of A(s) = 0 are,

$$s' = \pm j \quad \text{purely imaginary}$$

So two roots are having real part s = – 1 and other locating to the left of s = – 1 hence **largest time constant is 1.0 sec.**

Example 8.27 : *The open loop transfer function of a unity feedback system is given by,*

$$G(s) = \frac{K}{s(s+3)(s^2+s+1)}$$

Determine the value of K that will cause sustained oscillations in the closed loop system. Also find the oscillation frequency. **(M.U. : Dec.-2003, Dec.-2005)**

Solution : The characteristic equation is 1 + G(s)H(s) = 0

$$\therefore \quad 1 + \frac{K}{s(s+3)(s^2+s+1)} = 0 \quad \text{i.e. } s^4 + 4s^3 + 4s^2 + 3s + K = 0$$

The Routh's array is,

s^4	1	4	K
s^3	4	3	0
s^2	3.25	K	0
s^1	$\frac{9.75-4K}{3.25}$	0	
s^0	K		

For marginal value of K, row of s^1 must be row of zeros.

$\therefore \quad 9.75 - 4\,K_{mar} = 0$ i.e. K_{mar} = **2.4375**

For this value of K, the auxiliary equation is,

$$A(s) = 3.25\,s^2 + K_{mar} = 0 \text{ i.e. } s^2 = \frac{-K_{mar}}{3.25} = \frac{-2.4375}{3.25}$$

$$\therefore \quad s = \pm j\,0.866$$

Thus the frequency of oscillations is **0.866 rad/sec.**

Example 8.28 : *Check for stability and mention how many roots lie in the right hand side of s-plane for a control system represented by a characteristic equation,*

$s^6+s^5+3s^4+3s^3+2s^2+s^1+s^0 = 0.$ **(M.U. : May- 2004)**

Solution : The characteristic equation is ,

$s^6+s^5+3s^4+3s^3+2s^2+s^1+1 = 0$

The Routh's array is ,

s^6	1	3	2	1	
s^5	1	3	1	0	
s^4	$\boxed{0}+\varepsilon$	1	1		← Special case 1
s^3	$\frac{3\varepsilon-1}{\varepsilon}$	$\frac{\varepsilon-1}{\varepsilon}$	0		
s^2	$\frac{\left(\frac{3\varepsilon-1}{\varepsilon}\right)-(\varepsilon-1)}{\frac{(3\varepsilon-1)}{\varepsilon X}}$	1			
s^1	$\frac{X\left(\frac{(\varepsilon-1)}{\varepsilon}\right)-\left(\frac{3\varepsilon-1}{\varepsilon}\right)}{X}$	0			
s^0	1				

For s^3, $\lim_{\varepsilon \to 0} \frac{3\varepsilon-1}{\varepsilon} \to$ Negative

For s^2, $\lim_{s\to 0} \dfrac{\left(\dfrac{3\varepsilon-1}{\varepsilon}\right)-(\varepsilon-1)}{\left(\dfrac{3\varepsilon-1}{\varepsilon}\right)}$ $\rightarrow$ Positive i.e. X $\rightarrow$ positive

For s^1, $\lim_{s\to 0} \dfrac{X\left(\dfrac{\varepsilon-1}{\varepsilon}\right)-\left(\dfrac{3\varepsilon-1}{\varepsilon}\right)}{X}$ $\rightarrow$ Negative

Thus there are **four sign changes** in the first column of the Routh's array.

Hence the system is **unstable** with **four** roots lying in the right hand side of s - plane.

Example 8.29 : *Find K marginal and frequency of oscillations for,*

$$1+\frac{K}{s(s^2+2s+2)(s^2+6s+10)}=0.$$

(M.U. : Dec-2004)

Solution : The characteristic equation is,

$$s(s^2+2s+2)(s^2+6s+10)+K = 0$$

i.e. $$s^5+8s^4+24s^3+32s^2+20s+K = 0$$

Routh's array is,

s^5	1	24	20
s^4	8	32	K
s^3	20	$\frac{160-K}{8}$	0
s^2	$\frac{480+K}{20}$	K	
s^1	$\dfrac{\left(\frac{480+K}{20}\right)\left(\frac{160-K}{8}\right)-20K}{\left(\frac{480+K}{20}\right)}$	0	
s^0	K		

For marginal K_1, row of s^1 must be row of zeros.

$$\therefore \quad \left(\frac{480+K}{20}\right)\left(\frac{160-K}{8}\right)-20K = 0$$

$$\therefore \quad 76800+160K-480K-K^2-3200K = 0 \text{ i.e. } -K^2-3520K+76800=0$$

$$\therefore \quad (K+3541.6845)(K-21.6845) = 0$$

$$\therefore \quad K_{mar} = +21.6845.$$

For this value, $$A(s) = \left(\frac{480+K_{mar}}{20}\right)s^2+K_{mar} = 0$$

$$\therefore \quad s^2 = -\frac{21.6845}{\frac{(480+21.6845)}{20}} = -0.8644$$

$$\therefore \quad s = \pm j\,0.9297 = \pm j\omega$$

Hence the frequency of oscillations is **0.9297** rad/sec.

Example 8.30: *Consider a third order system with the characteristic equation* $s^3+10.1s^2+21s+2 = 0$. *Is the system stable ? If we shift jω axis to the left by 0.2 units, analyse the relative stability.* **(M.U. : May-2006)**

Solution : For the given equation, Routh's array is,

s^3	1	21
s^2	10.1	2
s^1	20.802	0
s^0	2	

As there are no sign changes in the first column, the system is **stable** in nature. Now it is required to investigate relative stability about s = – 0.2. Thus replace s by s' – 0.2 in the characteristic equation.

$$\therefore (s'-0.2)^3 + 10.1\,(s'-0.2)^2 + 21\,(s'-0.2)+2 = 0$$

$$\therefore \quad (s')^3 + 9.5\,(s')^2 + 17.08s' - 1.804 = 0$$

$(s')^3$	1	17.08
$(s')^2$	9.5	– 1.804
$(s')^1$	17.2698	0
$(s')^0$	– 1.804	

As there is one sign change in the first column, there is **one root located to the right side of s = – 0.2.** Hence about s = – 0.2, **system is unstable.**

Review Questions

1. *Define the following terms*
   i) Stable system (ii) Unstable system (iii) Critically stable system
   (iv) Conditionally stable system.
2. *State and explain Hurwitz's criterion.*
3. *State and explain Routh's criterion.*
4. *What are the necessary conditions to have all the roots of a characteristic equation in the left half of s-plane ?*
5. *Write a note on special cases of Routh's criterion.*
6. *Explain the significance of an Auxiliary equation.*
7. *How Routh's criterion can be used to study the relative stability?*
8. *State the advantages and limitation of Routh's method.*

9. *The characteristic polynomial of a system is*

$P(s) = s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24$. **(M.U. : Nov.-1994 and 1995)**

(Ans. : Marginally stable, $\omega = 2$ rad/sec**)**

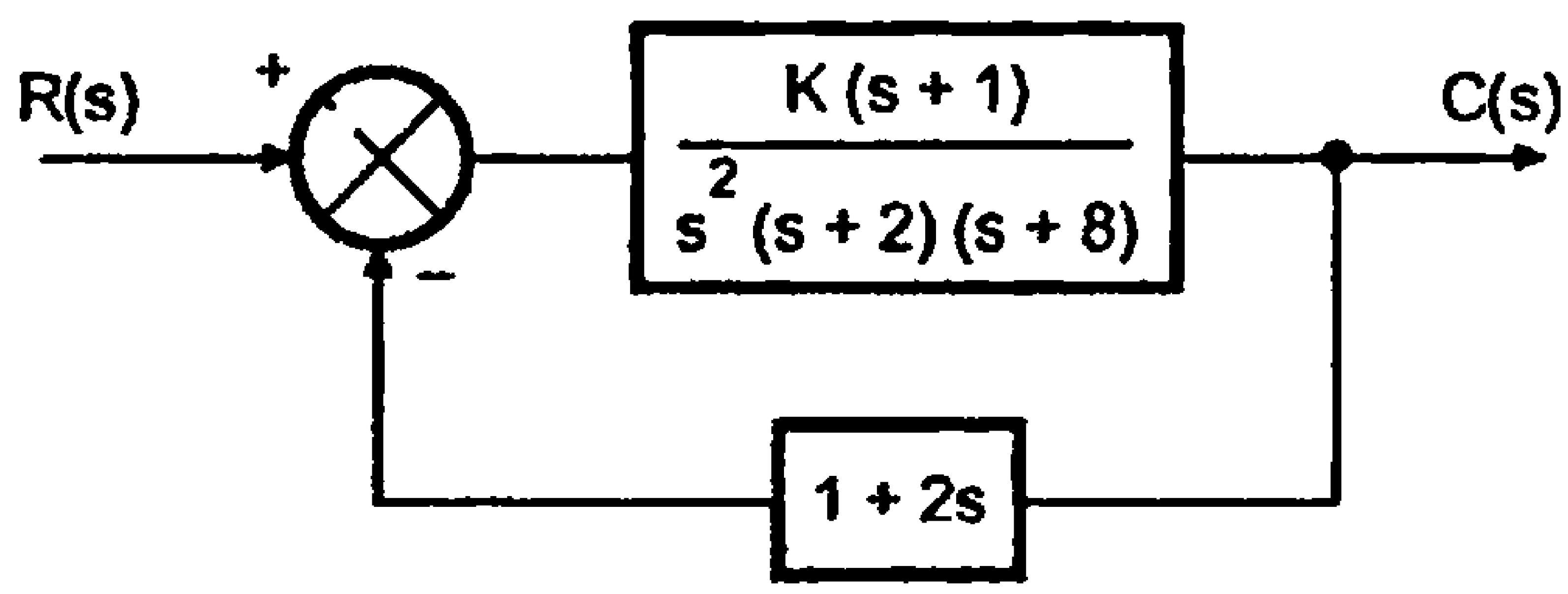
10. *Determine the stability of a system with characteristic equation.*

$s^5 + 4s^4 + 2s^3 + 8s^2 + s + 4 = 0$ **(M.U. : May-1996)**

(Ans. : Unstable, Multiple roots on imaginary axis at ± j**)**

11. *Determine the range of K for stability of the following system.* **(M.U. : May-1996)**

(Ans. : 0 < K < ∞)



12. *For K = 2, determine whether the following unity feedback system is stable. Use Routh criterion.*

$$G(s) = \frac{K(1 + 2s)(1 + 4s)}{s^2(s^2 + 2s + 10)}$$

(M.U. : Nov.-1995)

(Ans. : stable)

13. *Apply Routh criterion to check the stability of*

$s^6 + 9s^5 + 20s^4 + 12s^3 + 8s^2 + 16s + 16 = 0$ **(M.U. : May-1994)**

(Ans. : Unstable, 2 sign changes**)**

14. *Find the range of K for stability*

$s^4 + 2s^3 + 2s^2 + (3 + K)s + K = 0$, *for* $K > 0$ **(M.U. : May-1994)**

(Ans. : 0 < K < 0.464)

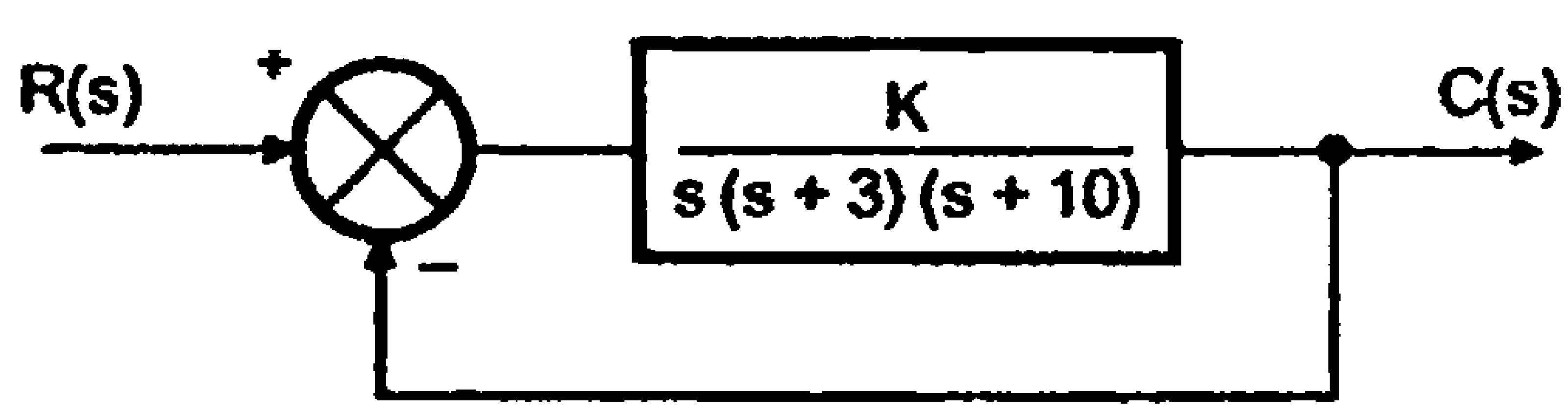
15. *Determine the range of K for stability*

$s^5 + 2s^4 + 3s^3 + 4s^2 + K(s + 1) + s + 5 = 0$ **(M.U. : May-1994)**

(Ans. : No real value of K makes the system stable**)**

16. *Find the range of K for which system given below is stable.* **(M.U. : Nov.-1993)**

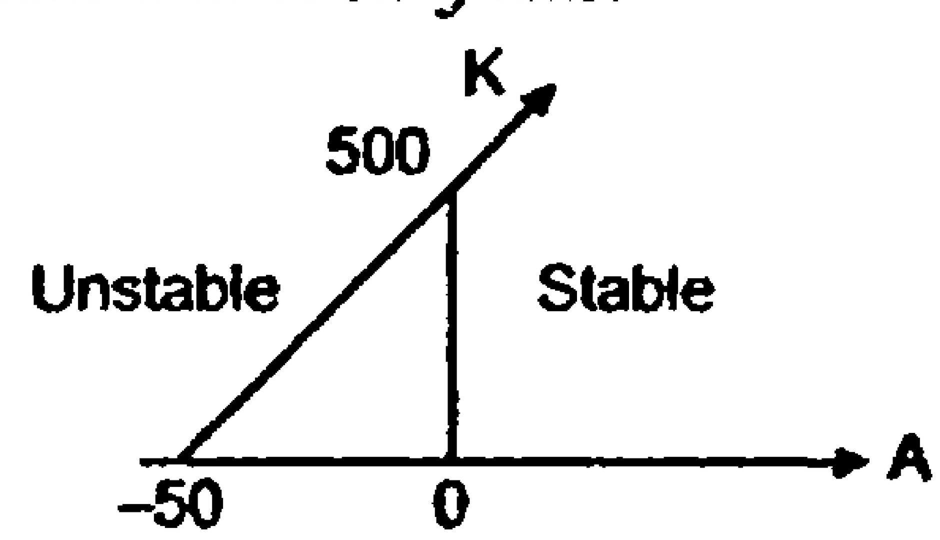
(Ans. : 0 < K < 390)



17. *For a system having characteristic equation* $s^3 + 10s^2 + (50 + A)s + K = 0$

Determine the stability boundary for stability by plotting A on x-axis and K on y-axis.

(Ans. : K > 0, K = 500 + 10Aand stable)



18. *Open loop transfer function of a unity feedback system is*

$$G(s) = \frac{K}{s(s^2 + s + 1)(s + 2)}$$

Find the range of K for stability. For what value of K system will oscillate and what is frequency of oscillations ? **(M.U. : Nov.-1993)**

(Ans. : $0 < K < 1.55$, $K_{critical} = 1.55$, $\omega = 0.816$ rad/sec**)**

19. *By means of Routh's criterion, determine stability of*

a) $s^6 + 5s^5 + 10s^4 + 24s^3 + 20s^2 + 15s + 10 = 0$

b) $s^4 + s^3 - 3s^2 - s + 2$ **(M.U. : Nov.-1993)**

(Ans. : (a) unstable (b) unstable **)**

20. *Determine the range of value of K(K > 0) such that the characteristic equation,*

$s^3 + 3(K+1)s^2 + (7K+5)s + (4K+7) = 0$ *has roots more negative that s = –1.*

(M.U. : Dec.-1998)

(Ans. : $0.5275 < K < \infty$**)**

21. *For a unity feedback system,*

$$G(s) = \frac{K(s+10)(s+20)}{s^2(s+2)}$$

Apply Routh's criterion and find range of values of K for stability. Find marginal K and corresponding frequency of oscillations. **(M.U. : May-99)**

(Ans. : $4.67 < K < \infty$, 11.834**)**

□□□

9 Root Locus

9.1 Background

In the previous chapters we have seen that the stability of any closed loop system depends on the locations of the roots of the characteristic equation i.e. the locations of closed loop poles. Nature of the transient response is closely related to the location of the poles in the s-plane. It is advantageous to know how the closed loop poles move in the s-plane if some parameters of the system are varied. The knowledge of such movement of the closed loop poles with small changes in the parameters of the system greatly helps in the design of any closed loop system.

Such movement of the poles can be known by the **Root Locus method,** introduced by W. R. Evans in 1948. This is a graphical method, in which movement of poles in the s-plane is sketched when a particular parameter of system is varied from zero to infinity. Note that the parameter is usually the gain but any other parameter may be varied. But for root locus method, **gain** is assumed to be a parameter which is to be varied from zero to infinity.

9.2 Basic Concept of Root Locus

In general, the characteristic equation of a closed loop system is given as,

$$1 + G(s)H(s) = 0$$

For root locus, the gain `K' is assumed to be a variable parameter and is a part of forward path of the closed loop system. Consider the system shown in the Fig. 9.1.

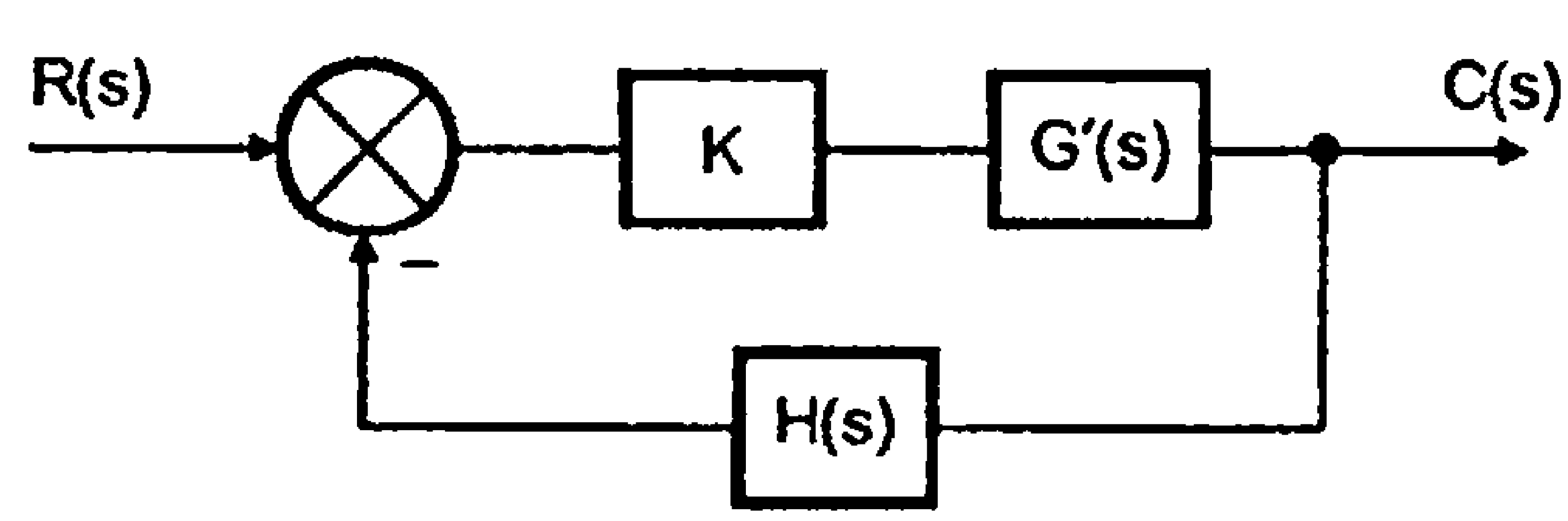


Fig. 9.1

$$G(s) = KG'(s)$$

where K = Gain of the amplifier in forward path or also called **System Gain.** The characteristic equation becomes,

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1 + KG'(s)H(s) = 0$$

which contains `K' as a variable parameter.

Key Point: *The closed loop poles i.e. the roots of the above equation are now dependent on the values of 'K'.*

If now gain 'K' is varied from $-\infty$ to $+\infty$ then for each separate value of 'K' we will get separate set of locations of the roots of the characteristic equation. If all such locations are joined, the resulting locus is called **Root Locus.** So we can define root locus as, **the locus of the closed loop poles obtained when system gain 'K' is varied from $-\infty$ to $+\infty$ is called Root Locus.**

Key Point: *When 'K' is varied from 0 to $+\infty$, the plot is called **Direct root locus** while when 'K' is varied from $-\infty$ to 0, the plot obtained is called **Inverse root locus.***

But generally the term root locus is used in the sense of Direct root locus. Unless otherwise stated, the variation in gain K is assumed to be 0 to $+\infty$ and plot is called Root Locus.

Example 9.1 : *Consider unity feedback system with $G(s) = \frac{K}{s}$. Obtain its roots locus.*

Solution : The characteristic equation becomes,

$$1 + G(s)H(s) = 0, \quad H(s) = 1$$

$$\therefore \quad 1 + \frac{K}{s} = 0$$

$$\therefore \quad s + K = 0$$

The root of this equation is located at $s = -K$

Now if gain 'K' is varied from 0 to $+\infty$, the location of this root is going to change.

The locus obtained by joining all such locations when K is varied from 0 to $+\infty$ is called Root Locus.

K	$s = -K$ Root location
0	0
1	-1
10	-10
⋮	⋮
$+\infty$	$-\infty$

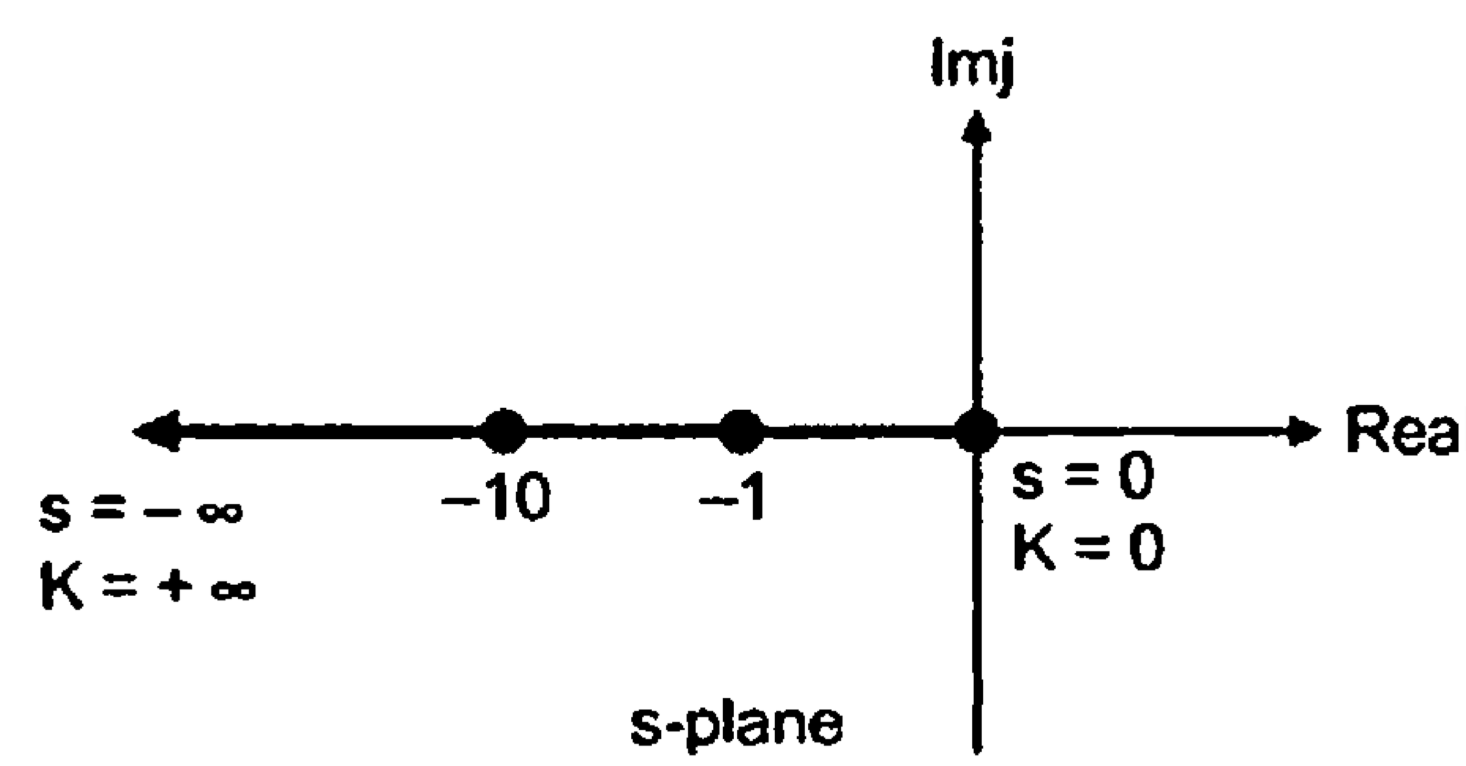


Fig. 9.2

The root locus is nothing but the negative real axis, for this system.

9.3 Angle and Magnitude Condition

For a general closed loop system the characteristic equation is,

$$1 + G(s)H(s) = 0$$

i.e. $$G(s)H(s) = -1$$

As s-plane is complex we can write above equation as,

$$\boxed{G(s)H(s) = -1 + j0}$$

All s-values can be expressed as '$\sigma + j\omega$' i.e. G(s)H(s) term is also complex one. **So for any value of 's' if it has to be on the root locus, it must satisfy the above equation.**

As both sides of the above equations are in rectangular form, we can convert both sides in polar form and then we can equate angle and magnitude of both sides. This gives us two conditions of root locus called (i) Magnitude condition and (ii) Angle condition.

9.3.1 Angle Condition

$$G(s)H(s) = -1 + j0$$

Equating angles of both sides,

$$\angle G(s)H(s) = \pm (2q + 1)\, 180^\circ \qquad q = 0, 1, 2 \ldots\ldots$$

Key Point: *$-1 + j0 = 1 \angle \pm 180^\circ$ but the point $-1 + j0$ is a point on negative real axis which can be traced as magnitude 1 at an angle $\pm 180^\circ$, $\pm 540^\circ$, $\pm 900^\circ$ ..... $\pm (2q + 1)\, 180^\circ$.*

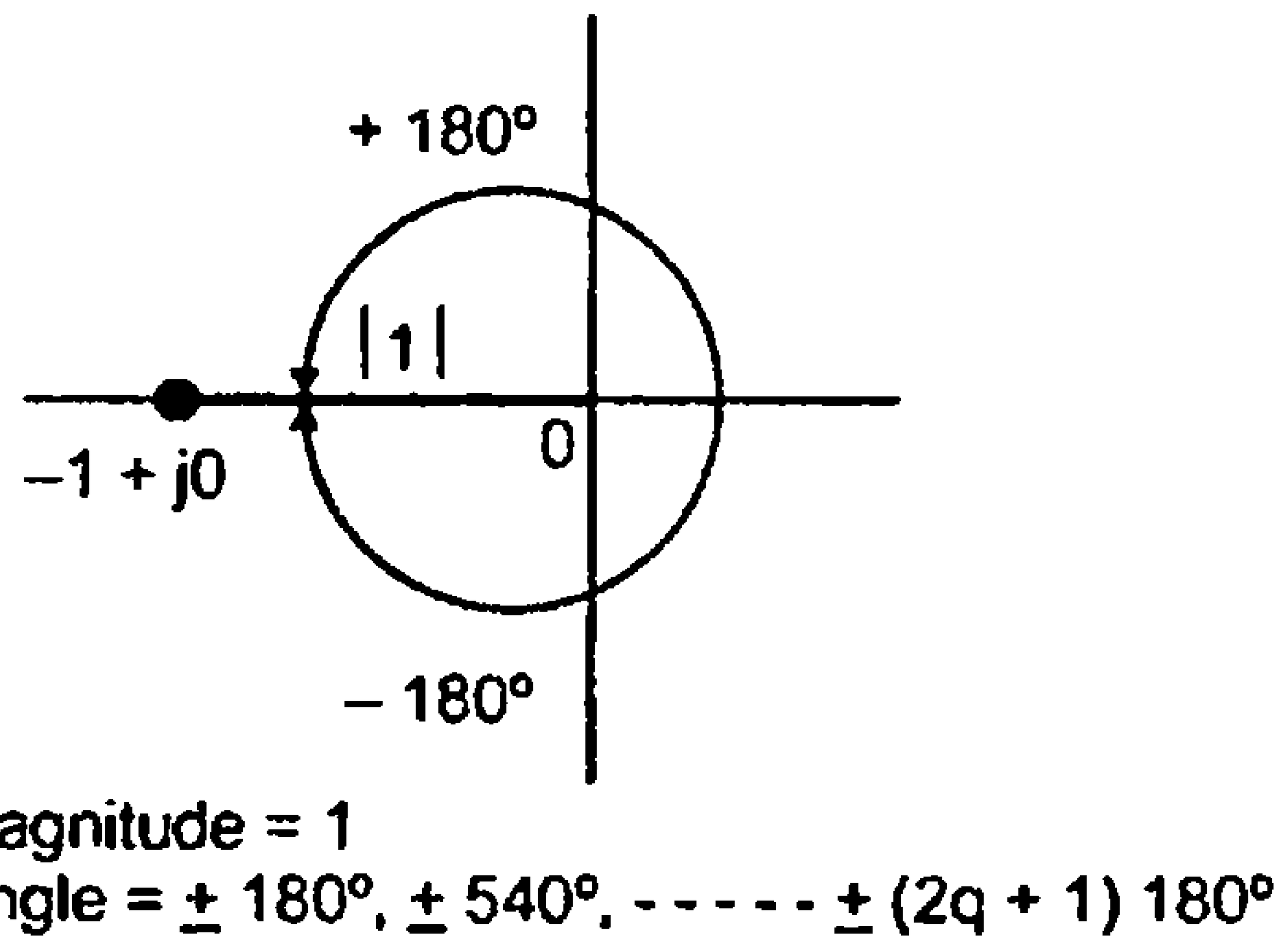


Fig. 9.3

∴ Angle condition can be stated as,

$\angle G(s)H(s)|$ for any value of 's' which is the root of equation [1 + G(s)H(s) = 0] is

$$= \pm (2q+1)\, 180^\circ \quad \text{where } q = 0, 1, 2 \ldots\ldots$$

$$= \text{Odd multiple of } 180^\circ$$

If any point in s-plane has to be on the root locus then it has to satisfy above angle condition. The angle of G(s)H(s) calculated at that point must be an odd multiple of ± 180°.

Key Point: *Any point in s-plane which satisfies the angle condition has to be on the root locus of the corresponding system.*

9.3.2 Use of Angle Condition

As all the points on the root locus must satisfy the angle condition, we can use the angle condition to test any point in s-plane for its existence on the root locus of the given system. This can be explained by taking an example.

Example 9.2 : *Consider the system with* $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$. *Find whether* $s = -0.75$ *is on the root locus or not using angle condition.*

Solution : Let us test whether s = – 0.75 is located on the root locus of above system i.e. whether s = – 0.75 is a root of the characteristic equation 1 + G(s)H(s) = 0 or not. Use Angle condition,

$$\angle G(s)H(s)\Big|_{\text{at point } s=-0.75} = \pm(2q+1)\,180^\circ \qquad q = 0, 1, 2, ...$$

Substituting s = – 0.75 in all the terms of G(s)H(s),

$$\angle G(s)H(s)\Big|_{\text{at } s=-0.75} = \frac{\angle K+j0}{\angle -0.75+j0 \cdot \angle\, 1.25+j0 \cdot \angle\, 3.25+j0}$$

Converting to polar form and considering angles, **(use calculator to obtain polar form from rectangular form and consider angle.)**

$$= \frac{0^\circ}{180^\circ \cdot 0^\circ \cdot 0^\circ} = -180^\circ$$

That is ∠ G(s)H(s) = – 180° at s = – 0.75 which satisfies angle condition and we can conclude that point s = – 0.75 is on the root locus of the given system.

Let us test, s = – 1 + j4 for its existence on the root locus of the same system,

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = \frac{\angle K+j0}{\angle -1+j4 \cdot \angle\, 1+j4 \cdot \angle\, 3+j4}$$

$$= \frac{0^\circ}{104.03^\circ \cdot 75.963^\circ \cdot 53.13^\circ}$$

$$= -233.123^\circ$$

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = -233.123^\circ$$

As this is not satisfying the angle condition, the point (–1 + j4) cannot be on the root locus of the given system.

9.3.3 Magnitude Condition

If magnitudes of both sides of the equation G(s)H(s) = –1 are equated then we get a magnitude condition.

$$|G(s)H(s)| = |-1 + j0| = 1$$

Now in the function G(s)H(s), K is unknown and hence we cannot find out |G(s)H(s)| at any point in s-plane. So this condition is not suitable to check the existence of a point on the root locus. But once we know that a point in s-plane is on the root locus then it must satisfy magnitude condition also. So at that point which is known to be on the root locus by angle condition, we can find out value of K by using magnitude condition. This 'K' is value of the gain for which a known point on root locus is the root of the characteristic equation.

So magnitude condition is, $|G(s)H(s)|_{\substack{\text{at a point in s-plane}\\ \text{which is on root locus}}} = 1$

At any point in s-plane, using magnitude condition we can find the value of K. **But use of magnitude condition totally depends on the existence of a point on the root locus.**

Key Point: *So magnitude condition can be used only when a point in s-plane is confirmed for its existence on the root locus by use of angle condition.*

9.3.4 Use of Magnitude Condition

Once a point is known to be on the root locus by angle condition, we can use magnitude condition. This gives us a value of K for which a tested point is one of the roots of the characteristic equation.

Example 9.3 : *Refer example 9.2 where* $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$ *and s = – 0.75 is confirmed to be on the root locus. Now we are interested in knowing that at what value of K, s = – 0.75 is one of the roots of 1 + G(s)H(s) = 0. Use the magnitude condition.*

Solution :

$$|G(s)H(s)|_{at\ s=-0.75} = 1$$

$$\frac{|K|}{|-0.75|\,|1.25|\,|3.25|} = 1$$

$$\therefore \qquad K = 3.0468$$

In this case, 1 + G(s)H(s) = 0 means $1 + \frac{K}{s(s+2)(s+4)} = 0$ i.e.

$s^3 + 6s^2 + 8s + K = 0$ is a cubic equation. But by use of angle and magnitude conditions one after the other we have decided that for K = 3.0468, one of the three roots is located at s = – 0.75. The remaining two roots then can be easily obtained.

Key Point: *So root locus method also helps us to solve higher order polynomial very quickly.*

9.4 Graphical Method of Determining 'K'

To use this method we must know the approximate nature of the root locus or location of the point which is known to be on the root locus. Then the value of K is given by,

$$K = \frac{\text{Product of phasor lengths drawn from open loop poles upto a point on root locus}}{\text{Product of phasor lengths drawn from open loop zeros upto a point on root locus}}$$

Key Point: *If open loop zeros are absent, then denominator in the above expression must be treated unity.*

Let us see use of this method for,

$$G(s)H(s) = \frac{K}{s(s+4)}. \text{ Find K for } s = -2 + j5$$

Open loop poles are at s = 0, – 4

Now join all open loop poles at a point – 2 + j5 which is confirmed to be on the root locus by angle condition.

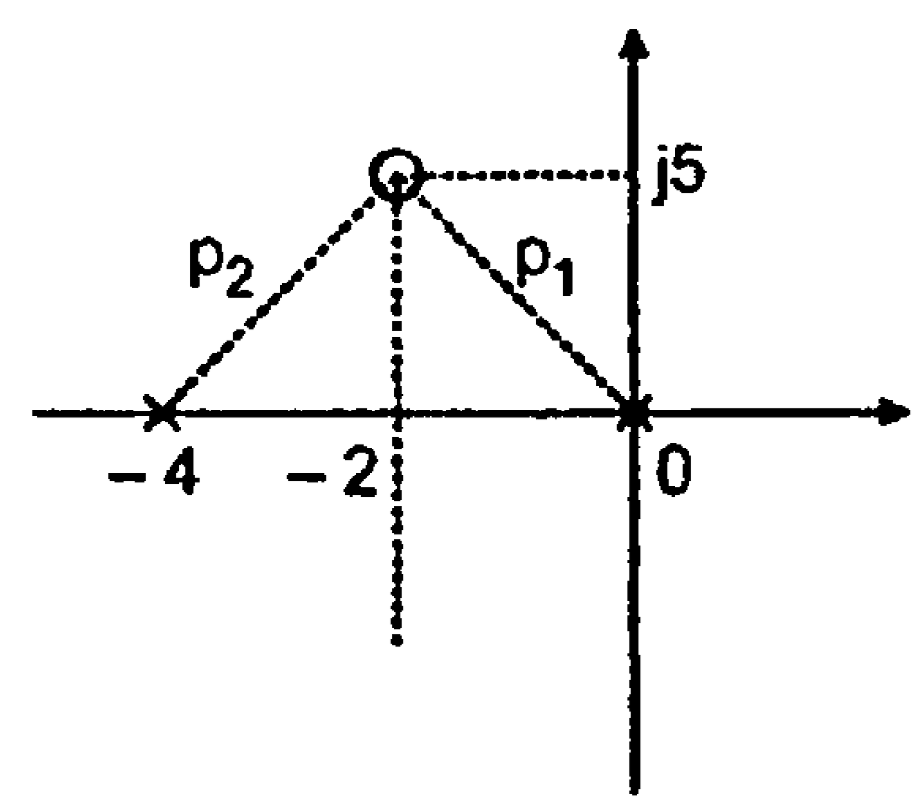


Fig. 9.4

Length from s = 0 upto a point = p_1

Length from s = – 4 upto a point = p_2

From geometry of the Fig. 9.4 we can decide,

$$p_1 = \sqrt{(2)^2+(5)^2} = \sqrt{29}$$

$$p_2 = \sqrt{(2)^2+(5)^2} = \sqrt{29}$$

Now as open loop zeros are absent, denominator is to be assumed as unity.

$$\therefore \quad K = p_1 \times p_2 = \sqrt{29}\,.\sqrt{29} = 29$$

Key Point: *Generally magnitude condition is used rather than the graphical method to determine K.*

9.5 Construction of Root Locus

To understand the rules of construction of root locus for higher order systems, let us examine some simple systems and draw some important conclusions.

In example 9.1, $G(s)H(s) = \frac{K}{s}$, we have seen that there is one branch of the root locus. Branch is starting at s = 0 which is open loop pole location while it is terminating at s = – ∞ .

Let us examine the root locus of the system having

$$G(s)H(s) = \frac{K}{s(s+2)}$$

Example 9.4 : *For $G(s)H(s) = \frac{K}{s(s+2)}$, obtain the nature of the root locus.*

Solution : $1 + G(s)H(s) = 1 + \frac{K}{s(s+2)} = 0$

i.e. $s^2 + 2s + K = 0$

Solving for its roots, roots $= \frac{-2 \pm \sqrt{4-4K}}{2} = -1 \pm \sqrt{1-K}$

Root 1 say $s_1 = -1 + \sqrt{1-K}$ and Root 2 say $s_2 = -1 - \sqrt{1-K}$. Let us see the locations for various values of K.

K	$s_1 = -1 + \sqrt{1-K}$	$s_2 = -1 - \sqrt{1-K}$
0	0	− 2
0.2	− 0.105	− 1.895
0.8	− 0.552	− 1.448
1	− 1	− 1
5	− 1 + j2	− 1 − j2
:	:	:
∞	− 1 + j∞	− 1 − j∞

This root locus has two branches, one showing locus or movement of s_1 while second showing movement of s_2. Both the branches are approaching to − 1 and then breaking into two, moving parallel to imaginary axis. This is shown in Fig. 9.5.

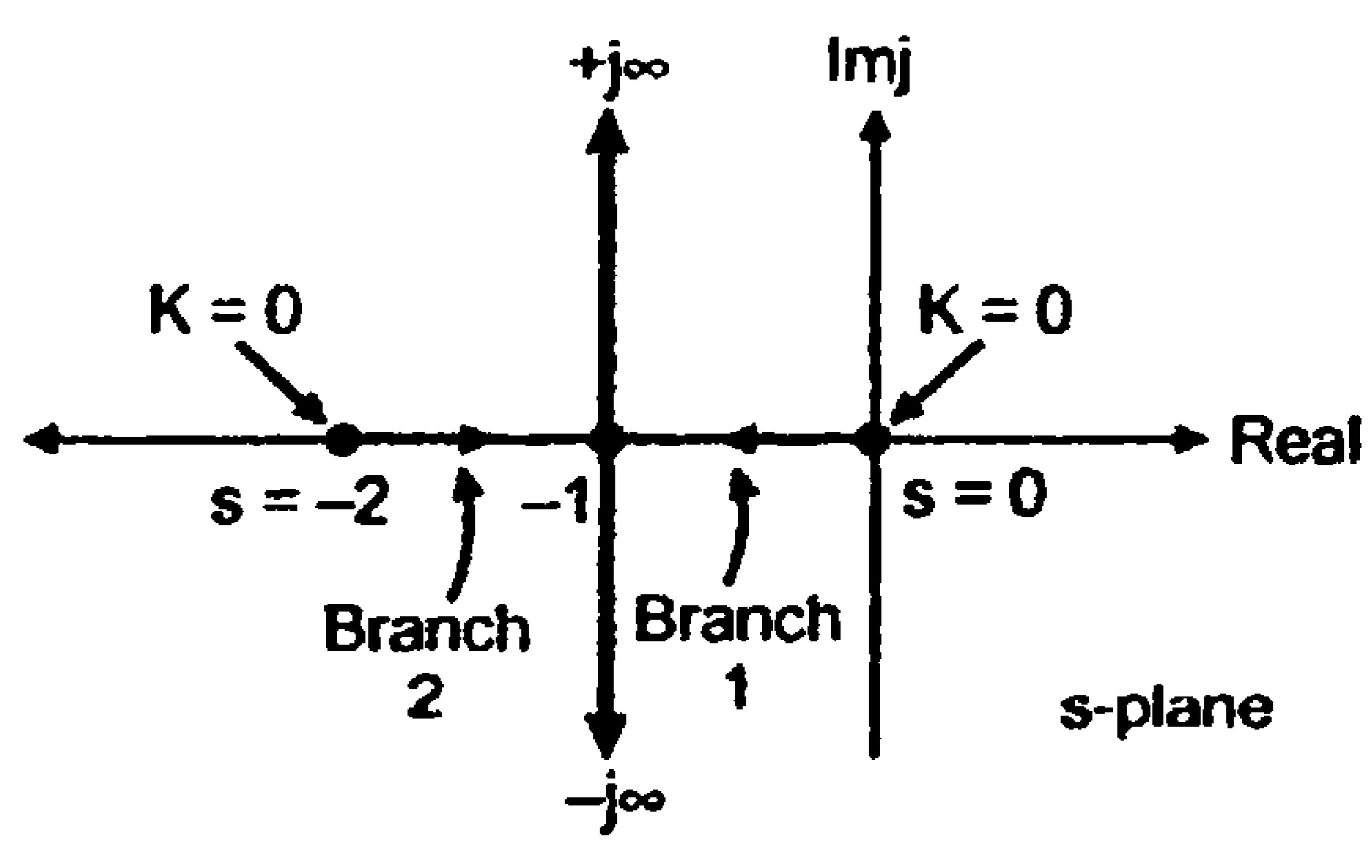


Fig. 9.5

From this we can conclude that there are two branches. Branches are starting from s = 0 and s = − 2 which are open loop poles of the system. Both the branches are approaching to infinity. Hence in general we can conclude that **number of branches equals number of open loop poles.** A separate branch starts from each location of open loop pole. Now to confirm whether branches always terminate at infinity or not let us see root locus of another system.

Example 9.5 : *Consider* $G(s)H(s)=\frac{K(s+1)}{s(s+5)}$. *Obtain the nature of its root locus.*

Solution : The characteristic equation is ,

$$1+G(s)H(s) = 1+\frac{K(s+1)}{s(s+5)}=0$$

$$\text{i.e. } s^2+5s+Ks+K = 0$$

$$\text{i.e. } s^2+s(K+5)+K = 0$$

$$\text{Roots are } \frac{-(K+5)\pm\sqrt{(K+5)^2-4K}}{2} = \frac{-(K+5)}{2}\pm\frac{\sqrt{K^2+6K+25}}{2}$$

Effect of variation in 'K' is

K	$s_1 = \frac{-(K+5)}{2}+\frac{\sqrt{K^2+6K+25}}{2}$	$s_2 = \frac{-(K+5)}{2}-\frac{\sqrt{K^2+6K+25}}{2}$
0	0	− 5
1	− 0.1715	− 5.828
5	− 0.527	− 9.472
:	:	:
∞	− 1	− ∞

We can observe, number of branches are again two i.e. number of open loop poles. Both branches are starting from s = 0 and s = − 5 which are open loop poles. **But important observation is, one of the branches terminates at s = − 1 which is open loop zero,** while other branch is terminating at infinity.

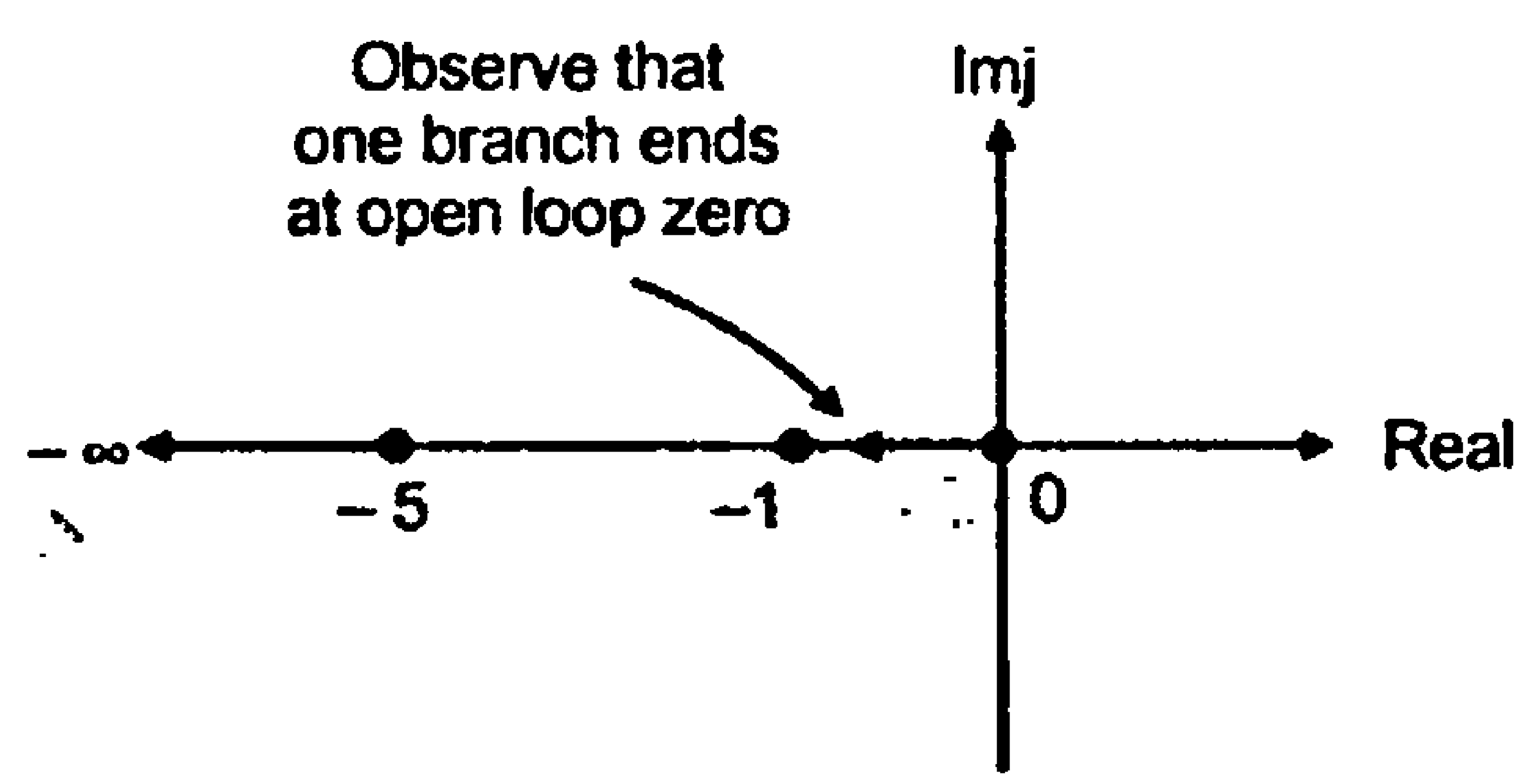


Fig. 9.6

It is very difficult to plot the root locus for higher order systems by the method of actually substituting different values of 'K' in the roots of characteristic equation as used above.

To simplify the construction of the root locus for higher order systems certain rules are specified based on the observations made earlier.

9.6 Rules for Construction of Root Locus

Rule No. 1 : The root locus is always symmetrical about the real axis. The roots of the characteristic equation are either real or complex conjugates or combination of both. Therefore their locus must be symmetrical about the real axis of the s-plane.

Rule No. 2 : Let G(s)H(s) = Open loop T.F. of the system

$$P = \text{Number of open loop poles}$$

$$Z = \text{Number of open loop zeros}$$

Then we can confirm basic information about the root locus as,

Case (i) P > Z Number of branches equal to number of open loop poles N = P	Case (ii) Z > P Number of branches equal to the number of open loop zeros. N = Z
Branches will start from each of the location of open loop pole. Out of 'P' number of branches, 'Z' number of branches will terminate at the locations of open loop zeros. The remaining 'P – Z' branches will approach to infinity e.g. : If P = 4 and Z = 1 then number of root locus branches = 4, number P – Z = 3. 4 branches will start from locations of open loop poles, out of this only one will terminate at the available finite open loop zero location. The remaining P – Z = 3 branches will approach to ∞.	Branches will terminate at each of the finite location of open loop zero. But out of 'Z' number of branches, 'P' number of branches will start from each of the finite open loop pole locations while remaining Z – P number of branches will originate from infinity and will approach to finite zeros. e.g. : If P = 1 and Z = 4 then number of separate branches = Z = 4, number of Z– P branches = 3. 3 branches will start from infinity while 1 branch will start from location of open loop pole and all 4 branches will terminate at available 4 finite locations of zeros.

Whatever may be the case, **branch direction always remains from open loop poles towards open loop zeros.** When P = Z, the number of branches N = P = Z. A separate branch will start from each of the open loop pole while will terminate at available each open loop zero. No branch will start or terminate at infinity when P = Z.

Rule No. 3 : A point on the real axis lies on the root locus if the sum of the number of open loop poles and the open loop zeros, on the real axis, to the right hand side of this point is odd.

To understand this rule consider the following example 9.6.

Example 9.6 : $G(s)H(s) = \dfrac{K(s+1)(s+4)}{s(s+3)(s+5)}$. *Using Rule No. 3, find on which sections of real axis the root locus exists.*

Solution : Sketch the pole-zero plot to use the above rule.

Consider point 'P' on real axis. Then consider right half of real axis to this point P. On right side there is 1 pole and 1 zero. Sum of number of poles + zeros on right side is 2 which is even. Hence according to rule, point P cannot be on the root locus. Similarly for point 'Q', sum of number of open loop poles and zeros on real axis to right side of point Q is now 3 i.e. two poles and one zero. This sum is odd hence point Q is on the root locus. For point R again sum of poles and zeros on real axis to right hand side is 5 i.e. 3 poles and 2 zeros. Hence 'R' is also on the root locus.

Now it can be observed that condition of sum of poles and zeros on right side of any point remains same, till point shifts in next section. For example, condition of sum = 2 for point P remains as it is for all points in section between s = –1 and s = –3. Hence the complete section from s = –1 to s = –3 cannot be the part of the root locus as indicated. When point shifts in next section condition changes. Hence rule can be effectively applied for the different sections between poles and zeros and can be judged whether root locus will exist in different sections or not.

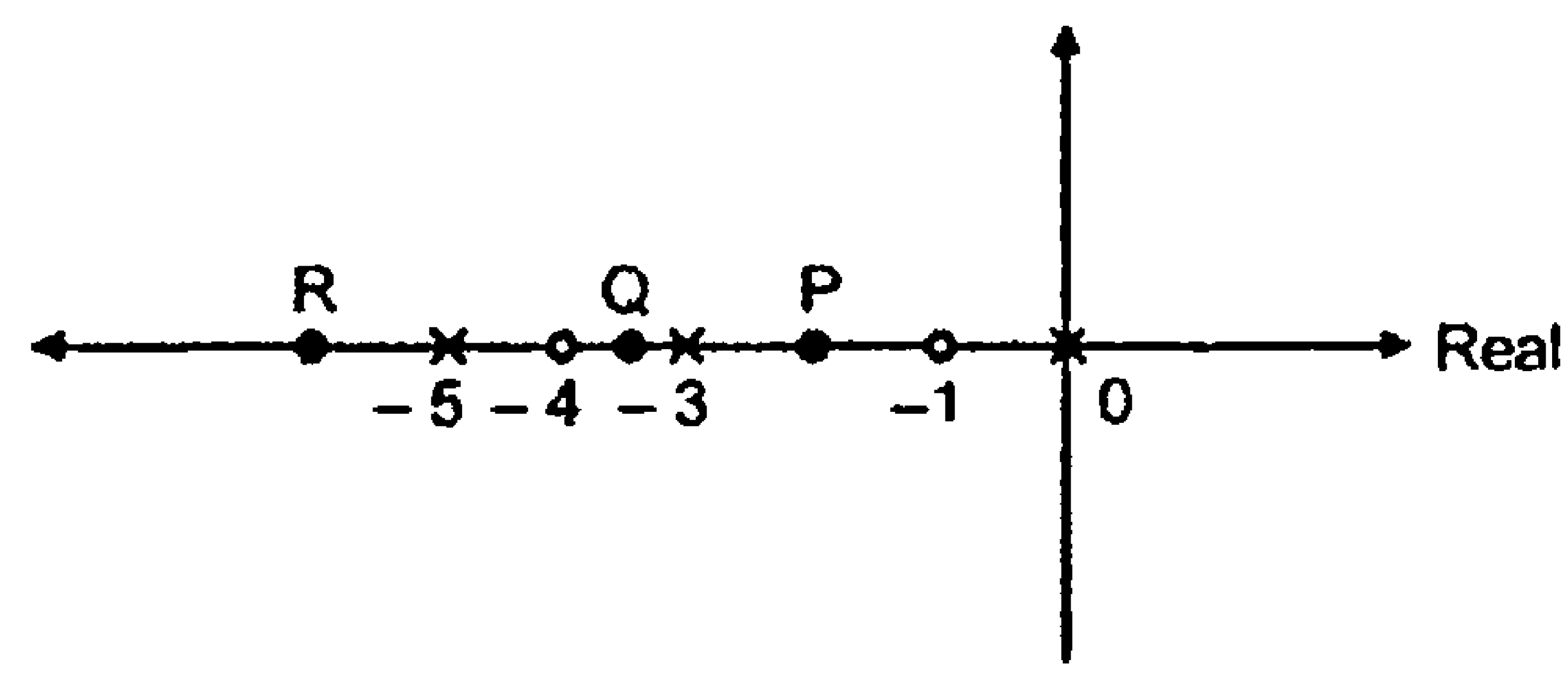


Fig. 9.7

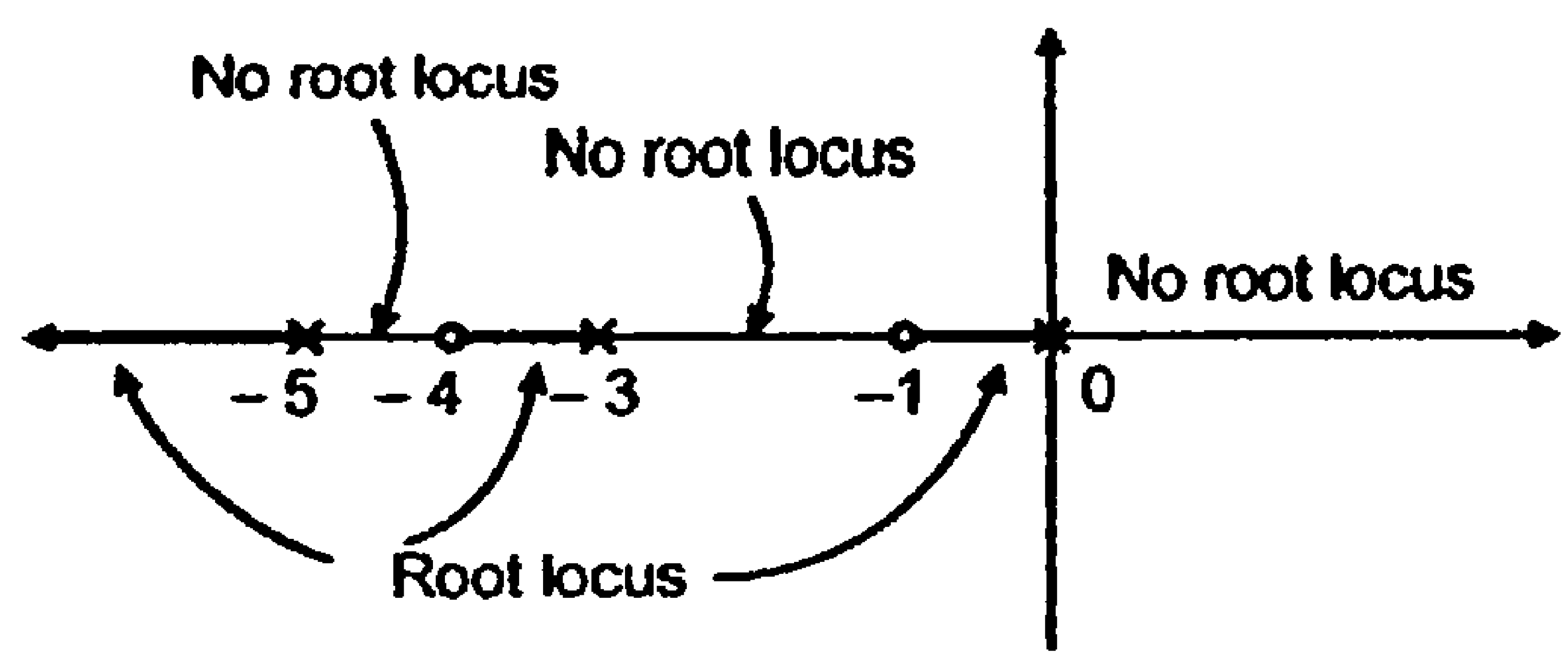


Fig. 9.8

So for example 9.6 we can conclude existence of root locus on the real axis as shown in the Fig. 9.8 by using rule 3.

Example 9.7 : $G(s)H(s) = \dfrac{K(s+2)}{s^2(s^2+2s+2)(s+3)}$. *Find the sections of real axis which belongs to the root locus.*

Solution : Poles are at 0, 0, –1 ± j, –3. [Zero is at –2.]

Pole - Zero plot is as shown in the Fig. 9.9.

For positive real axis, there is no pole and zero to right hand side so sum is zero and hence there is no root locus.

For next section between s = 0 to s = –2, to the right hand side sum is 2 which is even hence there is no root locus. Complex conjugate roots should not be considered while using this rule. For next section between s = – 2 and s = –3, to right hand side sum is 3, odd hence full section is part of the root locus. While section to left of s = – 3 there is no root locus.

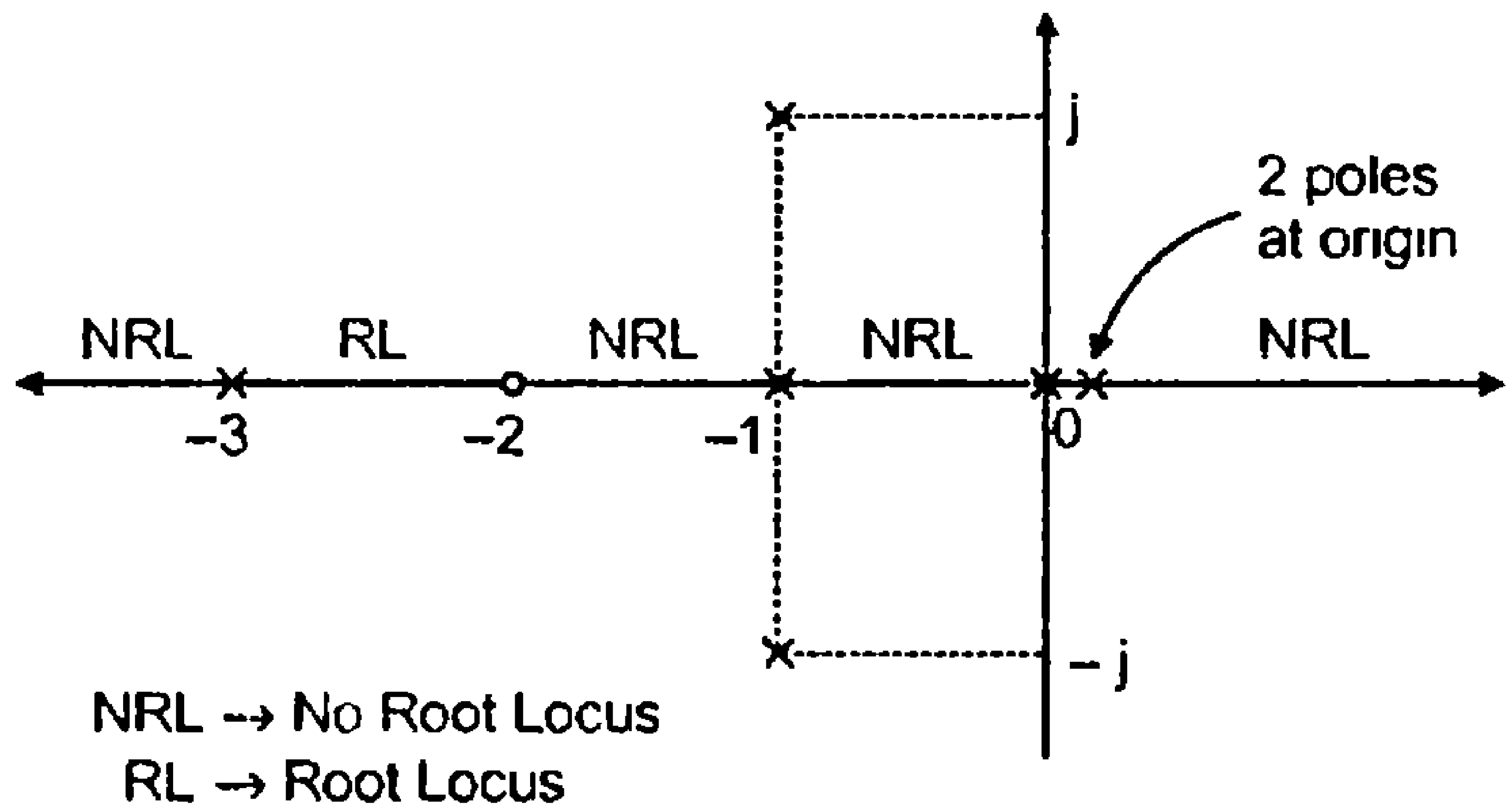


Fig. 9.9

Key Point: ***This rule should not be applied to actual open loop poles and zeros as these are always on the root locus. Rule should be applied to sections in between them. Similarly complex conjugate poles or zeros occur in pair hence do not affect the condition of sum. Hence complex poles or zeros need not be considered while applying this rule.***

Rule No. 4 : Generally number of poles are more than number of zeros and in such case 'P–Z' branches will approach to infinity. This rule gives us information about how these branches approach to infinity.

The branches which are approaching to infinity, do so along the straight lines called **Asymptotes** of the root locus. Asymptotes are the guidelines for the branches approaching to infinity. Angles of such asymptotes are given by ,

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \qquad \text{where } q = 0, 1, 2 \,.....\, (P-Z-1)$$

Asymptotes are always symmetrically located about real axis.

Rule No. 5 : Now only the angles of asymptotes are not sufficient but where the asymptotes are located in s-plane is equally important. Location of asymptotes in s – plane is given by this rule.

All the asymptotes intersect the real axis at a common point known as **centroid** denoted by σ. The co-ordinates of centroid can be calculated as,

$$\sigma = \frac{\sum \text{Real parts of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P-Z}$$

Key Point: *Centroid is always real, it may be located on negative or positive real axis. It may or may not be the part of the root locus.*

Let us see use of the above two rules.

Example 9.8 : *For* $G(s)H(s) = \frac{K}{(s+1)(s+2+j2)(s+2-j2)}$, *calculate angles of asymptotes and the centroid.*

Solution : P = 3, Z = 0, N = P = 3

$$P - Z = 3 \text{ branches are approaching to infinity.}$$

Poles located at $s = -1, -2 \pm j2$

Angles of asymptotes are given by,

$$\theta = \frac{(2q+1)\,180°}{P-Z} \qquad q = 0, 1, 2$$

Number of asymptotes = Number of branches approaching to infinity.

For q = 0, $\theta = \frac{180°}{3} = 60°$

q = 1, $\theta_1 = \frac{(2+1)\,180°}{3} = 180°$

q = 2, $\theta_2 = \frac{(2\times 2+1)\,180°}{3} = 300°$

All these asymptotes are going to intersect at a common point on real axis called centroid.

$$\sigma = \frac{\sum \text{Real parts of poles} - \sum \text{Real parts of zeros}}{P-Z}$$

$$= \frac{-1-2-2-0}{3} = \frac{-5}{3} = -1.67$$

Centroid and angles of asymptotes are shown in the Fig. 9.10.

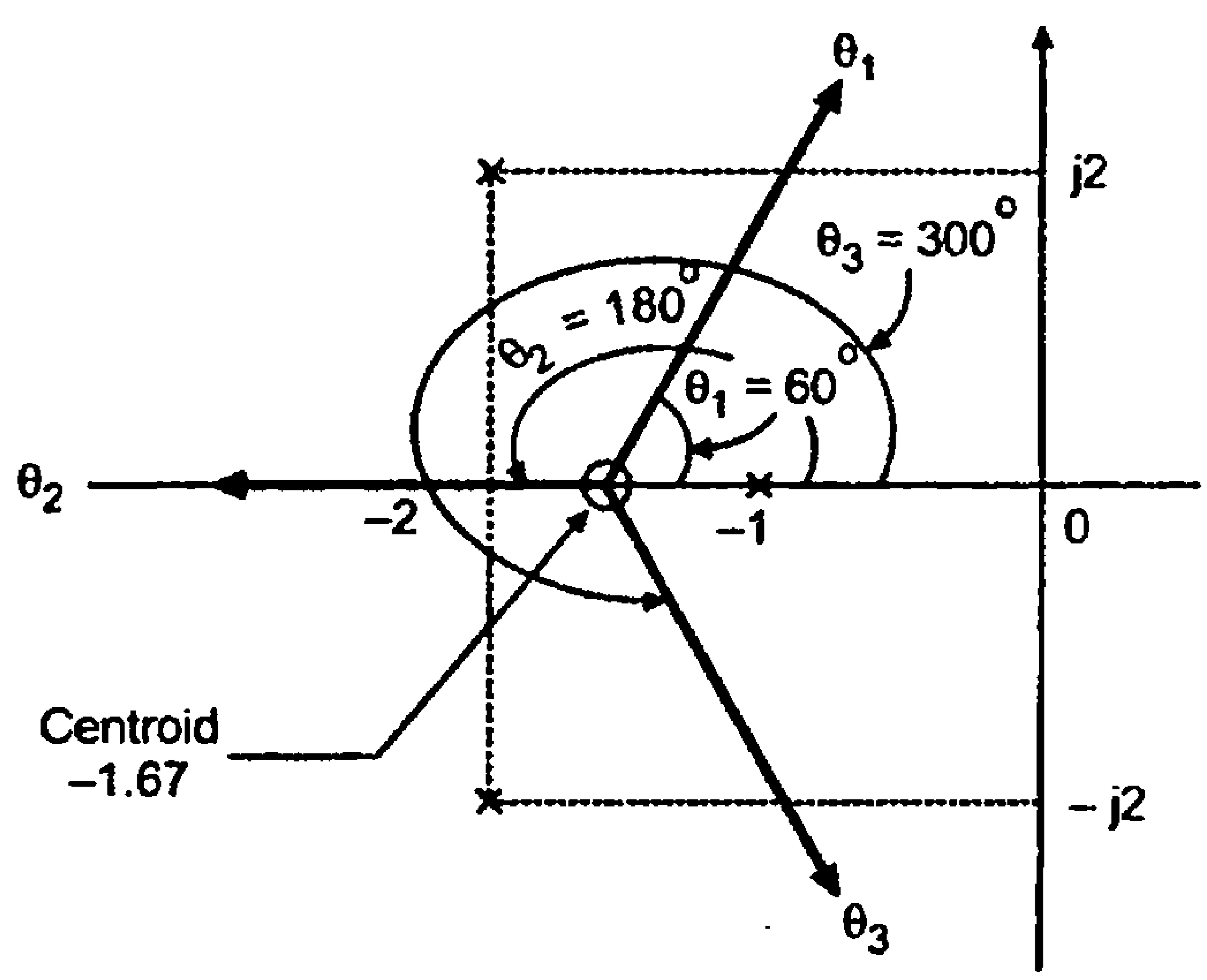


Fig. 9.10

The angles of asymptotes are fixed for fixed value of P–Z. The values of P and Z may be different but for particular values of P – Z, angles of asymptotes are fixed. The Table 9.1 gives the angles of asymptotes for few values of P – Z.

P – Z	Number of asymptotes required	Angles of asymptotes
0	0	–
1	1	$\theta_1 = 180°$
2	2	$\theta_1 = 90°$, $\theta_2 = 270°$
3	3	$\theta_1 = 60°$, $\theta_2 = 180°$, $\theta_3 = 300°$
4	4	$\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$

Table 9.1

Though the angles are repeated for certain value of P–Z, centroid decides their locations. Hence nature of root locus gets decided by the centroid i.e. locations of asymptotes in s-plane.

***Rule No. 6 :* Breakaway Point :**

Breakaway point is a point on the root locus where multiple roots of the characteristic equation occurs, for a particular value of K.

Consider example 9.4,

$$G(s)H(s) = \frac{K}{s(s+2)}$$

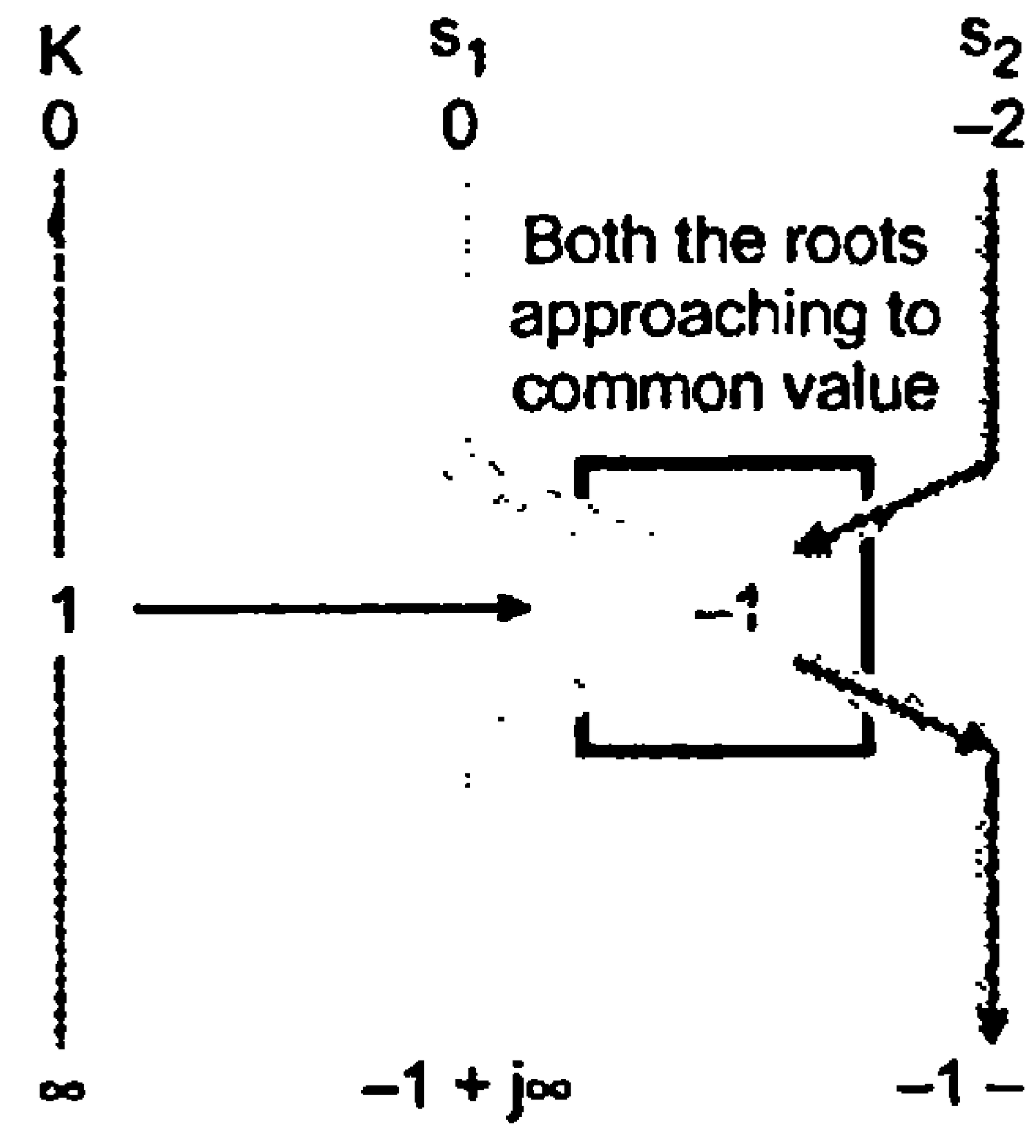


Fig. 9.11 Concept of breakaway point

In this example it can be observed that the roots s_1 and s_2 starts from open loop pole locations and both approach to common value of s = –1 as gain K increases from 0 to 1 and then both roots breakaway from s = –1 as pair of complex conjugate roots when K is increased from 1 to infinity. At K = 1, both the roots take same value as s = –1 from where roots are breaking away.

Such a point where two or more roots occur for a particular value of K is called **Breakaway point.** The root locus branches always leave breakaway points at an angle of $\pm\frac{180°}{n}$ where n = number of branches approaching at breakaway point.

As breakaway point indicates values of multiple root, it is **always on the root locus.**

General predictions about existence of breakaway points :

1) *If there are adjacently placed poles on the real axis and the real axis between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed poles.*

Consider example 9.4, $G(s)H(s) = \frac{K}{s(s+2)}$

$s = 0$ and $s = -2$ are adjacently placed poles on real axis and according to rule 3, section between them is a part of the root locus hence there must exist at least one breakaway point in between them. How to calculate co-ordinates of such breakaway point will be discussed later.

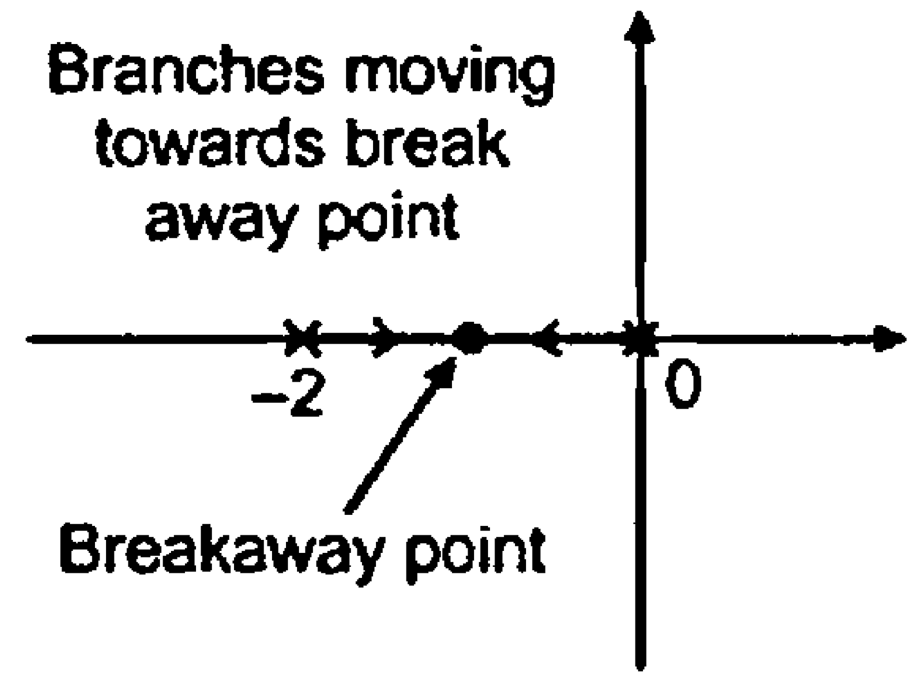


Fig. 9.12

Example 9.9 : *For* $G(s)H(s) = \frac{K(s+6)}{s(s+2)(s+4)}$ *how many minimum breakaway points exist ?*

Solution : In this case there are two pairs of adjacently placed poles on real axis.

$s = 0$ and $s = -2$, section between them is a part of root locus hence minimum one breakaway point exists in between them.

Now $s = -2$ and $s = -4$ also forms a pair of adjacently placed poles but section between them is not the part of root locus and hence there cannot be a breakaway point in between them. So minimum one breakaway point exists between $s = 0$ and $s = -2$. In such case, **branches are approaching towards breakaway point from the poles.**

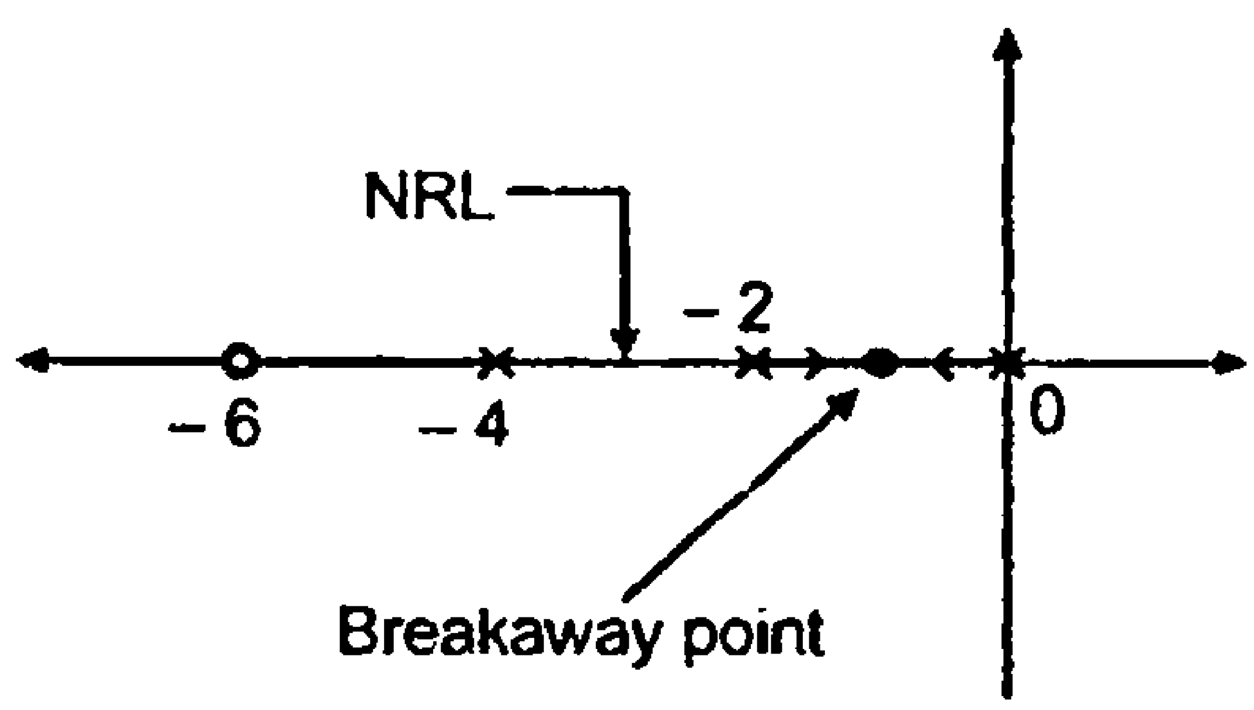


Fig. 9.13

2) *If there are two adjacently placed zeros on real axis and section of real axis in between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed zeros.*

Example 9.10 : *For* $G(s)H(s) = \frac{K(s+2)(s+4)}{s^2(s+6)}$, *how many minimum breakaway points exist ?*

Solution : Consider the pole-zero plot as shown in the Fig. 9.14.

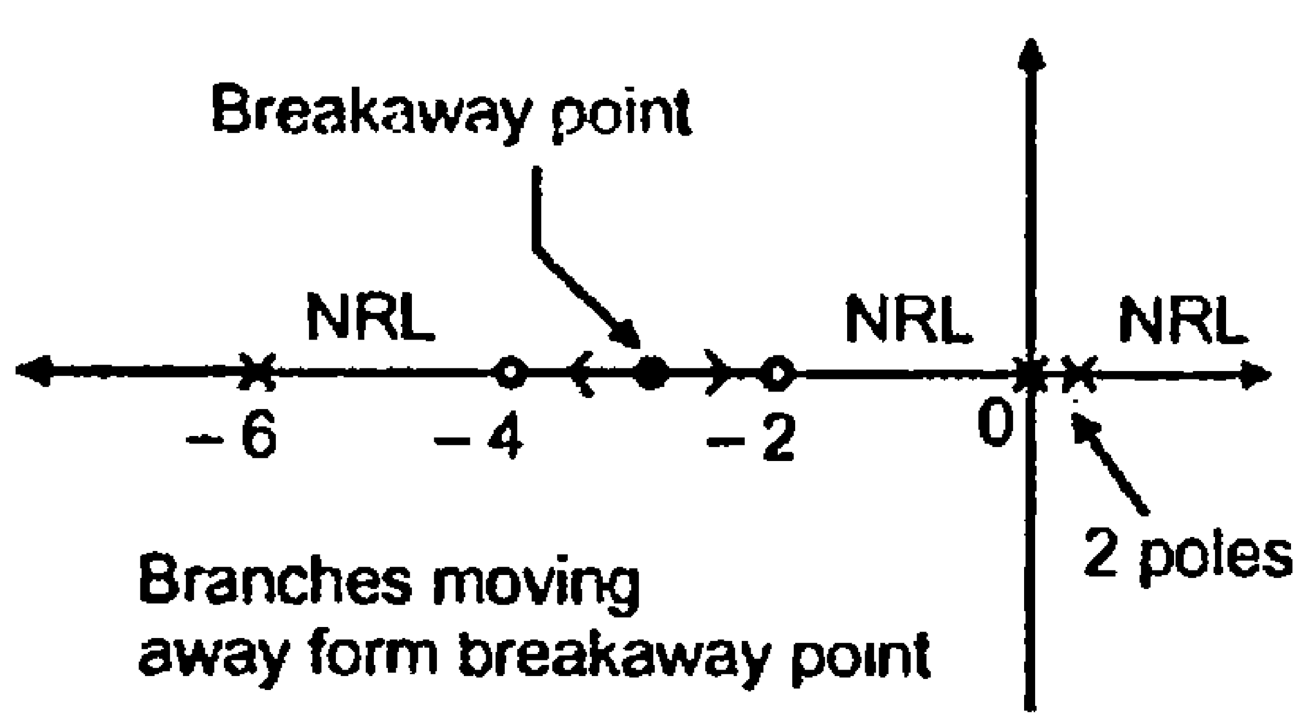


Fig. 9.14

s = –2 and s = – 4 are adjacently placed zeros and section between them is a part of the root locus so there exists minimum one breakaway point in between them.

In such case, **branches move away from the breakaway point** towards the open loop zeros, between which it is located.

Thus in such a case complex branches meet and break into real branches, approaching to zeros. Such a point is called **breakin point**. In general the point from where branches break into complex from real is called **breakaway** while the point from where branches break into real from complex is called **breakin** point.

3) *If there is a zero on the real axis and to the left of that zero there is no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one breakaway point to the left of that zero.*

Consider open loop transfer function,

$$G(s)H(s) = \frac{K(s+2)(s+4)}{s(s^2+2s+20)}$$

An open loop zero s = – 4 satisfies all the conditions and hence minimum one breakaway point exists to the left of this zero.

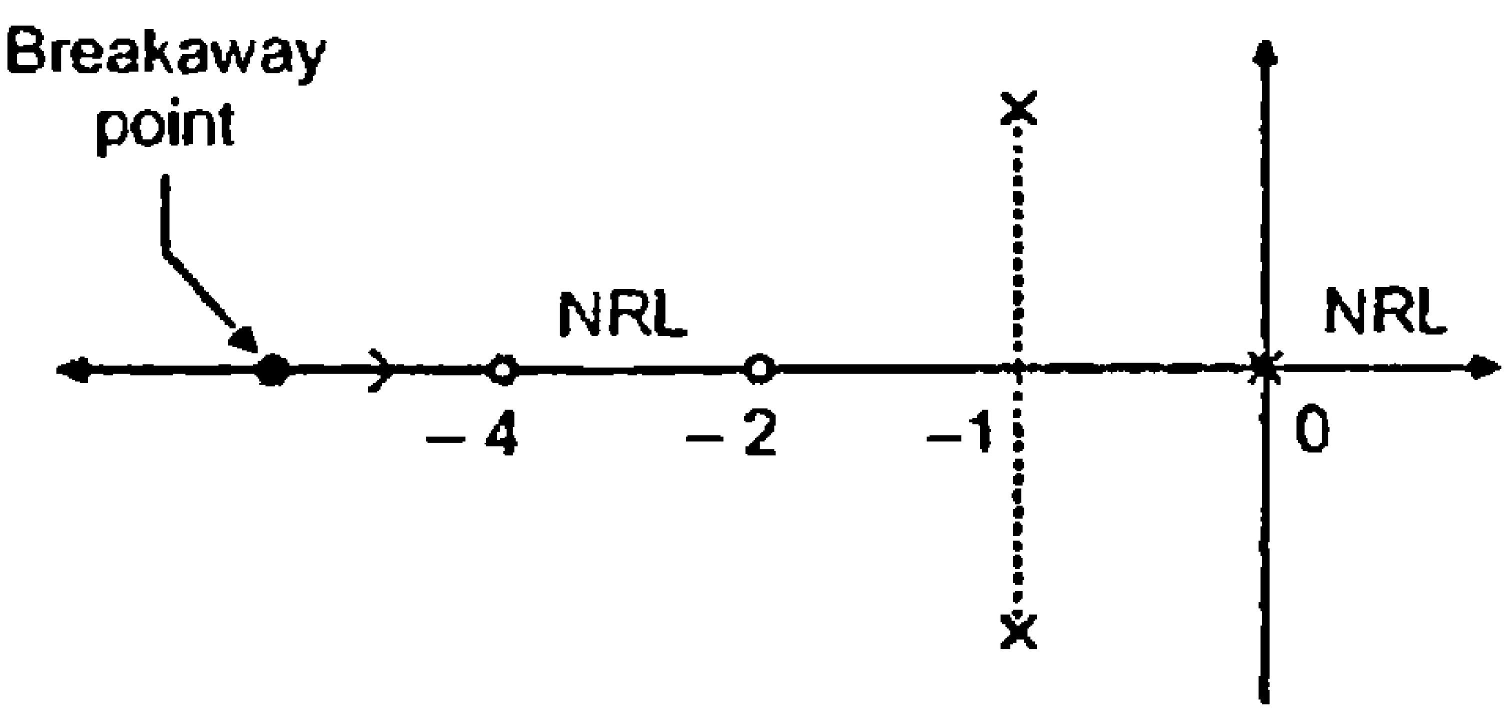


Fig. 9.14 (a)

Example 9.11 : *For* $G(s)H(s) = \frac{K(s+4)}{(s+2)(s^2+2s+2)}$, *how many minimum breakaway points exist?*

Solution : In this case, zero at s = – 4 satisfies the part of the above condition but to the left of that there is no root locus hence there cannot be a breakaway point to the left of this zero. There is minimum one breakaway point possible between s = –2 and s = –4 as per prediction number 2.

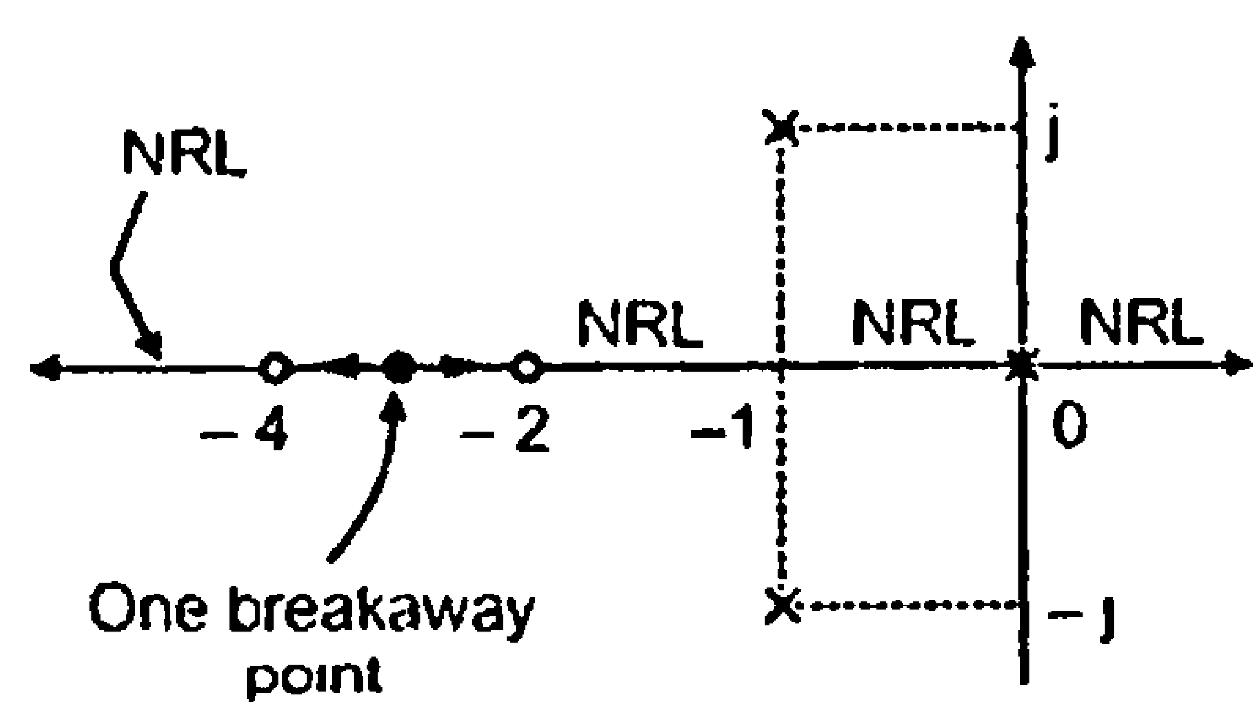


Fig. 9.15

Determination of breakaway point :

Steps to determine the co-ordinates of breakaway points are,

Step 1 : Construct the characteristic equation $1 + G(s)H(s) = 0$ of the system.

Step 2 : From this equation, separate the terms involving 'K' and terms involving 's'. Write the value of K in terms of s.

$$K = f(s)$$

Step 3 : Differentiate above equation w.r.t. 's', equate it to zero.

$$\frac{dK}{ds} = 0$$

Step 4 : Roots of the equation $\frac{dK}{ds} = 0$ gives us the breakaway points.

By this method we get all the breakaway and breakin points applicable for range of K from $-\infty$ to $+\infty$. So to decide valid breakaway points, substitute breakaway point value in the equation of K to get value of K .

Key Point: *If value of K is positive that breakaway point is valid for the root locus. The breakaway points for which values of K are negative, are invalid for direct root locus but are valid for inverse root locus.*

Example 9.12 : *For* $G(s)H(s) = \frac{K}{s(s+1)(s+4)}$, *determine the co-ordinates of valid breakaway points.*

Solution : Characteristic equation $1 + G(s)H(s) = 0$

Step 1 : $1 + \frac{K}{s(s+1)(s+4)} = 0$ i.e. $s^3 + 5s^2 + 4s + K = 0$

Step 2 : $K = -s^3 - 5s^2 - 4s$

Step 3 : $\frac{dK}{ds} = -3s^2 - 10s - 4 = 0$

Step 4 : $3s^2 + 10s + 4 = 0$

$$\therefore \text{Breakaway points} = \frac{-10 \pm \sqrt{100 - 4\times 4\times 3}}{2\times 3} = -0.46, -2.86$$

Substituting in expression for K

For $s = -0.46$, $K = +0.8793$

For $s = -2.86$, $K = -6.064$

$\therefore$ For s = – 0.46, K is positive.

i.e. s = – 0.46 is valid breakaway point for the root locus.

Root locus approaches and leaves breakaway point at an angle $\pm\frac{180^o}{n}$.

Here number of branches approaching = 2

$\therefore$ Angle of approaching = $\pm 90^o$ i.e. $\pm\frac{\pi}{2}$

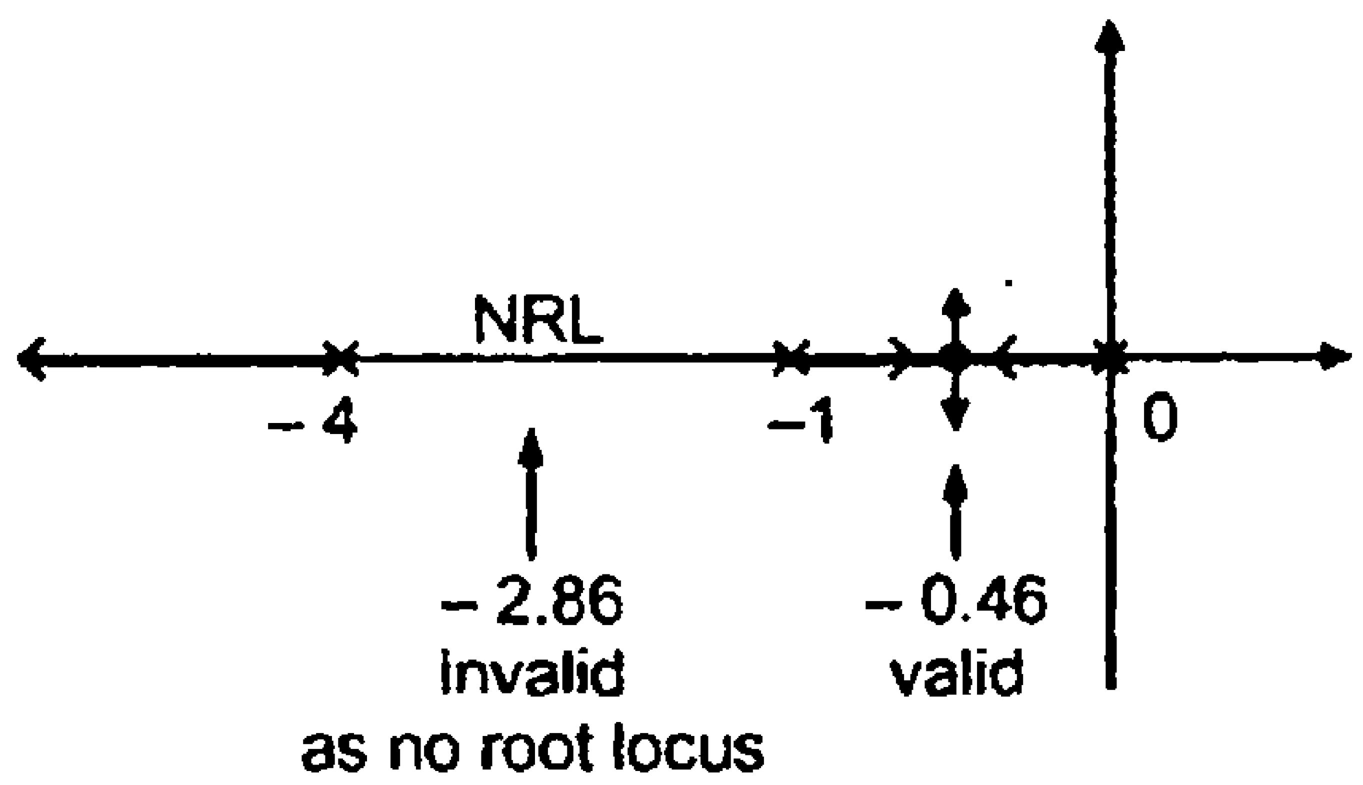


Fig. 9.16

Rule No. 7 : Intersection of root locus with imaginary axis. This can be determined by following procedure.

Step 1 :	Consider characteristic equation 1 + G(s)H(s) = 0 as obtained in Rule 6.
Step 2 :	Construct Routh's array in terms of "K".
Step 3 :	Determine $K_{marginal}$ i.e. value of K which creates one of the rows of Routh's array as row of zeros, except the row of s^0.
Step 4 :	Construct auxiliary equation A(s) = 0 by using coefficients of a row which is just above the row of zeros.
Step 5 :	Roots of auxiliary equation A(s) = 0 for K = K_{mar} are nothing but the intersection points of the root locus with imaginary axis.

Consider example 9.12 :

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

Characteristic equation is given by,

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+4)} = 0$$

i.e. $s^3 + 5s^2 + 4s + K = 0$

Routh's array,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

$$K_{mar} = 20 \text{ that makes row corresponding to } s^1 \text{ as row of zeros.}$$

$$\therefore \quad A(s) = 5s^2 + K = 0$$

$$K = K_{mar} = 20$$

$$5s^2 + 20 = 0$$

$$s^2 = -4 \quad \therefore \quad s = \pm j2$$

So $s = \pm j2$ are the points of intersection of root locus with imaginary axis. If K_{mar} is positive there is vaild intersection of root locus with imaginary axis. Such problems are already discussed in the previous chapter.

> **Key Point**: *If K_{mar} is positive, root locus intersects with imaginary axis. But if K_{mar} is negative root locus does not intersect with imaginary axis and lies totally in left half of s-plane.*

Rule No. 8 : Angle of departure at complex conjugate poles and angle of arrival at complex conjugate zeros.

Angle of departure at complex pole :

As branch always leaves from an open loop pole, it is advantageous to know at what angle it departs from complex conjugate pole. This angle at which it departs from complex pole is called **angle of departure denoted as ϕ_d.**

$$\phi_d = 180° - \phi \qquad \text{where } \phi = \sum \phi_P - \sum \phi_Z$$

where $\sum \phi_P$ = Contributions by the angles made by remaining open loop poles at the pole at which ϕ_d is to be calculated.

$\sum \phi_Z$ = Contributions by the angles made by the open loop zeros at the pole at which ϕ_d is to be calculated.

To calculate $\Sigma \phi_P$, join all the remaining poles to the complex pole under consideration. Add all the angles subtended by phasors joining poles to pole under consideration. Similarly join all zeros to pole under consideration and adding all angles determine $\Sigma \phi_Z$.

This is illustrated by the following example.

Example 9.13 : *For* $G(s)H(s) = \dfrac{K(s+2)}{s(s+4)(s^2+2s+2)}$ *, calculate angles of departures at complex conjugate poles.*

Solution : P = 4 , Z = 1

Poles are at s = 0 , −4 , − 1 ± j

Zero at s = −2.

Draw Pole-Zero plot.

Let us calculate ϕ_d at the pole s = −1 + j.

Join all other poles to this pole and measure or calculate the angles ϕ_{P1}, ϕ_{P2}, ϕ_{P3} as shown in the Fig. 9.17.

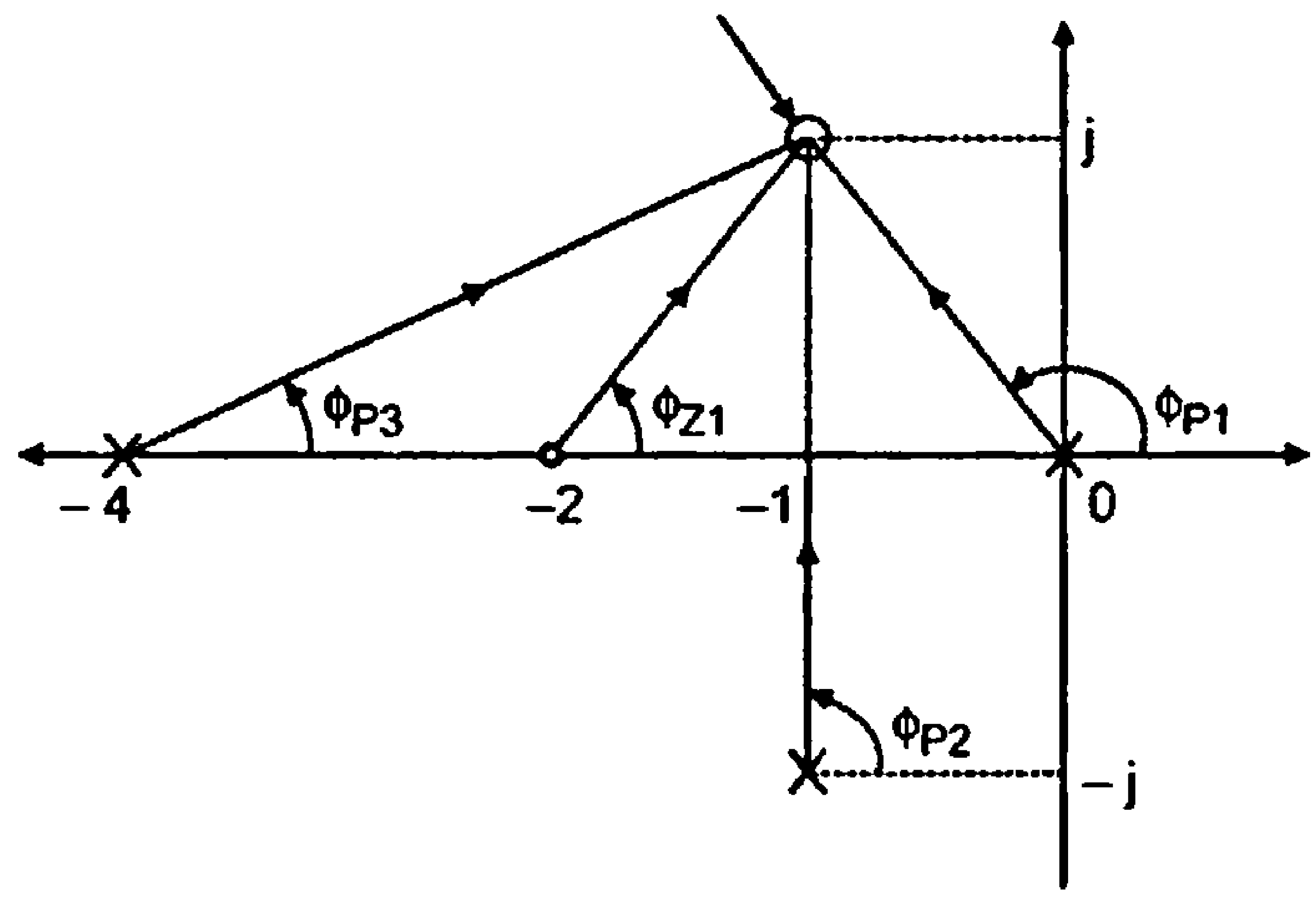


Fig. 9.17

Join all zeros to this pole and calculate ϕ_{Z1}.

Then , $\sum\phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3}$ while

$$\sum\phi_Z = \phi_{Z1}$$

From geometry of the Fig. 9.17 we can calculate,

$$\phi_{P1} = 135^\circ, \quad \phi_{P2} = 90^\circ, \quad \phi_{P3} = 18.43^\circ$$

$$\therefore \quad \sum\phi_P = 135^\circ + 90^\circ + 18.43^\circ = 243.43^\circ$$

$$\sum\phi_Z = \phi_{Z1} = 45^\circ$$

$$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 243.43^\circ - 45^\circ = 198.43^\circ$$

$$\phi_d = 180^\circ - \phi = 180^\circ - 198.43^\circ = -18.43^\circ$$

∴ Root locus branch leaving this pole will depart tangentially to the line whose angle is given by $\phi_d = -18.43^\circ$ as shown in the Fig. 9.18.

For second complex conjugate pole, sign of ϕ_d will be just opposite as root locus is always symmetrical about real axis. So root locus branch departing from s = − 1− j will depart tangentially to the line whose angle is given by $\phi_d = +18.43^\circ$.

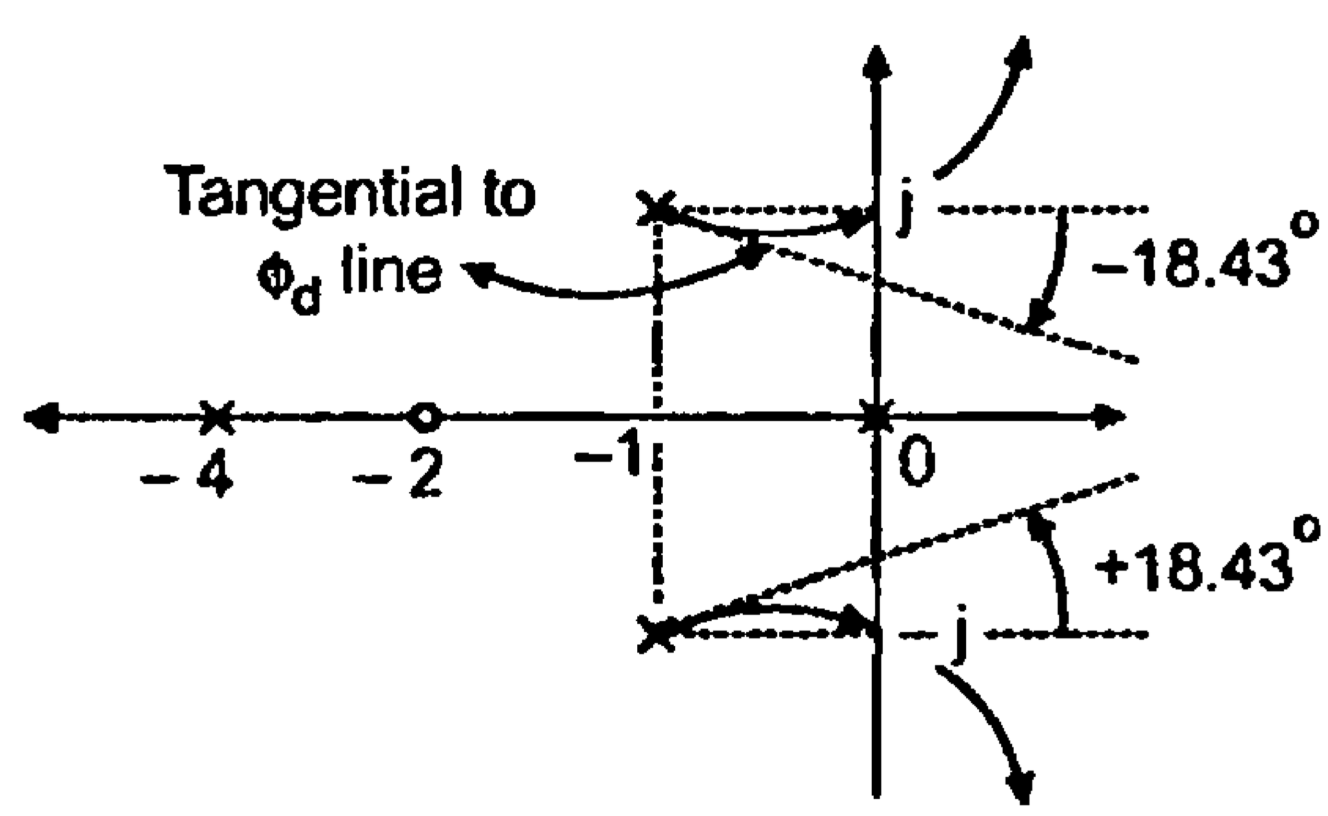


Fig. 9.18

Angle of arrival at a complex zero :

Angle of arrival at a complex zero can be calculated by the same method, which is denoted as ϕ_a. The only change to calculate the angle of arrival is,

$$\phi_a = 180^\circ + \phi$$

where $$\phi = \Sigma\,\phi_P - \Sigma\,\phi_Z$$

Such branches will arrive and terminate at the complex zeros running tangentially to the lines whose angles are given by ϕ_a as explained above.

9.7 Graphical Determination of 'K' for Specified Damping Ratio 'ξ'

In higher order systems, the transient response i.e. damping ratio ξ gets controlled by the dominant pair of roots. If these dominant roots are complex conjugates of each other then such roots can be represented as,

$$-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Consider the location of point P as, $(-\xi\omega_n + j\omega_n\sqrt{1-\xi^2})$ as shown in the Fig. 9.19 which is in left half of s-plane.

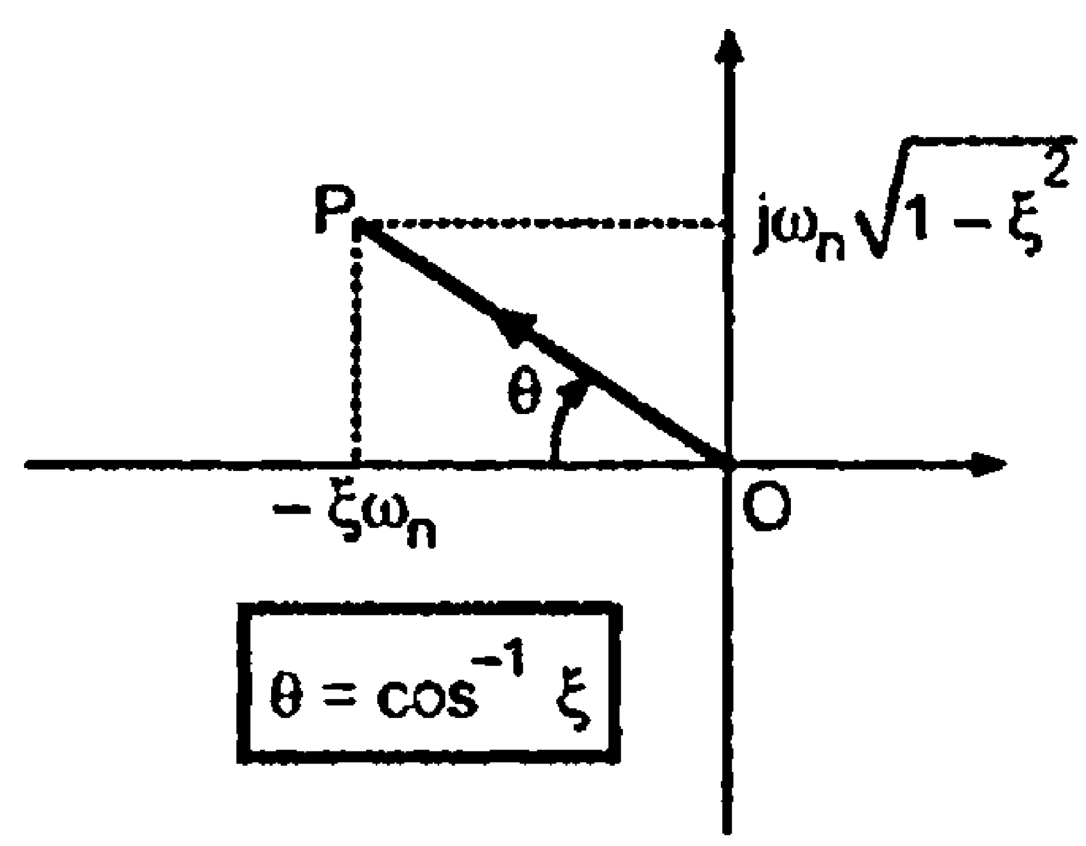


Fig. 9.19

If phasor is drawn from such point to origin then angle subtended by this phasor with respect to negative real axis is θ.

Then $$\cos\theta = \frac{\xi\omega_n}{l(OP)}$$

$$l(OP) = \sqrt{(-\xi\omega_n)^2 + (\omega_n\sqrt{1-\xi^2})^2} = \omega_n$$

$$\therefore \quad \cos\theta = \frac{\xi\omega_n}{\omega_n} = \xi \quad \text{i.e.} \quad \theta = \cos^{-1}\xi$$

Now it is desired to design a system with a certain damping ratio 'ξ' for which it requires to decide corresponding value of system gain 'K' by using root locus.

For this, use following steps.

Step 1 :	Draw the root locus to the scale on graph paper. Preferably choose same scale for X and Y axis.
Step 2 :	Get the value of $\theta = \cos^{-1} \xi$.
Step 3 :	Draw a line at angle 'θ' from origin such that 'θ' is **measured from negative real axis in clockwise direction.**
Step 4 :	Determine the intersection point of this line with the root locus sketched to the scale.
Step 5 :	For this point, apply the magnitude condition to decide the corresponding system gain 'K'.

9.8 General Steps to Solve the Problem on Root Locus

Step 1 : Get the general information about number of open loop poles, zeros, number of branches etc. from G(s)H(s) .

Step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions.

Step 3 : Calculate angles of asymptotes.

Step 4 : Determine the centroid. Sketch a separate sketch for step 3 and step 4.

Step 5 : Calculate the breakaway and breakin points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as breakaway points.

Step 6 : Calculate the intersection points of root locus with the imaginary axis.

Step 7 : Calculate the angles of departures or arrivals if applicable.

Step 8 : Combine steps 1 to 7 and draw the final sketch of the root locus.

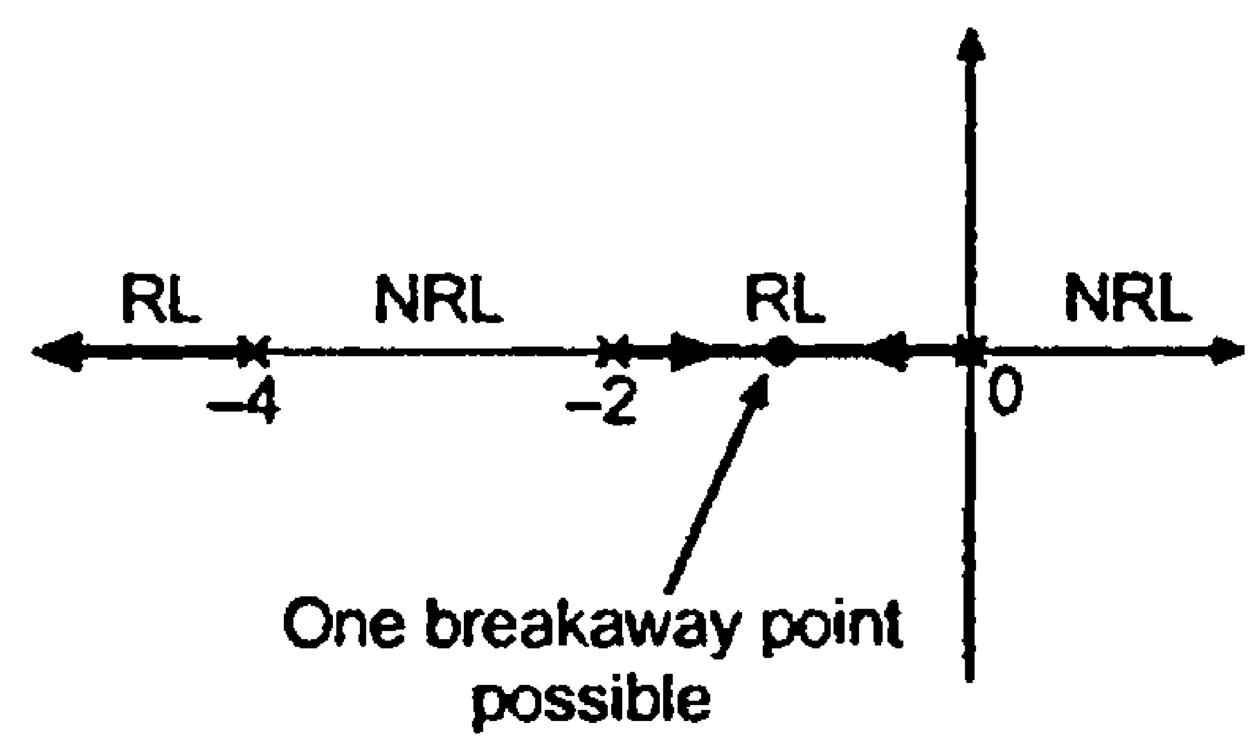
Step 9 : Predict the stability and performance of the given system by using the root locus.

Example 9.14 : *For a unity feedback system,* $G(s) = \frac{K}{s(s+4)(s+2)}$. *Sketch the rough nature of the root locus showing all details on it. Comment on the stability of the system.*

(M.U. : June-92)

Solution : Step 1 : General information from $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$

P = 3, Z = 0, number of branches N = P = 3. No finite zero so all P – Z = 3 branches will terminate at infinity. Starting points are locations of open loop poles i.e. 0, – 2, – 4.



Step 2 : Pole-Zero plot and sections of real axis.

Directions of branches away from poles. One breakaway point exists between 0 and – 2 according to general prediction.

Sections of real axis identified as a part of the root locus as to right side sum of poles and zeros is odd for those sections.

Step 3 : Angles of asymptotes.

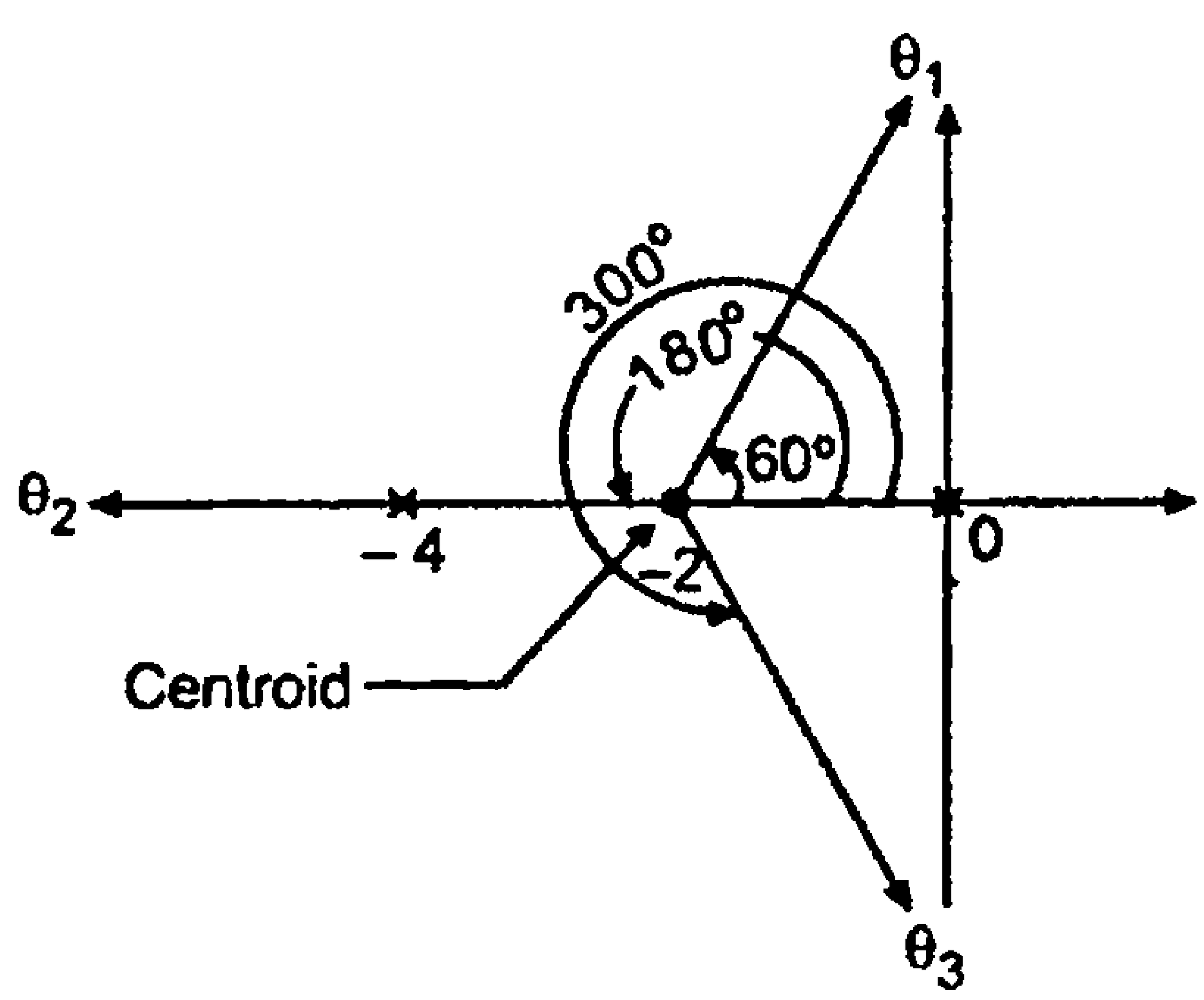
3 branches are approaching to ∞ , 3 asymptotes are required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \qquad q = 0, 1, 2$$

$$\therefore \ \theta_1 = \frac{180^\circ}{3} = 60^\circ, \ \theta_2 = \frac{(2+1)180^\circ}{3} = 180^\circ, \ \theta_3 = \frac{(2\times 2+1)180^\circ}{3} = 300^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0-2-4}{3} = -2$$



Branches will approach to ∞ along these lines which are asymptotes.

Step 5 : To find breakaway point (Refer Rule No. 6). Characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$\therefore \quad s^3 + 6s^2 + 8s + K = 0$$

$$\therefore \quad K = -s^3 - 6s^2 - 8s \quad \text{... (1)}$$

$$\frac{dK}{ds} = -3s^2 - 12s - 8 = 0$$

$$\text{i.e.} \quad 3s^2 + 12s + 8 = 0$$

Roots i.e. breakaway points $= \dfrac{-12 \pm \sqrt{144 - 4 \times 3 \times 8}}{2 \times 3} = -0.845, -3.15$

As there is no root locus between – 2 to – 4 , – 3.15 cannot be a breakaway point. It also can be confirmed by calculating `K' for s = – 3.15. It will be negative that confirms s = – 3.15 is not a breakaway point.

For s = – 3.15, K = – 3.079 (Substituting in equation for K)

But as there has to be breakaway point between `0' and `– 2' , s = – 0.845 is vaild breakaway point.

For s = – 0.845 K = + 3.079

As K is positive **s = – 0.845 is vaild breakaway point.**

Step 6 : Intersection point with imaginary axis.

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

Routh's array

s^3	1	8
s^2	6	K
s^1	$\frac{48-4}{6}$	0
s^0	K	

$K_{marginal}$ = 48 which makes row of s^1 as row of zeros.

$$A(s) = 6s^2 + K = 0$$

$$K_{mar} = 48$$

$$\therefore \quad 6s^2 + 48 = 0$$

$$s^2 = -8$$

$$\therefore \quad s = \pm j\sqrt{8} = \pm j2.828$$

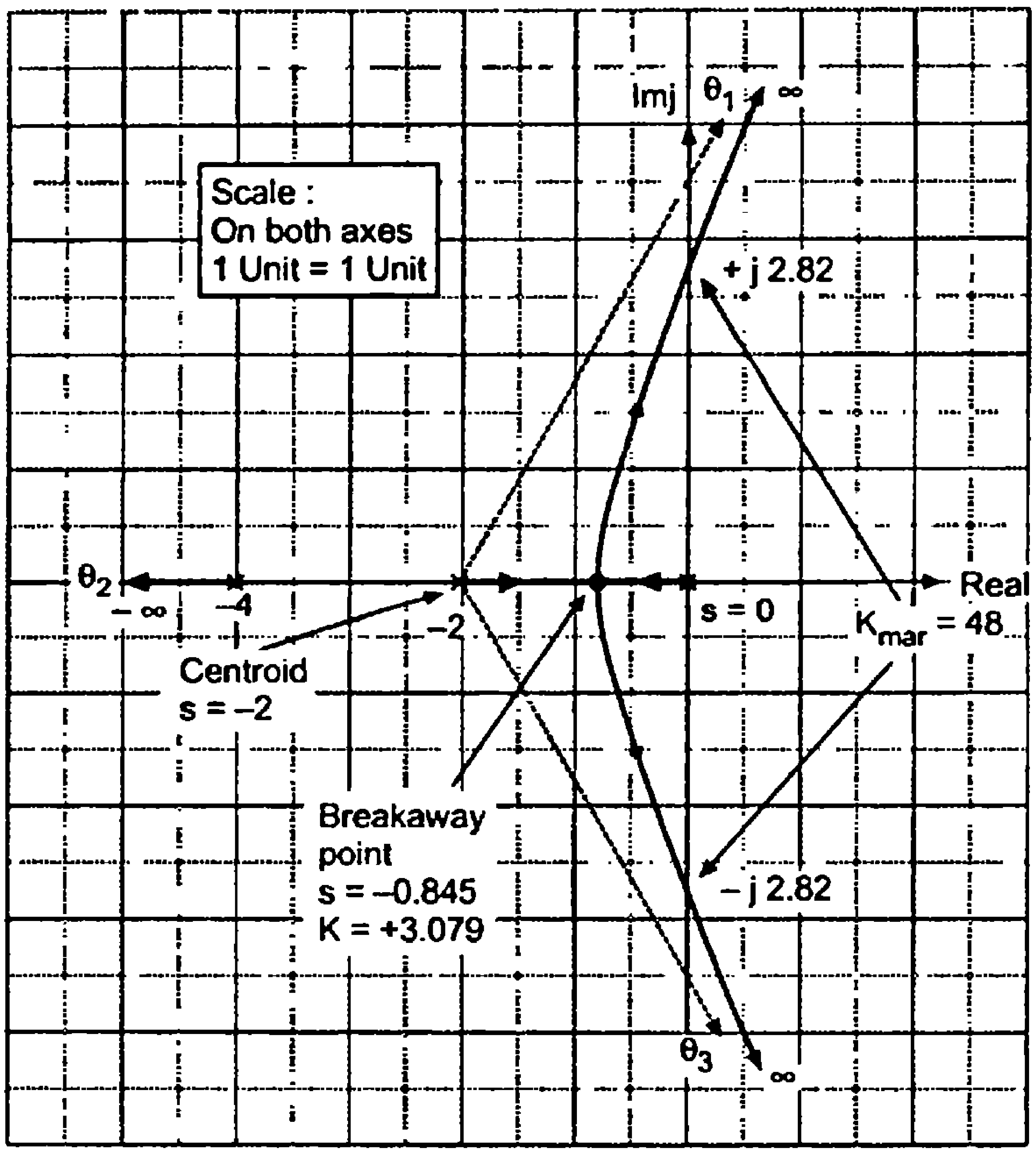
Intersection of root locus with imaginary axis is at ± j 2.828 and corresponding value of K_{mar} = 48.

Step 7 : As there are no complex conjugate poles or zeros, no angles of departures or arrivals are required to be calculated.

Step 8 : The complete root locus is as shown below.

Step 9 : Prediction about stability :

For 0 < K < 48, all the roots are in left half of s–plane hence system is absolutely stable. For K_{mar} = +48, a pair of dominant roots on imaginary axis with remaining root in left half. So system is marginally stable oscillating at 2.82 rad/sec. For 48 < K < ∞, dominant roots are located in right half of s-plane hence system is unstable.



Concept of Dominant Roots

Key Point : *Stability is predicted by the locations of the dominant roots. The dominant roots are those which are located closest to the imaginary axis. The branches starting from such roots which are dominant decide the stability.*

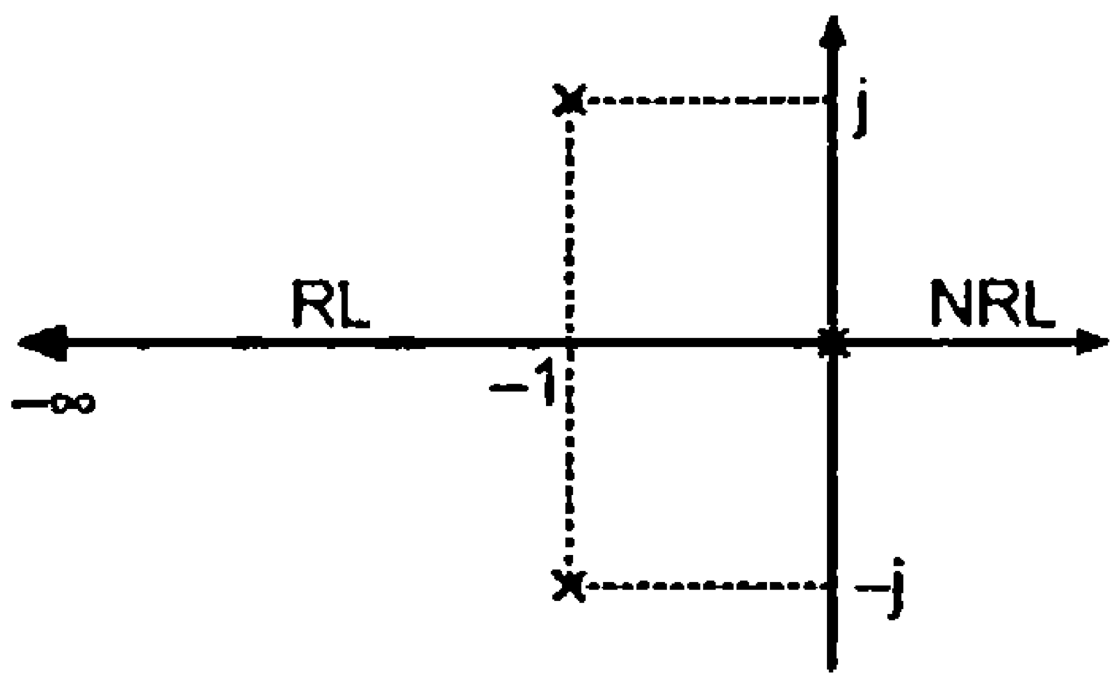
In this problem, branches starting from s = 0 and s = – 5 are dominant root locus branches. The branch starting from s = –10 is not dominant as it moves away form imaginary axis to the left. So stability of system will get decided by root locus branches starting from s = 0 and –5 which are dominant. Using the same concept, a **third order system having three roots can be approximated as a second order, considering only its dominant roots.**

Example 9.15 : *Sketch the root locus for the system having* $G(s)H(s) = \dfrac{K}{s(s^2+2s+2)}$.

Solution : Step 1 : P = 3, Z = 0, N = P = 3

P – Z = 3 branches approaching to ∞ . Starting points open loop poles,

s = 0, s = – 1 + j , s = –1 – j.



Step 2 : Pole-Zero plot and sections of real axis.

One branch is approaching to – ∞ starting from pole at s = 0. No breakaway point exists according to general predictions.

Step 3 : Angles of asymptotes : 3 branches approaching to ∞, 3 asymptotes required

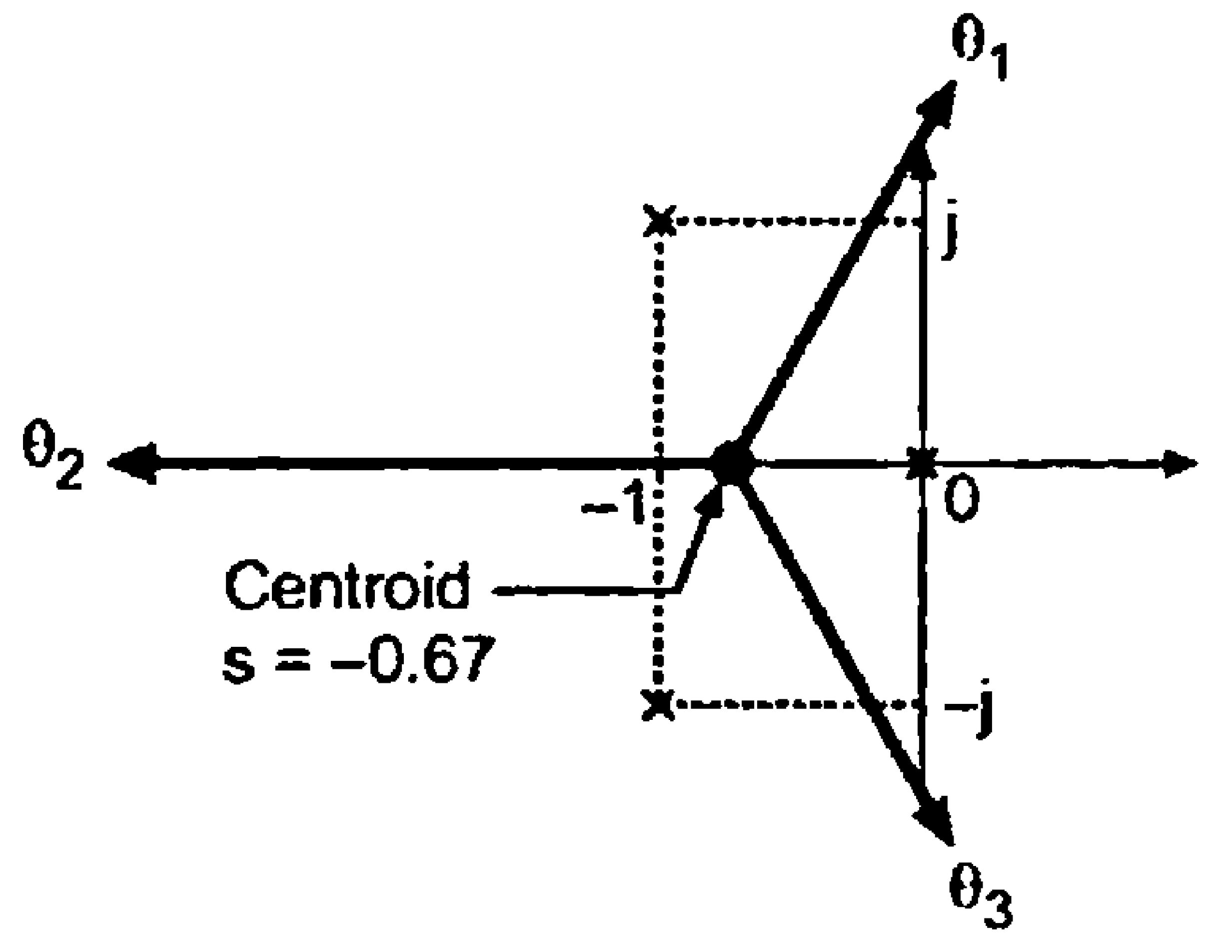
$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \quad q = 0, 1, 2.$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)\,180^\circ}{3} = 180^\circ,\ \theta_3 = \frac{(2\times2+1)\,180^\circ}{3} = 300^\circ$$

Step 4 : Centroid : $\sigma = \dfrac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P-Z} = \dfrac{0-1-1-0}{3}$

$$= -\frac{2}{3} = -0.67$$

One branch approaching to ∞ along θ_2 while remaining two branches starting from – 1 + j and – 1 – j will approach to ∞ along θ_1 and θ_3 respectively.



Step 5 : Breakaway point

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s^2+2s+2)} = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

$$\therefore \quad K = -s^3 - 2s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 4s - 2 = 0$$

$\therefore \quad 3s^2 + 4s + 2 = 0$

$\therefore$ Breakaway points are $\dfrac{-4 \pm \sqrt{16-24}}{2 \times 3} = -0.67 \pm j\,0.4714$

Now instead of substituting $-0.67 \pm j\,0.4714$ in expression for K we can check validity of these points as breakaway points by using angle condition.

The point which satisfies angle condition is on the root locus and point on root locus satisfying $\dfrac{dK}{ds} = 0$ is nothing but breakaway point.

Let us test $\quad s = -0.67 + j\,0.4714$

$$\angle G(s)H(s) = \pm(2q+1)\,180^\circ, \quad q = 0, 1, 2 \ldots\ldots$$

$$G(s)H(s) = \frac{K}{s(s+1+j)(s+1-j)}$$

at $\quad s = -0.67 + j\,0.4714$

$$\angle G(s)H(s) = \frac{\angle K + j0}{\angle -0.67 + j0.4714 \;\; \angle -0.67 + j0.4714 + 1 + j \;\; \angle -0.67 + j0.4714 + 1 - j}$$

$$= \frac{\angle K + j0}{\angle -0.67 + j0.4714 \;\; \angle 0.33 + 1.47j \;\; \angle 0.33 - j0.53}$$

$$= \frac{0^\circ}{\angle 144.87^\circ \;\; \angle 77.34^\circ \;\; \angle -58.09^\circ} = -164.11^\circ$$

This is not odd multiple of 180°. Hence **point is not on the root locus** and hence there **is no breakaway point existing for this system.**

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^3 + 2s^2 + 2s + K = 0$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\dfrac{4-K}{2}$	0
s^0	K	

$K_{mar} = +4$ makes row of $s^1 = 0$

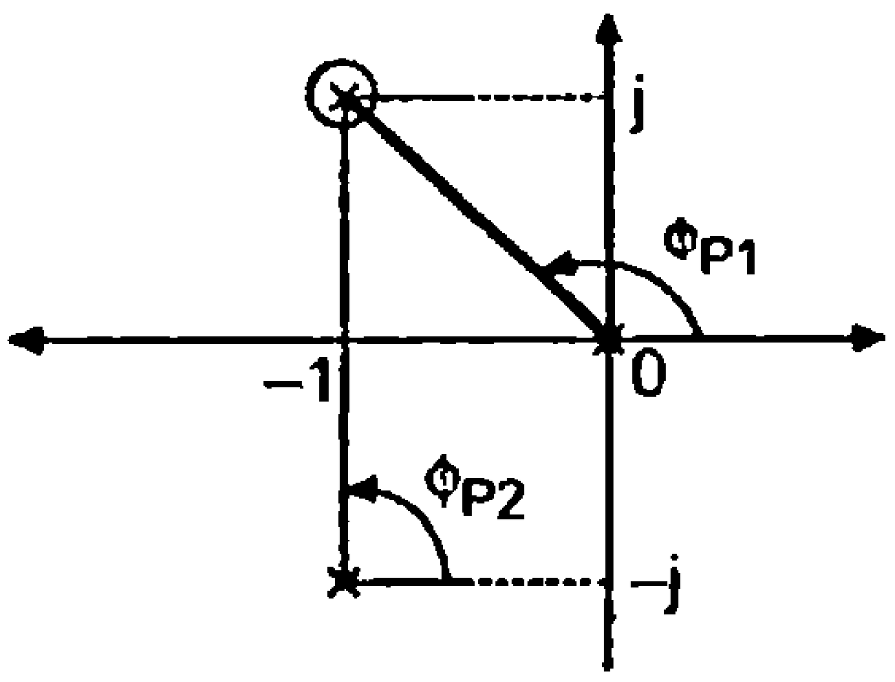
$A(s) = 2s^2 + K = 0$

At $K_{mar} = 4$

$2s^2 + 4 = 0$

$s^2 = -2 \quad \therefore \; s = \pm j\,1.414$

Step 7 : Angle of departure : As branch is departing at $-1+j$ let us calculate angle of departure, at $-1+j$.



$$\phi_{P1} = 135^\circ , \ \phi_{P2} = 90^\circ$$

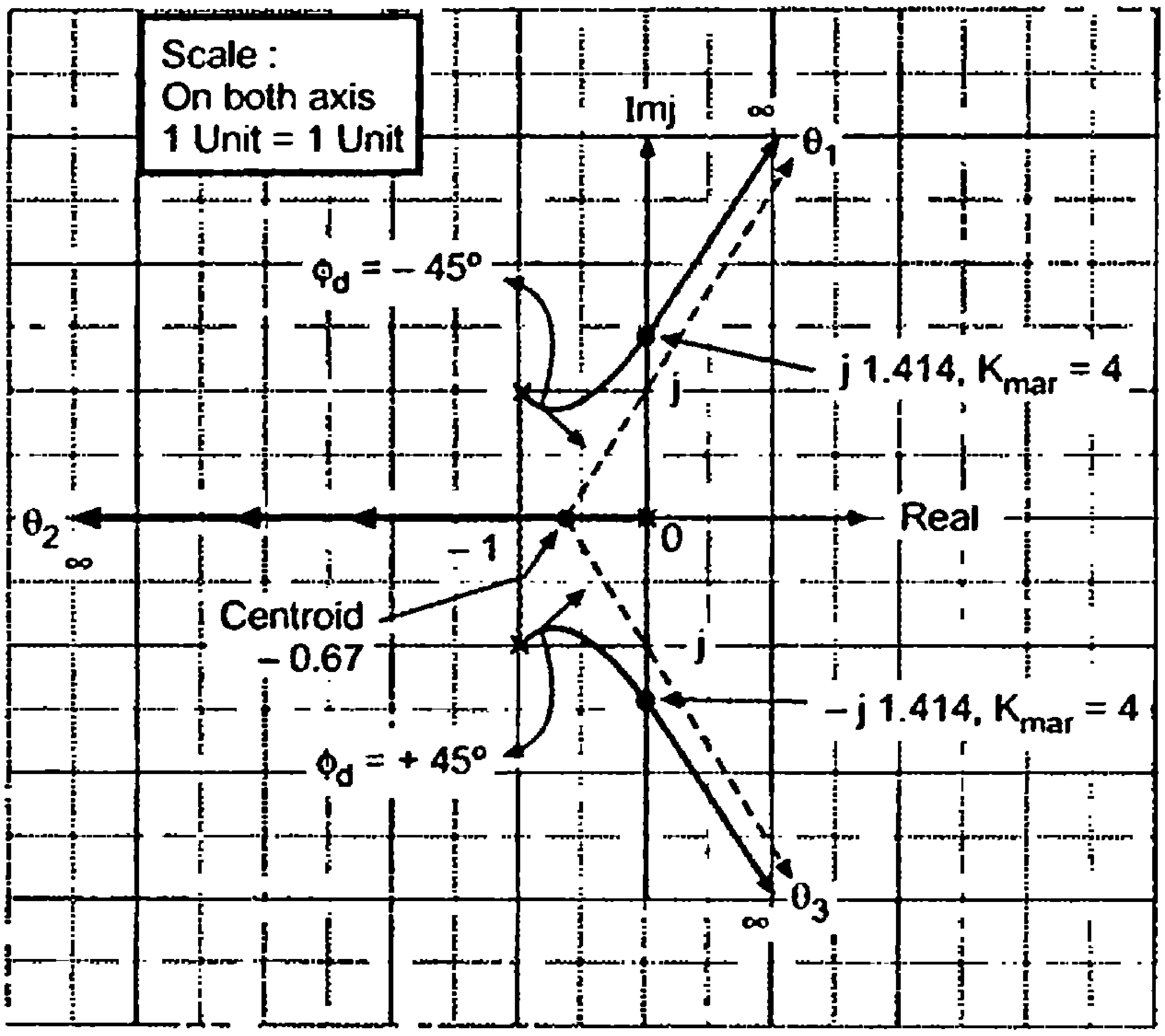
$$\Sigma\phi_P = \phi_{P1} + \phi_{P2} = 225^\circ, \ \Sigma\phi_Z = 0$$

$$\therefore \quad \phi = \Sigma\,\phi_P - \Sigma\phi_Z = 225^\circ$$

$$\therefore \quad \varphi_d = 180^\circ - \phi = 180^\circ - 225^\circ = -45^\circ$$

At $-1-j$, $\phi_d = +45^\circ$

Step 8 : Complete Root Locus is :



Step 9 : Comment on stability :

For $0 < K < 4$ all roots are in left half of s-plane. System is absolutely stable.

At $K = +4$, dominant roots are on imaginary axis, system is marginally stable, oscillating with 1.414 rad/sec.

At $K > 4$, dominant roots are in right half of s-plane and hence system becomes unstable in nature.

9.9 Effect of Addition of Open Loop Poles and Zeros

9.9.1 Addition of Pole

In general we can state that adding a pole to the function G(s)H(s) in the left half of the s-plane has the effect of pushing original root locus towards right half of s - plane. This can be proved by following examples.

Consider, $G(s)H(s) = \frac{K}{s(s+2)}$

Corresponding root locus is shown in the Fig. 9.20.

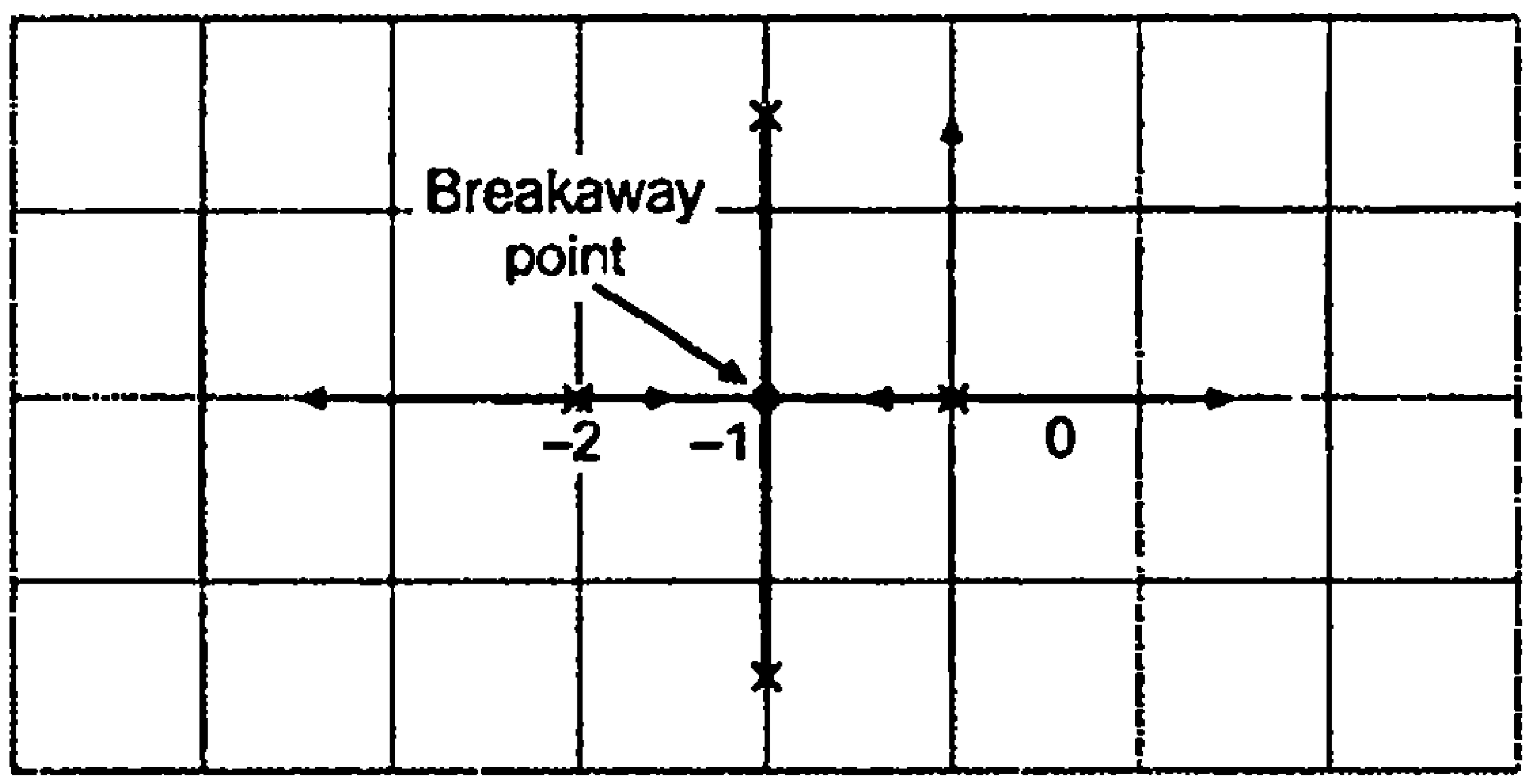


Fig. 9.20

Now if pole at s = −4 is added to G(s)H(s) root locus becomes as shown in the Fig. 9.21, so for any value of 'K' before addition of pole in left half, system is totally stable but after addition of pole in left half two branches of root locus after some value of 'K' moves in right half of s-plane so system up to this value of 'K' is stable. After this value 'K' system becomes unstable so stability of system gets restricted.

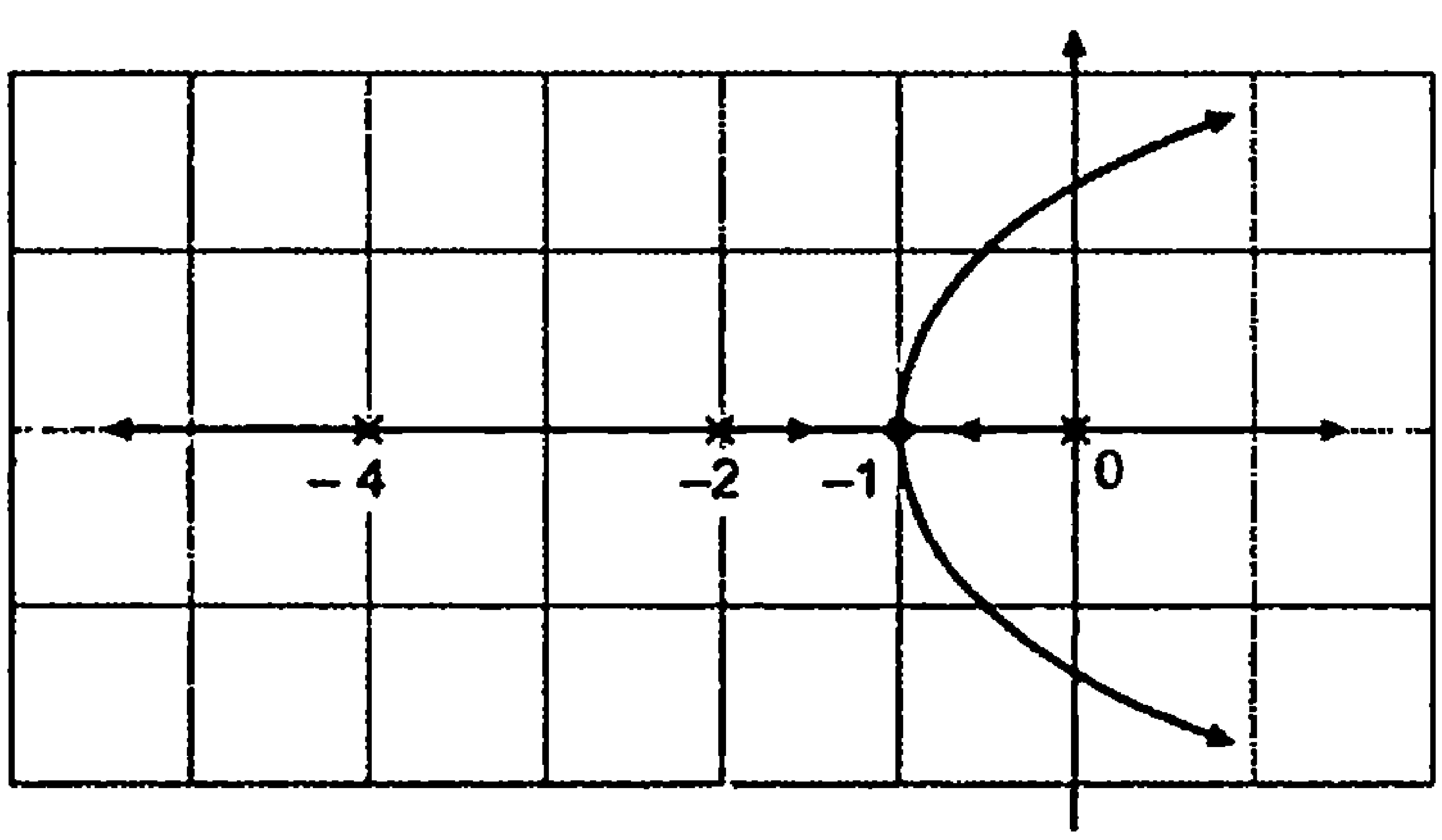


Fig. 9.21

If now one more pole at s = – 6 is added to the system,

$$G(s)H(s) = \frac{K}{s(s+2)(s+4)(s+6)}$$

Breakaway point in section s = 0 and s = –2 gets shifted towards right as compared to previous case. So system stability further gets restricted. This is shown in the Fig. 9.22.

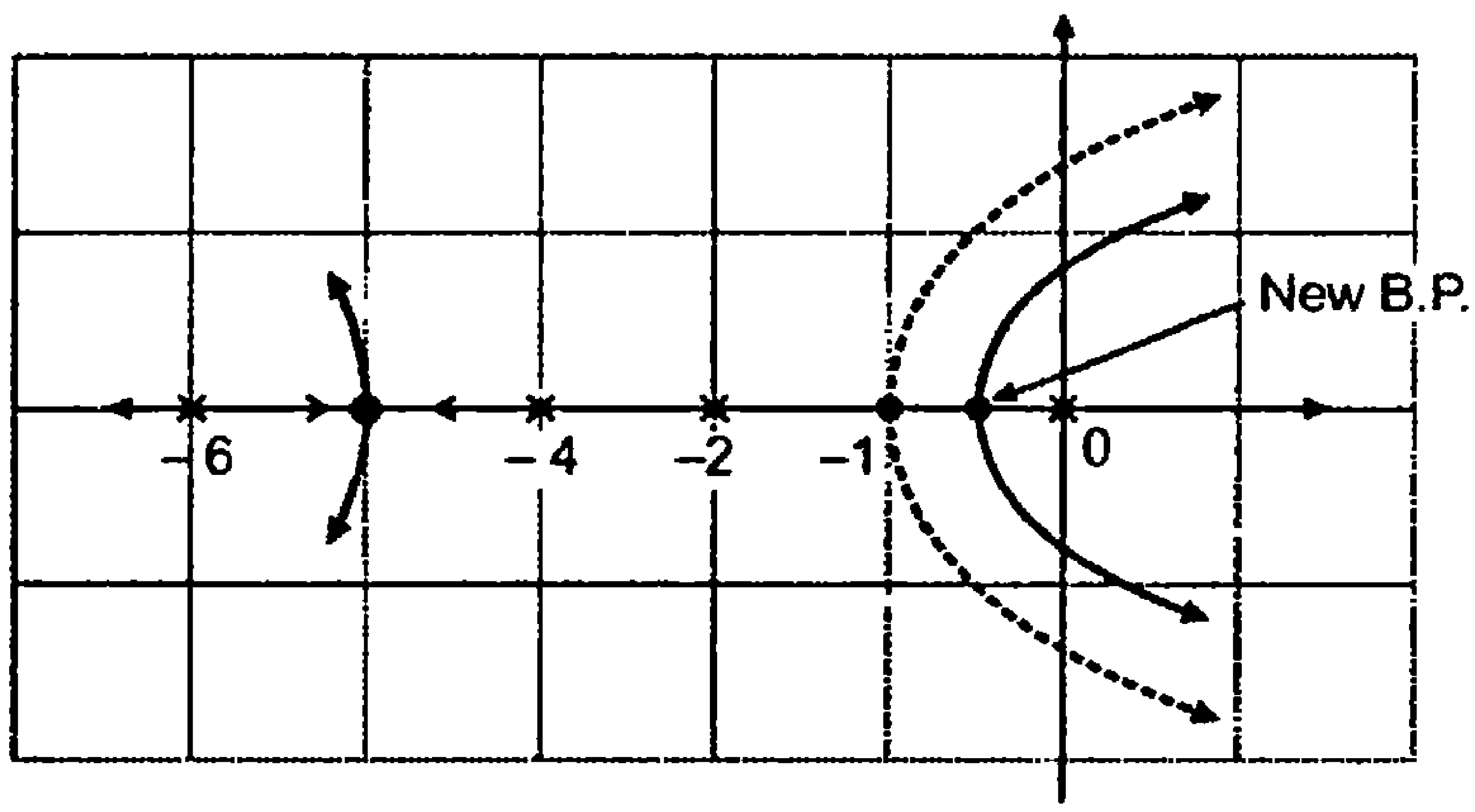


Fig. 9.22

From this we can conclude that due to addition of poles, root locus shifts towards R.H.S. of s - plane so system stability decreases.

Effects of addition of open loop poles can be summarized as :

1) Root locus shifts towards imaginary axis.
2) System stability relatively decreases.
3) System becomes more oscillatory in nature.
4) Range of operating values of 'K' for stability of the system decreases.

9.9.2 Addition of Zeros

Let us introduce a zero at

$s = -4$ to $G(s)H(s) = \frac{K}{s(s+2)}$

Root loci for $G(s)H(s) = \frac{K}{s(s+2)}$ is shown in the Fig. 9.23.

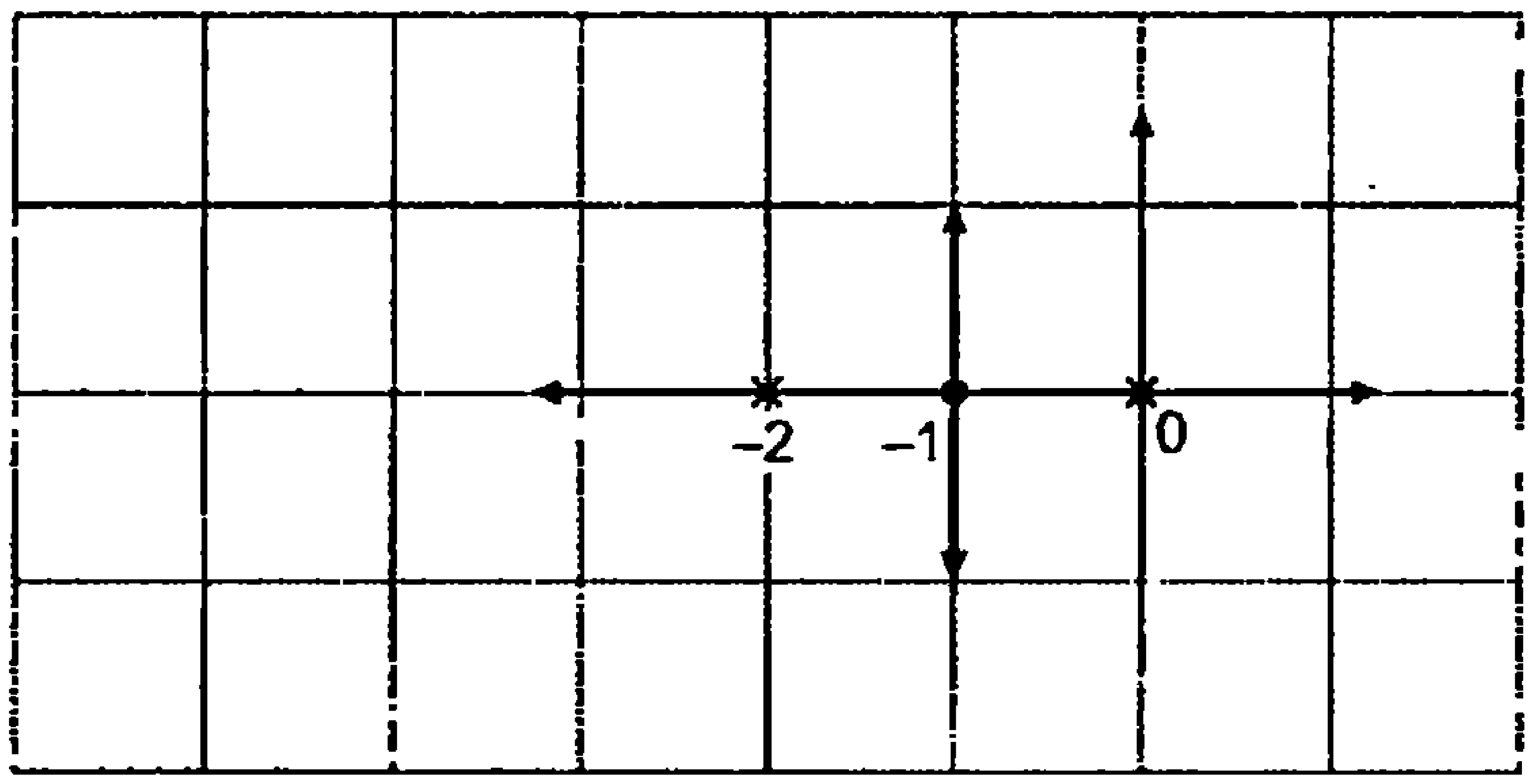


Fig. 9.23

Root loci for $G(s)H(s) = \frac{K(s+4)}{s(s+2)}$ is shown in the Fig. 9.24.

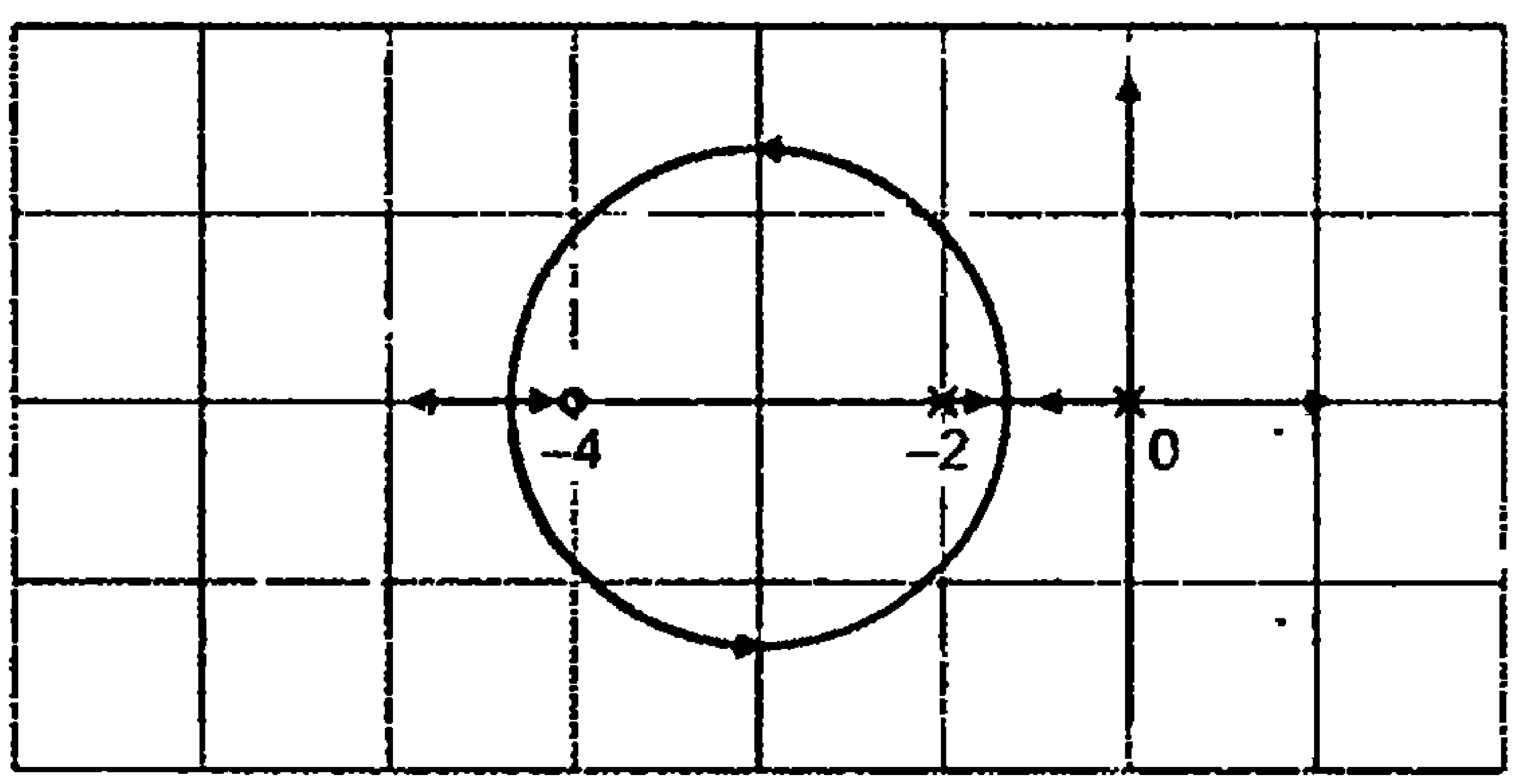


Fig. 9.24

It can be seen that root locus shift towards left i.e. towards zero which is added. So as roots move towards left half of s-plane relative stability increases as compared to previous case. Let us introduce one more zero at s = – 6.

$$G(s)H(s) = \frac{K(s+4)(s+6)}{s(s+2)}$$

In this case breakaway points shifts towards left half of s-plane as shown. So relative stability of system gets further increased.

As we go on adding zeros to left, resultant root loci bend towards left half of s-plane.

As roots move towards left half of s-plane, relative stability of system

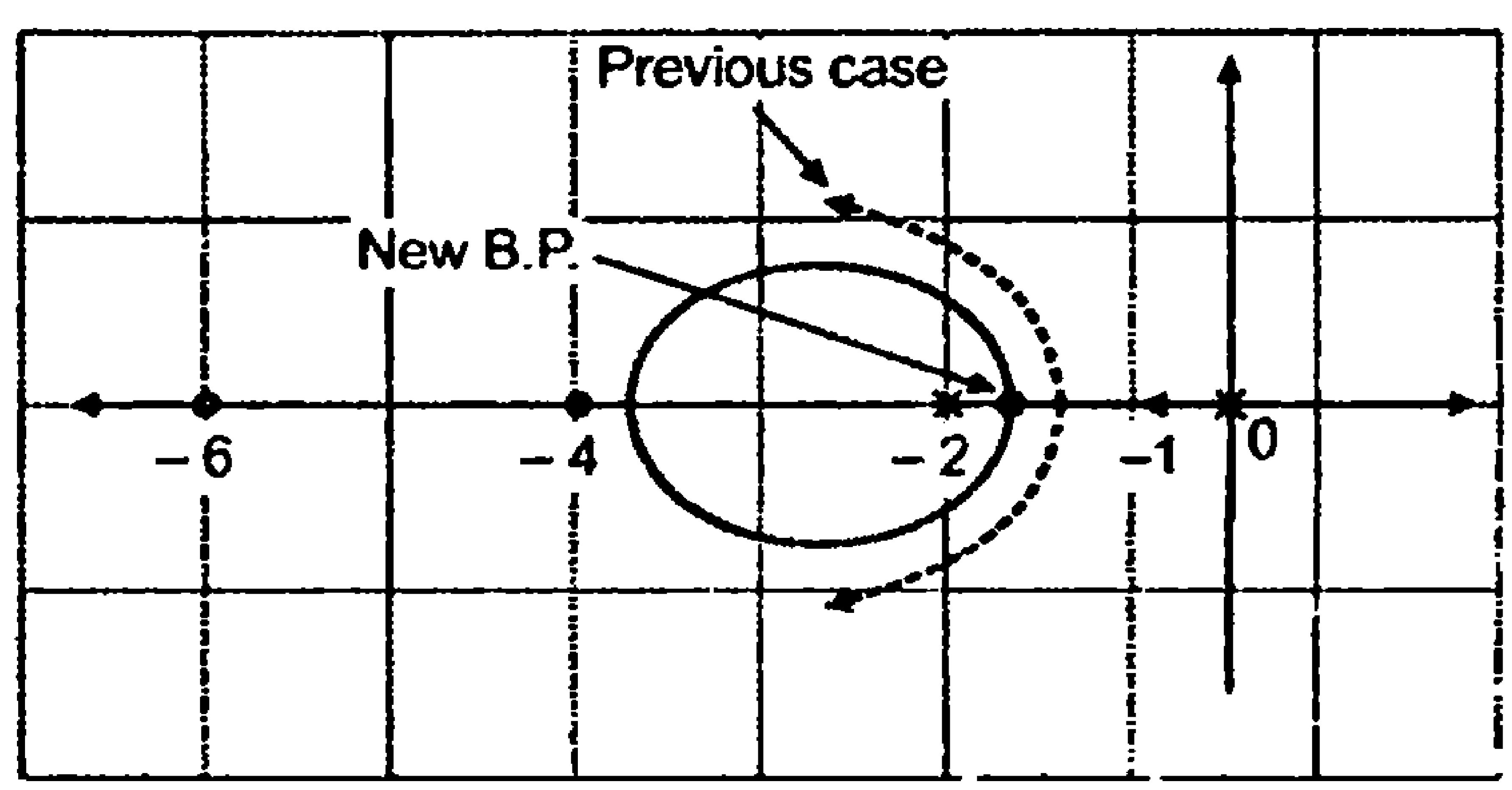


Fig. 9.25

improves. Due to addition of zeros towards left half of s-plane system stability increases. Also it increases the range of operating values of `K' for system stability.

In short effect of addition of zeros are :

1) Root locus shifts to left away from imaginary axis.
2) Relative stability of the system increases.
3) System becomes less oscillatory.
4) Range of operating values of `K' for system stability increases.

9.10 Advantages of Root Locus Method

The root locus technique is more advantageous as it gives us following information :

i) The absolute stability of the system can be predicted from the locations of the roots in the s-plane.

ii) Limiting range of the values of the system gain `K' can be decided for absolute stability of the system.

iii) Marginal value of the system gain `K' which makes the system marginally stable can be determined and the corresponding value of the frequency of oscillations can be determined from intersection of root locus with imaginary axis.

iv) Using root locus, value of system gain `K' for any point on the root locus can be determined, by using magnitude condition.

v) For particular damping ratio of the system, gain `K' can be determined which helps to design system more correctly.

vi) Root locus analysis also helps in deciding the stability of the control systems with time-delay.

vii) Gain margin of the system can be determined from root locus.

viii) Phase margin of the system can be determined from root locus.

ix) Relative stability about a particular value of '$s = -\sigma$' can be determined.

x) Information about settling time of the system also can be determined from the root locus.

9.11 Obtaining G(s)H(s) from Characteristic Equation

Sometimes the characteristic equation of the system is known from which, it is required to sketch the root locus. But for it, it is necessary to get open loop transfer function G(s)H(s) to start the sketching of root locus.

Now the characteristic equation is $1 + G(s)H(s) = 0$. But there is a specific way to rewrite given characteristic equation in such a form which can be compared with
$1 + G(s)H(s) = 0$ to get the open loop transfer function G(s)H(s).

Now the gain 'K' which is to be varied from $0 \rightarrow \infty$ is always in the numerator of G(s)H(s) and hence the way in which the characteristic equation is to be adjusted is :

i) Collect the terms of s without K together.

ii) Collect the terms of K together.

iii) Divide the entire equation by polynomial containing the terms of s without K.

This gives the form of equation as 1 + G(s)H(s) = 0.

For example, if the characteristic equation is given as,

$$s^3 + 7s^2 + 12s + Ks + 10K = 0$$

Then rewrite the equation as,

$$(s^3 + 7s^2 + 12s) + K(s + 10) = 0$$

Then divide entire equation by polynomial in s without K i.e.

$$1 + \frac{K(s+10)}{s^3 + 7s^2 + 12s} = 0$$

$$\text{i.e.}\ 1 + \frac{K(s+10)}{s(s+3)(s+4)} = 0$$

Comparing this with 1 + G(s)H(s) = 0 we get,

$$G(s)H(s) = \frac{K(s+10)}{s(s+3)(s+4)}$$

From this root locus can be obtained.

9.12 Cancellation of Poles of G(s) with Zeros of H(s)

It is important to note that if the numerator of H(s) and denominator of G(s) contain some common factors then the corresponding factor cancellation takes place and order of characteristic equation gets reduced by one or more.

Consider, $$G(s) = \frac{K}{s(s+2)(s+4)}$$

$$H(s) = (s + 2)$$

then $$G(s)H(s) = \frac{K}{s(s+2)(s+4)} \cdot (s + 2) = \frac{K}{s(s+4)}$$

$\therefore$ Characteristic equation 1 + G(s)H(s)

$$= 1 + \frac{K}{s(s+4)} = s^2 + 4s + K = 0$$

But actually $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$= \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}(s+2)} = \frac{K}{s(s+2)(s+4)+K(s+2)}$$

$\therefore$ Characteristic equation is, $(s+2)[s(s+4)+K] = 0$

So due to cancellation, order of characteristic equation reduces by one and root locus of such equation cannot provide appropriate information about variations in actual closed loop poles.

Hence to obtain complete set of closed loop poles the cancelled pole must be added to G(s)H(s) to sketch the root locus. So in above example, poles must be considered to be s = 0, − 2 , and − 4 and not only 0 and − 4.

Key Point: ***Cancelled pole of G(s)H(s) is always closed loop pole of the system.***

9.12.1 Gain Margin

Gain margin is the margin in gain K allowed to be increased till system reaches on the verge of instability. From the root locus, gain margin can be obtained as,

$$G.M = \frac{\text{Value of K at intersection of root locus with imaginary axis}}{\text{Designed value of K}}$$

If root locus is not intersecting with imaginary axis, then G.M. is ∞. The concept of gain margin will be more clear in frequency domain analysis of control systems.

9.13 Root Sensitivity

The root sensitivity is defined from the equation $\frac{dK}{ds} = 0$ which gives breakaway points. It is defined as,

$$S_K = \frac{ds/s}{dK/K} = \frac{K}{s}\frac{ds}{dK}$$

The equation shows that at the breakaway points, root sensitivity is infinite. Thus system becomes very much sensitive to the small changes in K. The system must be insensitive to the parameter variations. Hence from root sensitivity point of view, the system should not be operated with K at the breakaway points.

9.14 Root Contour

The root locus technique for the study of stability of closed loop system from open loop transfer function can be extended by varying parameter other than K in system from 0 to ∞.

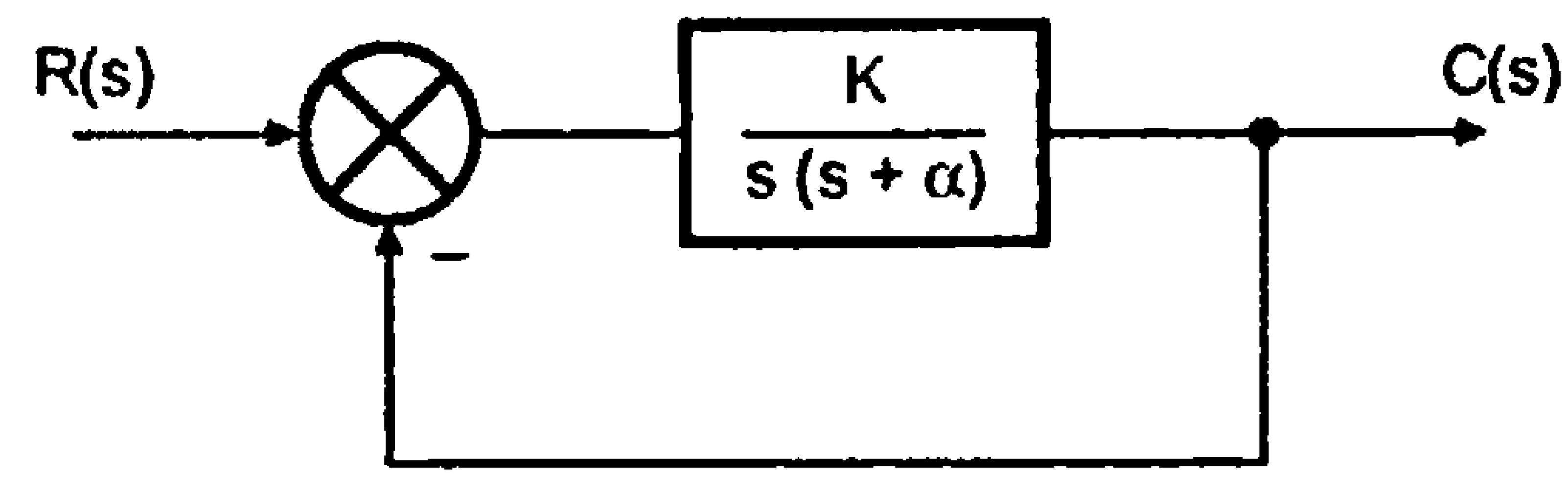


Fig. 9.26

Consider a system shown in the Fig. 9.26.

The open loop transfer function is $G(s)H(s) = \frac{K}{s(s+\alpha)}$

Key Point: *The locus of roots of characteristic equation obtained by varying parameter of the system other than K from 0 to ∞ e.g. like α is called* ***root contour*** *of the system.*

The parameters like α, K are to be varied simultaneously, while sketching the root contours.

9.15 Inverse Root Locus

When gain 'K' is varied from − ∞ to 0 i.e. in the negative region then root locus obtained is called as Inverse root locus.

For this characteristic equation becomes

$$1 - K\,G'(s)H(s) = 0$$

i.e. $$1 - G(s)H(s) = 0$$

i.e. $$G(s)H(s) = +1$$

So angle and magnitudes conditions will get modified as,

$\lvert G(s)H(s) \rvert = 1$	Magnitude condition	
$\angle G(s)H(s) = \pm q(360^\circ)$	where	$q = 0, 1, 2, \ldots$

Thus root locus follows 0° locus as compared to the 180° locus considered for direct root locus.

Due to this there are few modifications in the rules.

Rule 3 : If the total number of real poles and zeros to the right hand side of a point on the real axis is **even**, then that point lies on the root locus.

Rule 4 : Angles of asymptotes

$$\theta = \frac{q\,360^\circ}{P-Z}, \quad q = 0, 1, 2 \ldots\ldots (P-Z-1)$$

In Rule 5 and 6, if K is negative for the breakaway points and intersection points with imaginary axis, they are valid for inverse root locus.

Rule 7 : Angle of departure at complex pole $\phi_d = 0° - \phi$

where $\phi = \Sigma\phi_P - \Sigma\phi_Z$

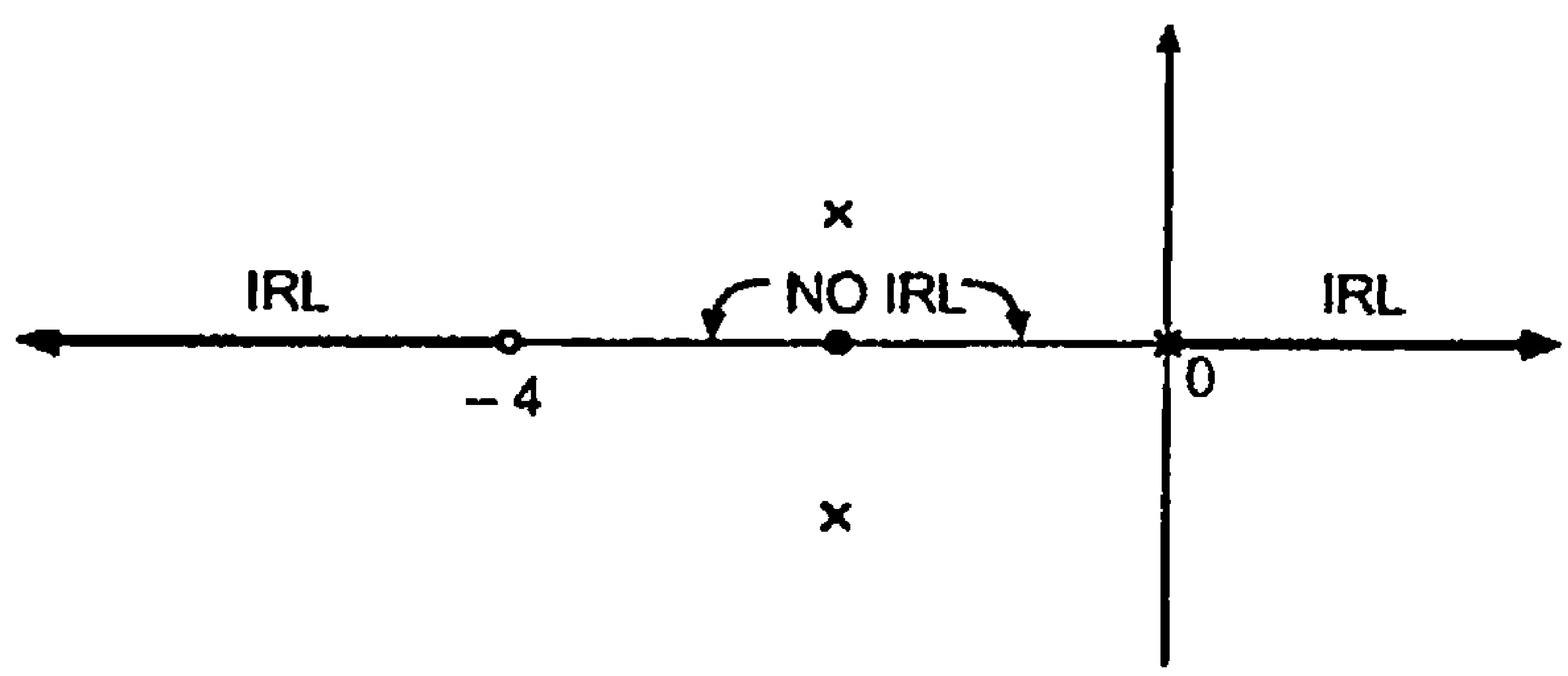
Example 9.16 : *Sketch the inverse root locus for system having*

$$G(s)H(s) = \frac{K(s+4)}{s(s^2+2s+2)}$$

Solution : Step 1 : P = 3, Z = 1, N = 3.

Out of 3 branches 1 will terminate at finite zero at s = – 4 while remaining P– Z = 2 branches will approach to ∞ starting points are s = 0, –1 ± j

Step 2 : Pole-Zero plot.



As the rule is modified for inverse the part of real axis which is not on direct root locus becomes part of inverse root locus. One breakaway point to the left of s = – 4 is now possible, according to prediction no3.

Step 3 : Angles of asymptotes

$$\theta = \frac{q\,360°}{P-Z}, \quad q = 0, 1.2 \text{ asymptotes required.}$$

$$\therefore \quad \theta_1 = 0°, \quad \theta_2 = \frac{360°}{2} = 180°$$

Step 4 : Centroid

$$\sigma = \frac{\sum R.P. \text{ of poles} - \sum R.P. \text{ of zeros}}{P-Z}$$

$$= \frac{0-1-1-(-4)}{2} = +1$$

But as angles of asymptotes are 0° and 180°, centroid is not required.

Step 5 : Breakaway point :

Characteristic equation

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+4)}{s(s^2+2s+2)} = 0$$

$$\therefore \quad s^3 + 2s^2 + 2s + K(s+4) = 0$$

$$\therefore \quad K = \frac{-s^3 - 2s^2 - 2s}{(s+4)} = \frac{u}{v}$$

$$\therefore \quad \frac{dK}{ds} = \frac{vu' - uv'}{v^2} = 0$$

$$\therefore \quad (s+4)(-3s^2 - 4s - 2) - (-s^3 - 2s^2 - 2s)(1) = 0$$

$$\therefore \quad -3s^3 - 16s^2 - 18s - 8 + s^3 + 2s^2 + 2s = 0$$

$$\therefore \quad -2s^3 - 14s^2 - 16s - 8 = 0$$

i.e.

$$s^3 + 7s^2 + 8s + 4 = 0$$

Approximate breakaway points are ,

$$s = -5.735 \text{ and } -0.632 \pm j\,0.587$$

Valid breakaway point is $s = -5.735$

Step 6 : There is no intersection of inverse root locus with imaginary axis.

Characteristic equation : $s^3 + 2s^2 + s(K+2) + 4K = 0$

Routh's array,

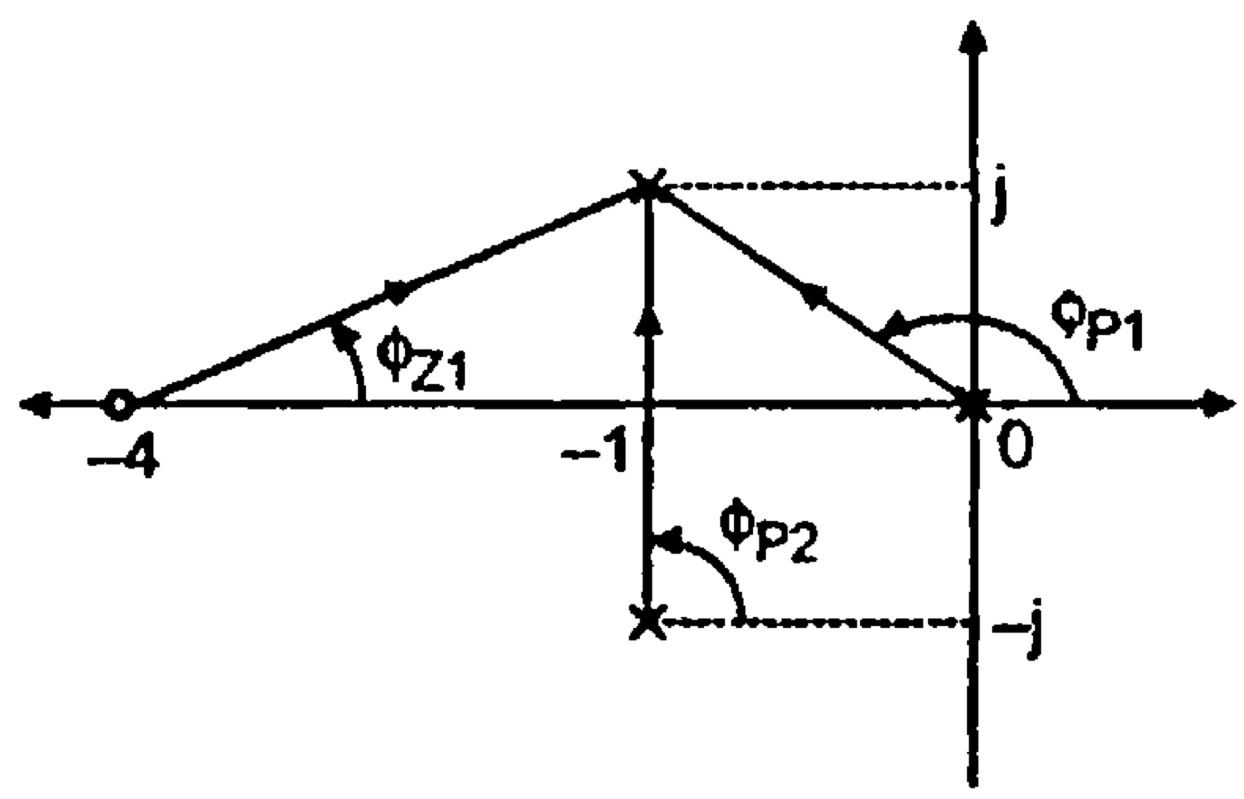
s^3	1	K + 2
s^2	2	4 K
s^1	$\frac{4-2K}{2}$	0
s^0	4 K	

$$4 - 2K = 0$$

$$\therefore \quad K_{mar} = +2$$

As K_{mar} is positive and we are plotting for negative values of K, for positive K.

Step 7 : Angle of departure



Consider –1 + j

$$\phi_{P1} = +135^\circ$$

$$\phi_{P2} = 90^\circ$$

$$\phi_{Z1} = \tan^{-1}\frac{1}{2} = 18.43^\circ = \sum\phi_Z$$

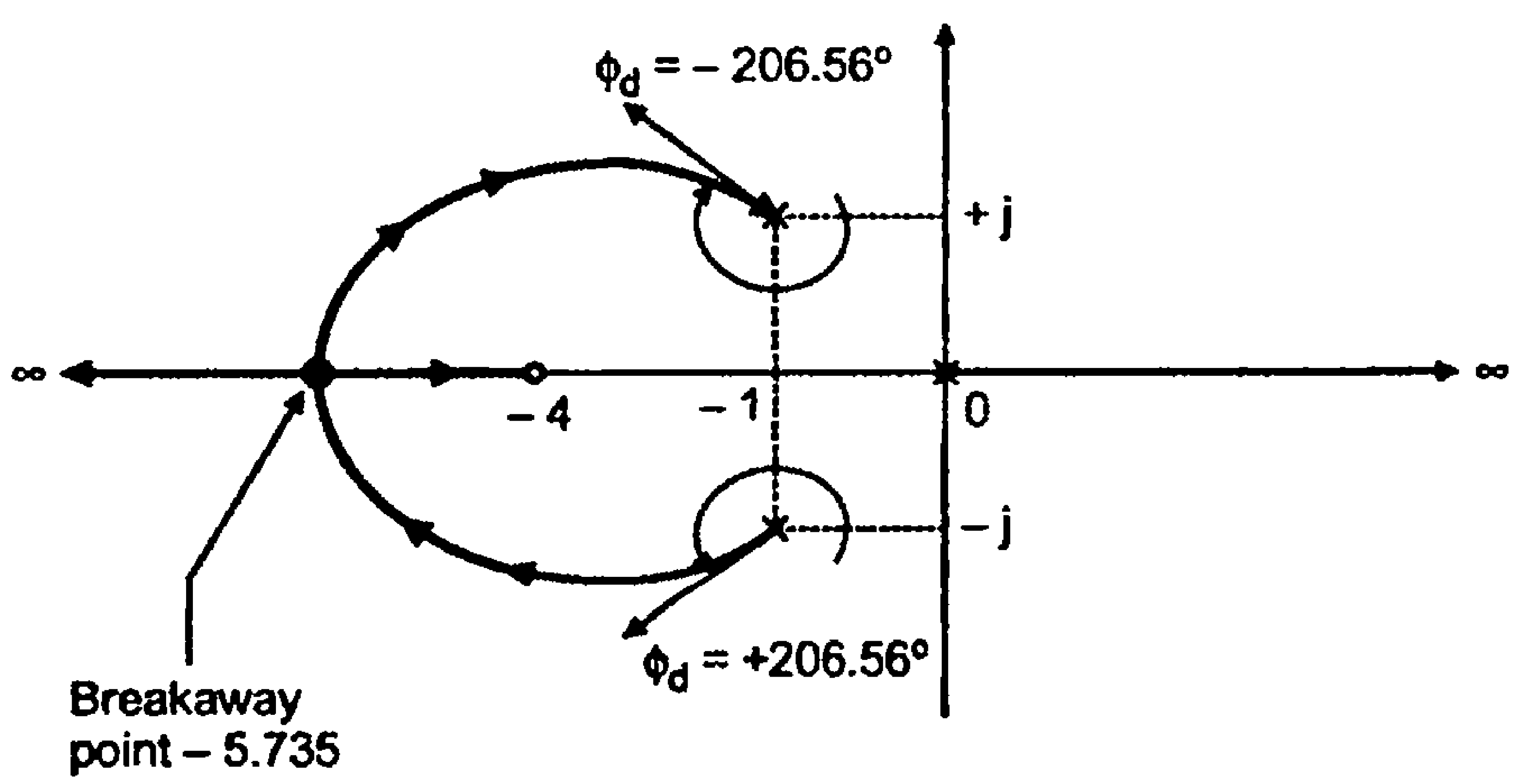
$$\therefore \quad \sum\phi_P = 225^\circ$$

$$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 206.56^\circ$$

$$\therefore \quad \phi_d = 0^\circ - \phi = -206.56^\circ \quad \text{at } -1+j$$

$$\phi_d = +206.56^\circ \quad \text{at } -1-j$$

Step 8 : Complete inverse root locus is shown as below.



9.16 System with Positive Feedback (K is Positive)

Characteristic equation :

$$1 - G(s)H(s) = 0$$

$$\therefore \quad G(s)H(s) = 1$$

∴ When K is positive, $0 < K < \infty$ then to sketch the root locus for such positive feedback system, rules must be modified as explained in earlier section. Root locus follows 0° locus as against 180° for negative feedback system.

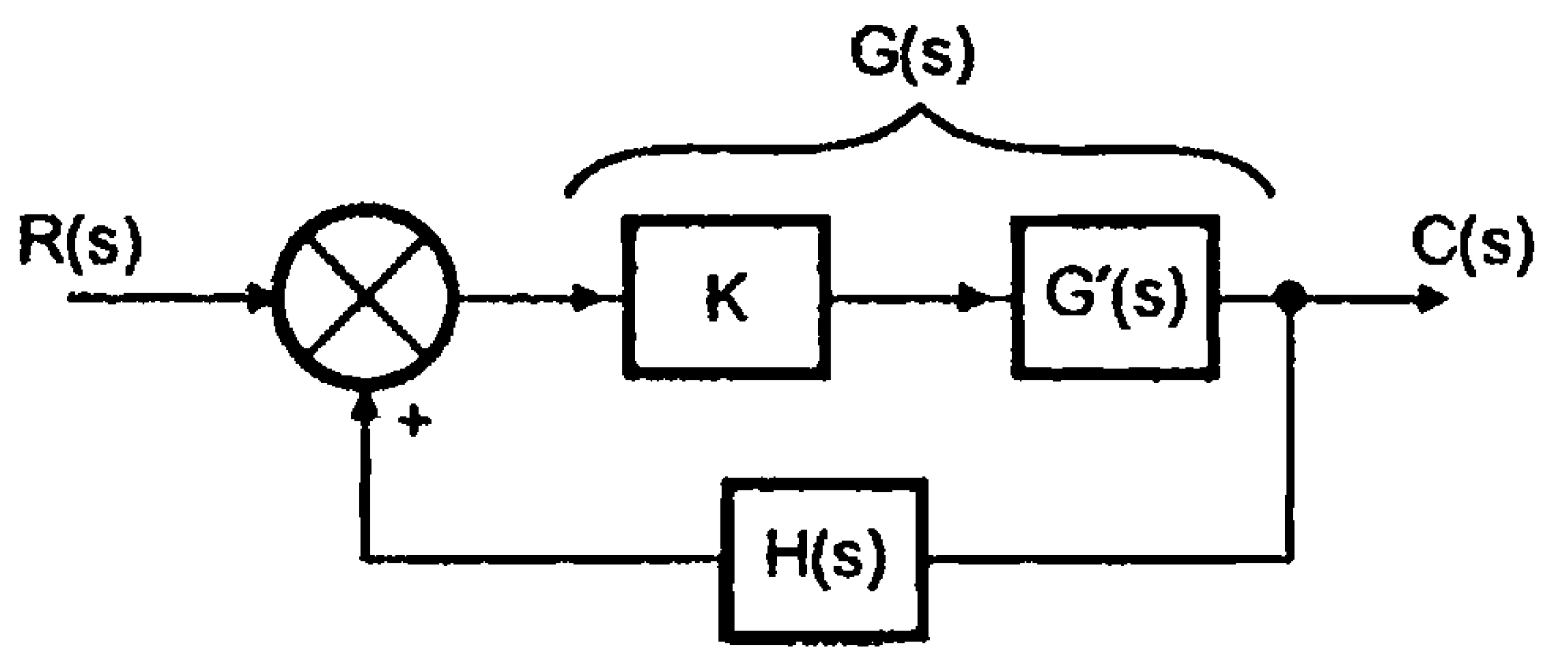
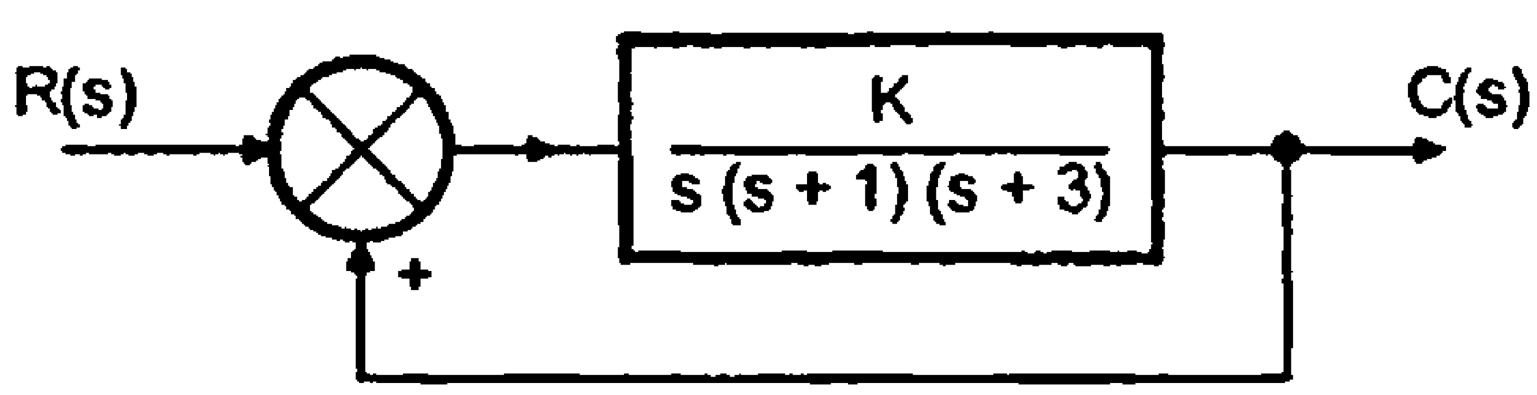


Fig. 9.27

So in general :

		Sign of feedback	
		Positive	**Negative**
Range of K	$0 < K < \infty$ positive	Modified rules (0°)	Original rules (180°)
	$-\infty < K < 0$ negative	Original rules (180°)	Modified rules (0°)

Example 9.17 : *Sketch the root locus for positive feedback system shown below.*



Solution : Unless and until specified always assume variation in 'K' in positive region. i.e. $0 < K < +\infty$

$$G(s)H(s) = \frac{K}{s\,(s+1)\,(s+3)}$$

We have to use modified rules as K is positive and feedback is also positive.

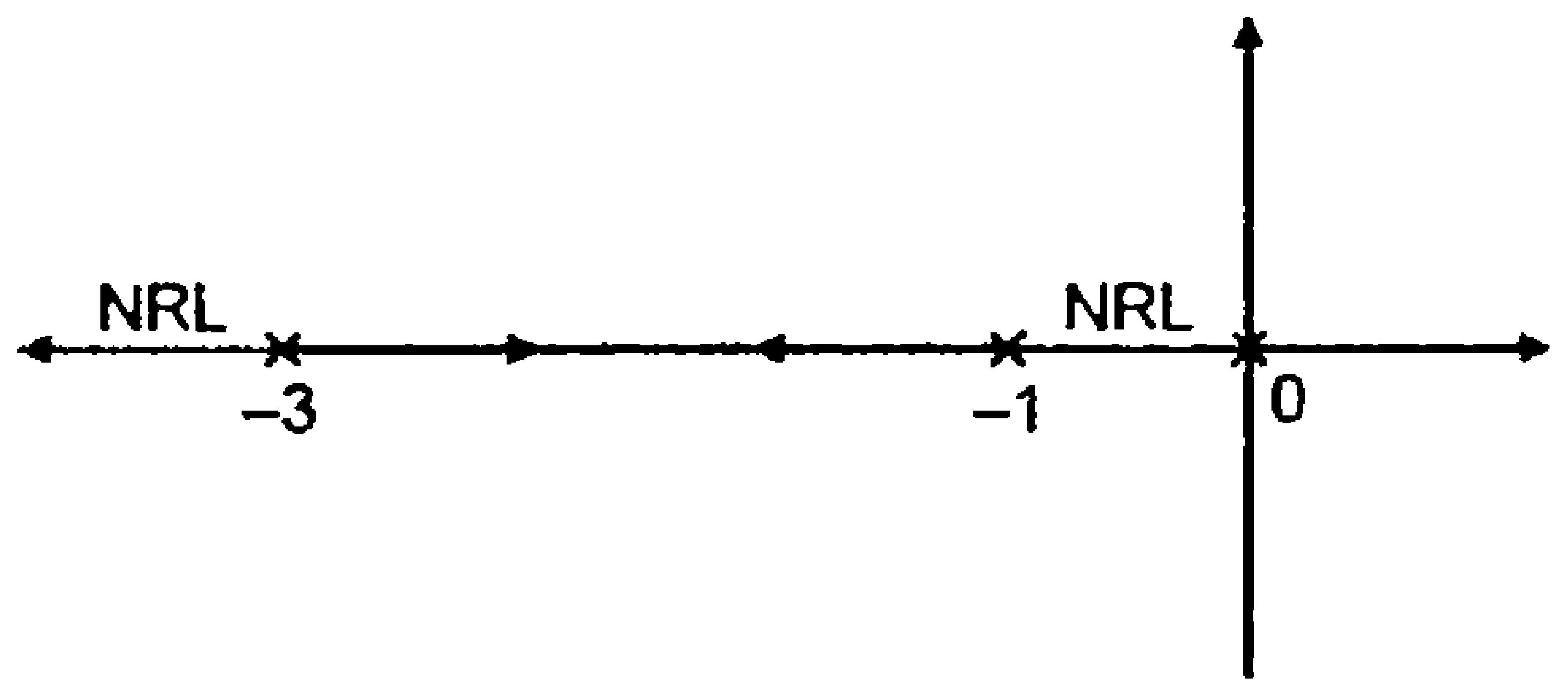
Characteristic equation : $1 - G(s)\,H(s) = 0$

Step 1 : P = 3, Z = 0, N = 3. All branch approaching to ∞. Starting points are $s = 0, -1, -3$.

Step 2 : Pole-Zero plot

Sections of real axis are identified by modified rule as discussed in section 9.15.

One breakaway point exists between s = – 1 and s = – 3

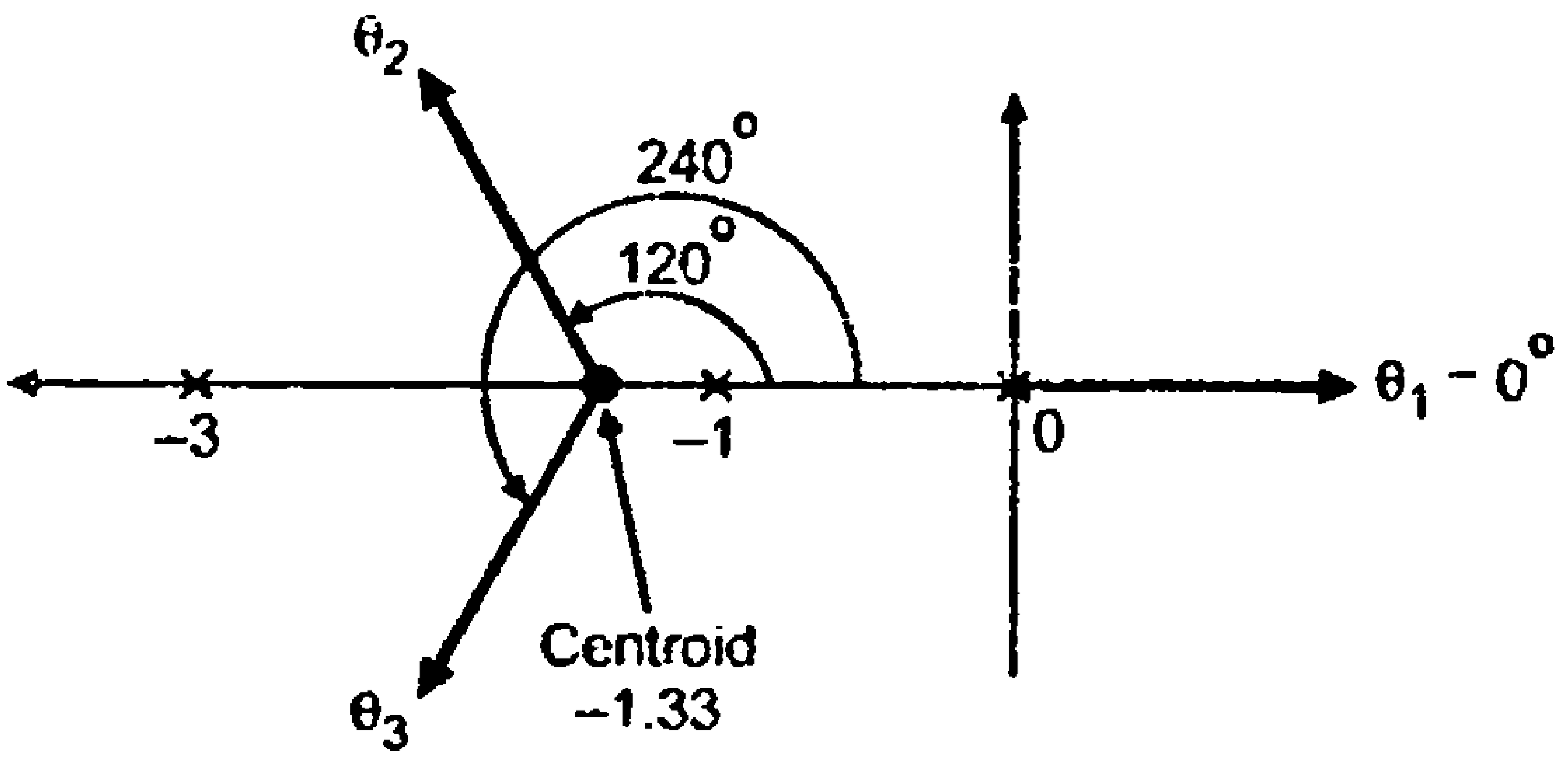


Step 3 : Angles of asymptotes

$$\theta = \frac{q\,360^\circ}{P-Z}, \quad q = 0, 1, 2. \quad \text{3 asymptotes are required}$$

$$\theta_1 = 0^\circ, \ \theta_2 = \frac{1\times 360^\circ}{3} = 120^\circ, \theta_3 = \frac{2\times 360^\circ}{3} = 240^\circ$$

Step 4 : Centroid



$$\sigma = \frac{\sum R.P.\text{ of poles} - \sum R.P.\text{ of zeros}}{P-Z}$$

$$= \frac{0-1-3}{3} = -1.33.$$

Step 5 : Breakaway point

Characteristic equation

$$1 - G(s)H(s) = 0$$

$$1 - \frac{K}{s(s+1)(s+3)} = 0$$

$$\therefore s^3 + 4s^2 + 3s - K = 0 \quad \therefore K = s^3 + 4s^2 + 3s$$

$$\frac{dK}{ds} = 0 = 3s^2 + 8s + 3$$

∴ Breakaway points $s = -0.45, -2.21$

Between s = 0 and − 1 there is no Root Locus so s = − 0.45 is not valid.

At s = − 2.21, K = + 2.11 ∴ it is valid breakaway point.

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^3 + 4s^2 + 3s - K = 0$

Routh's array :

s^3	1	3
s^2	−4	−K
s^1	$\frac{12+K}{4}$	0
s^0	−K	

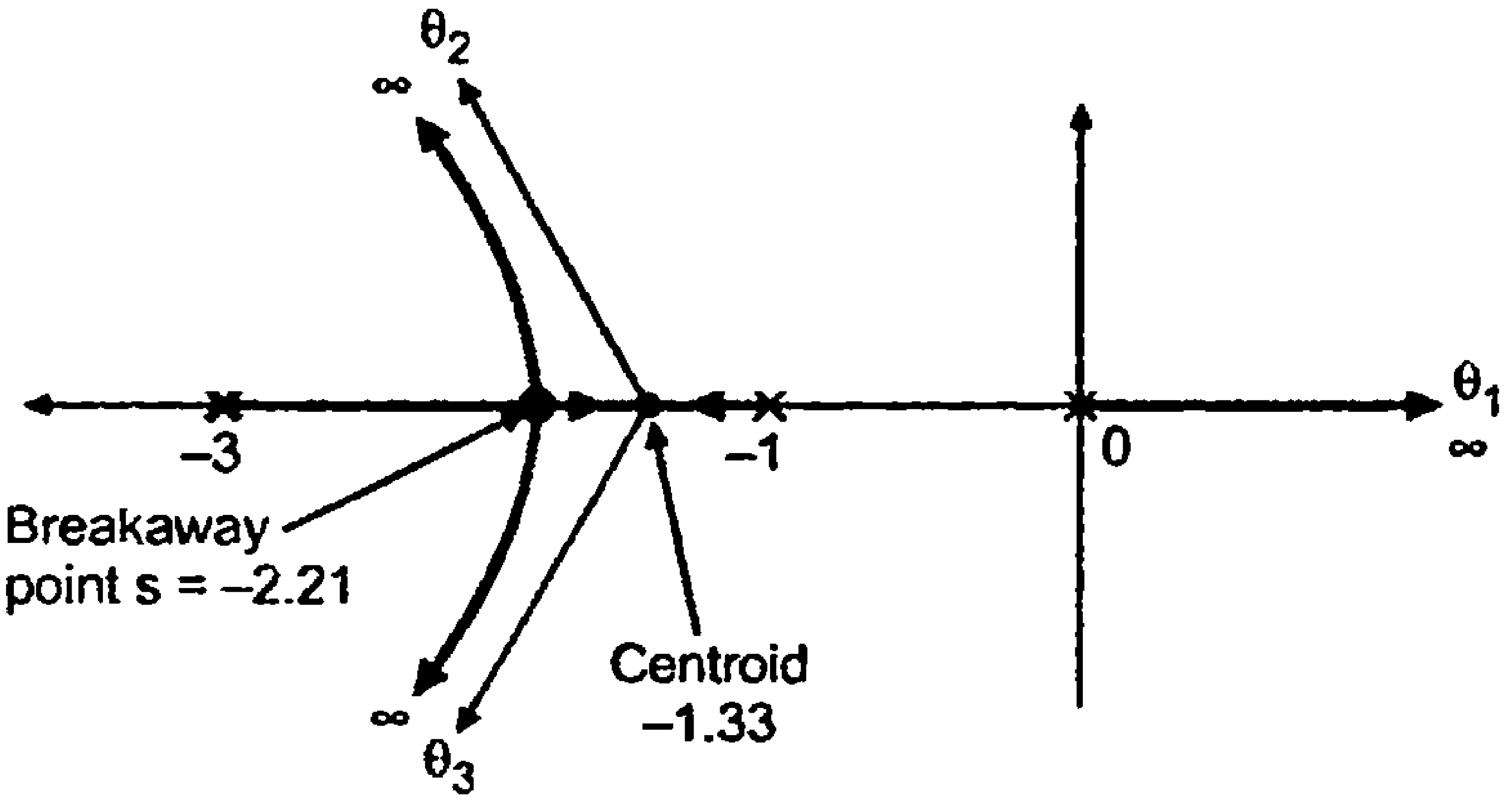
∴ $12 + K = 0$

∴ $K_{mar} = -12$

As it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : No complex poles, this step of angle of departure is not required.

Step 8 : Complete root locus



Step 9 : For any K > 0 , one root is always in right half of s-plane and hence system is always unstable.

Note : The compensation techniques are covered separately in the chapter 14.

Examples with Solutions

Example 9.18 : *Sketch the complete root locus for the system having* $G(s)H(s) = \frac{K(s+5)}{(s^2+4s+20)}$. **(M.U. : Dec.-2007)**

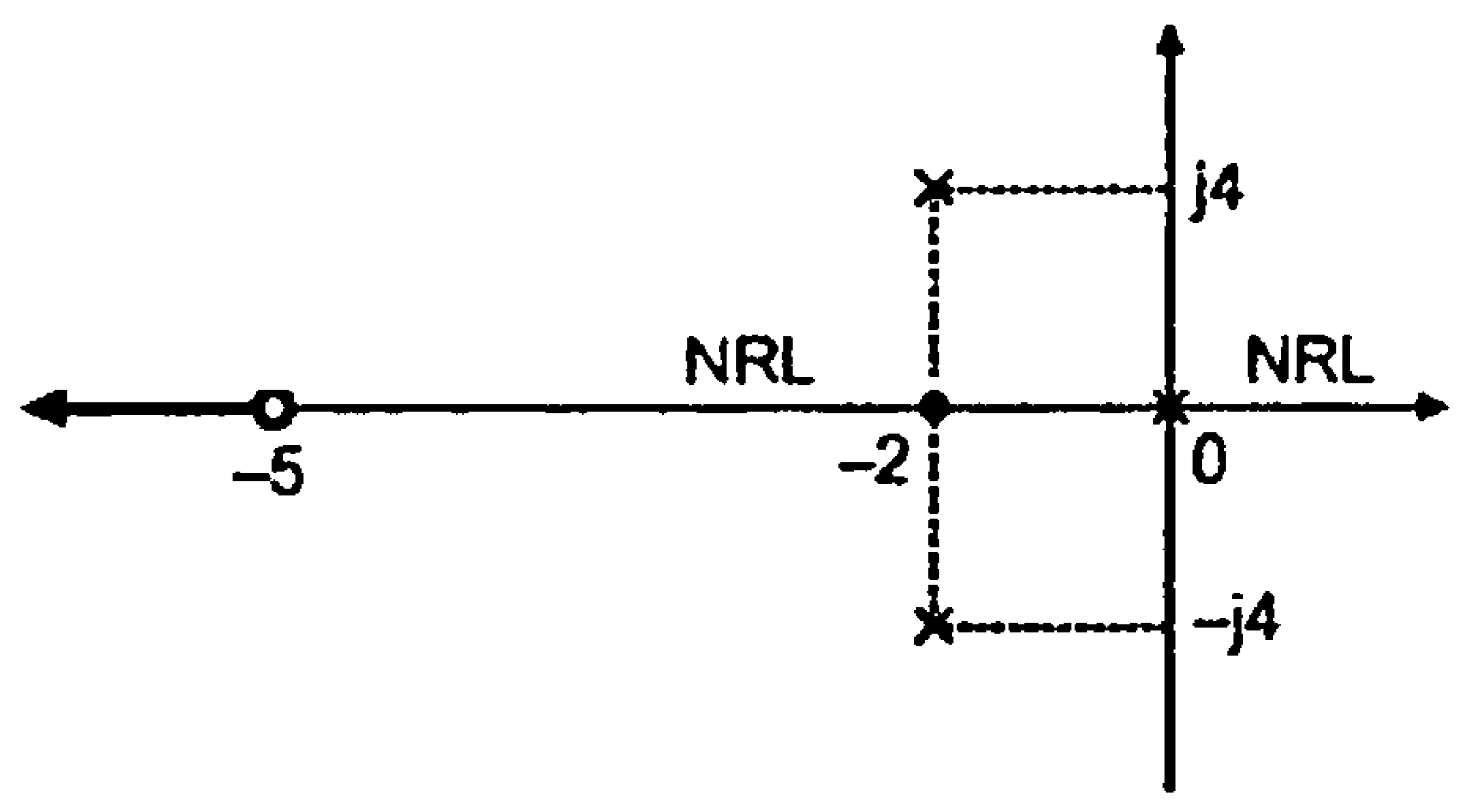
Solution : Step 1 : Number of poles P = 2, Z = 1 , N = P = Z

One branch has to terminate at finite zero s = − 5 while P − Z = 1 branch has to terminate at ∞. Starting points of branches are,

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j4$$

Step 2 : Pole - Zero plot and sections of real axis are as in following figure.

Section to the left of s = – 5 is a part of the root locus. s = – 5 is a zero on real axis to the left of which there is no pole or zero and to the left of it, complete root locus exists hence atleast one breakaway point exists to the left of s = – 5.



Step 3 : Angles of asymptotes

One branch approaches to ∞ so one asymptote is required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \quad , q = 0$$

$$\therefore \quad \theta_1 = 180^\circ$$

Branch approaches to ∞ along + 180° i.e. negative real axis.

Step 4 : Centroid

As there is only one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway point

Characteristic equation : 1 + G(s)H(s) = 0

$$1 + \frac{K(s+5)}{(s^2+4s+20)} = 0$$

$$\therefore \quad s^2+4s+20+Ks+5K = 0$$

$$\therefore \quad s^2+4s+20+K(s+5) = 0$$

$$\therefore \quad K = \frac{-s^2-4s-20}{(s+5)} \quad \text{... (1)}$$

Now $$\frac{dK}{ds} = \frac{vu'-uv'}{v^2} = 0$$

$$= (s+5)(-2s-4) - (-s^2-4s-20)(1) = 0$$

$$= -2s^2-14s-20+s^2+4s+20 = 0$$

i.e. $$-s^2 - 10s = 0$$

$$\therefore \quad -s(s+10) = 0$$

s = 0 and s = − 10 are breakaway points. But s = 0 cannot be breakaway point as for s = 0, K = − 4.

For s = − 10, $$K = \frac{-100+40-20}{-10+5} = +16$$

Hence s = − 10 is valid breakaway point.

Step 6 : Intersection with imaginary axis.

Characteristic equation

$$s^2 + 4s + 20 + Ks + 5K = 0$$

$$s^2 + s(K+4) + (20+5K) = 0$$

Routh's array

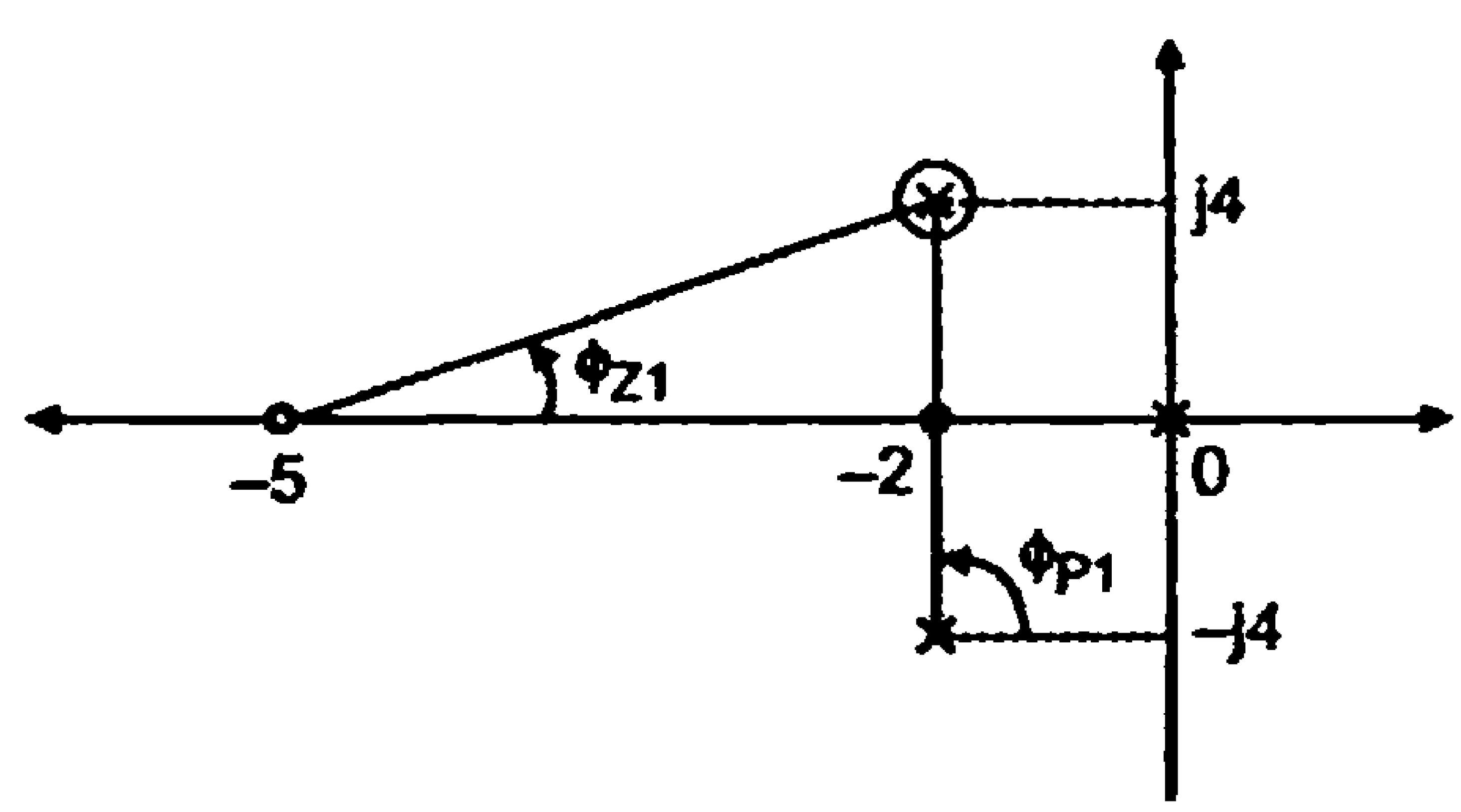
s^2	1	20 + 5 K
s^1	K + 4	0
s^0	20 + 5 K	

K_{mar} = **− 4 makes** s^1 **row as row of zeros.**

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure

Consider − 2 + j4 join remaining pole and zero to it.



$$\phi_{P1} = 90°, \quad \phi_{Z1} = \tan^{-1}\frac{4}{3} = 53.13°$$

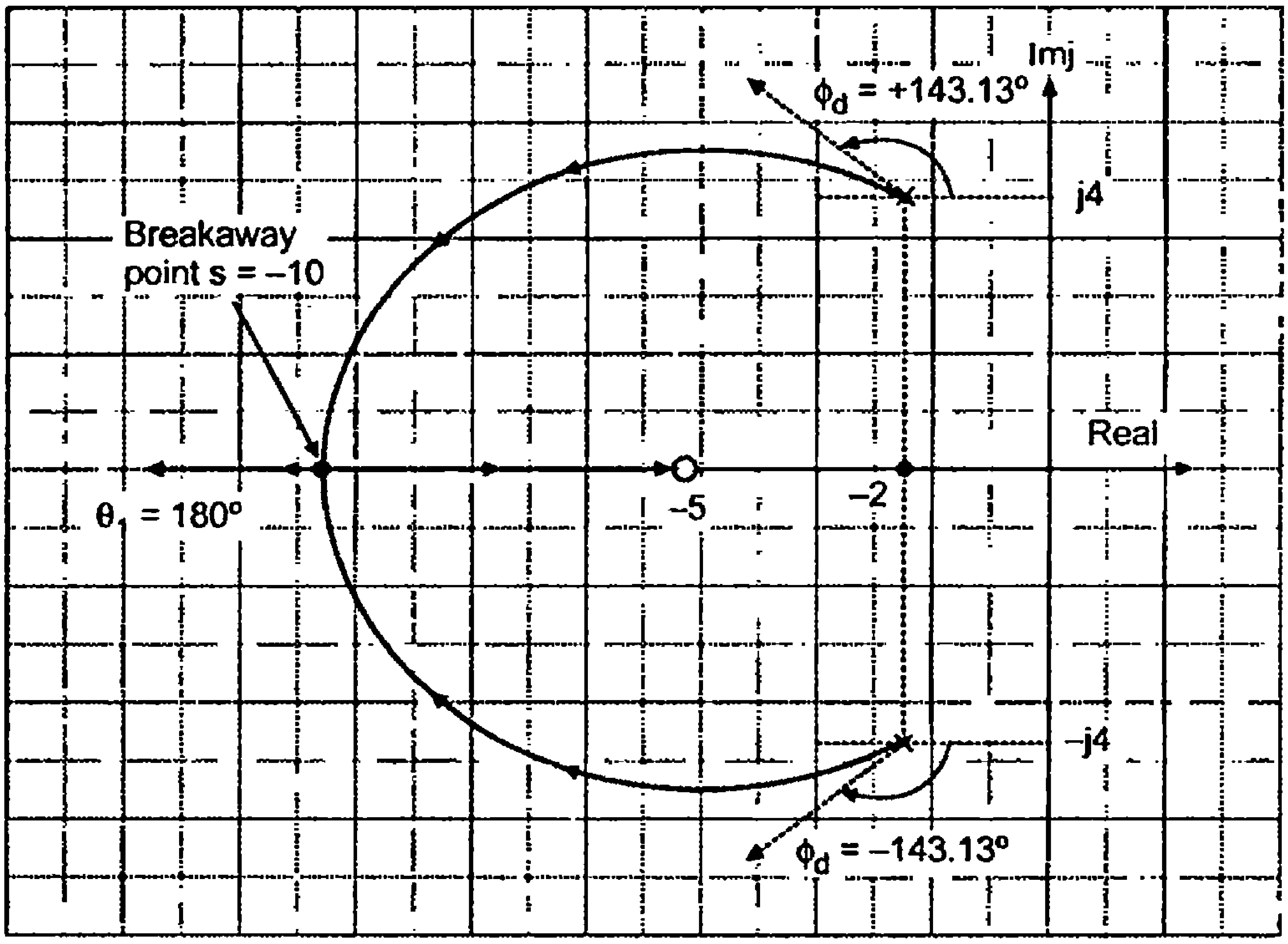
$$\Sigma\phi_P = 90°, \quad \Sigma\phi_Z = 53.13°$$

$$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 36.86°$$

$$\therefore \quad \phi_d = 180° - \phi = +143.13° \quad \text{at } -2+j4 \text{ pole}$$

$$\phi_d = -143.13° \quad \text{at } -2-j4 \text{ pole.}$$

Step 8 : Complete Root Locus is using following figure.



Step 9 : Prediction of stability

For all ranges of K i.e. $0 < K < \infty$, both the roots are always in left half of s-plane. So system is inherently stable.

Example 9.19 : *Sketch the complete root locus of system having*

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$$

Solution : Step 1 : P = 4, Z = 0, N = 4.

All branches approaching to ∞. Starting points are s = 0, – 1, – 2, – 3.

Step 2 : Pole-Zero plot and sections of real axis are as shown in Fig. 9.28.

Sections between 0 and –1 and – 2 and – 3 are the parts of root locus.

According to general predictions minimum 2 breakaway points exists. One between s = 0 and –1 and second between s = – 2 and s = – 3.

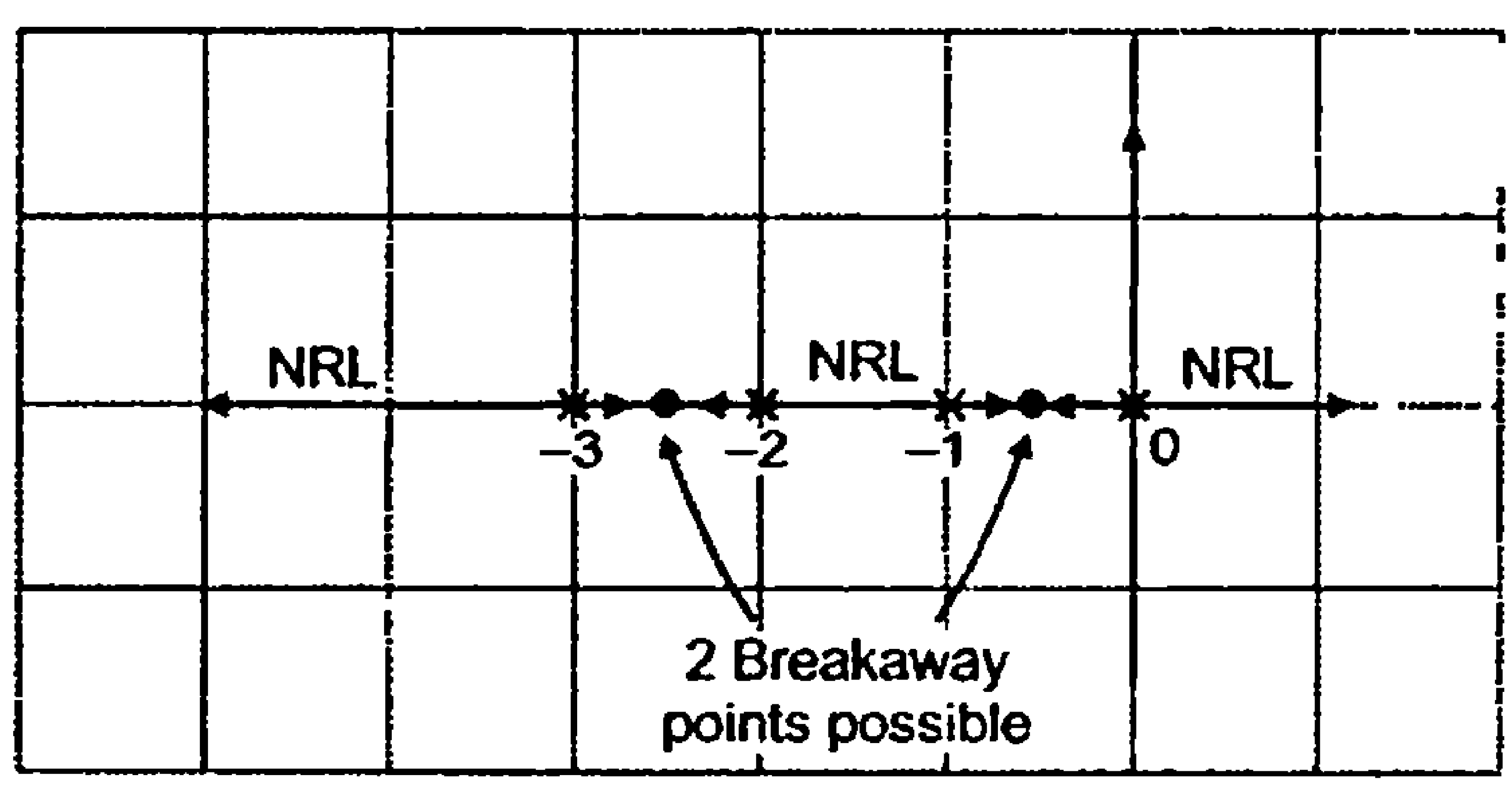


Fig. 9.28

Step 3 : Angles of asymptotes

4 branches approaching to ∞ hence 4 asymptotes are required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \quad q = 0, 1, 2, 3.$$

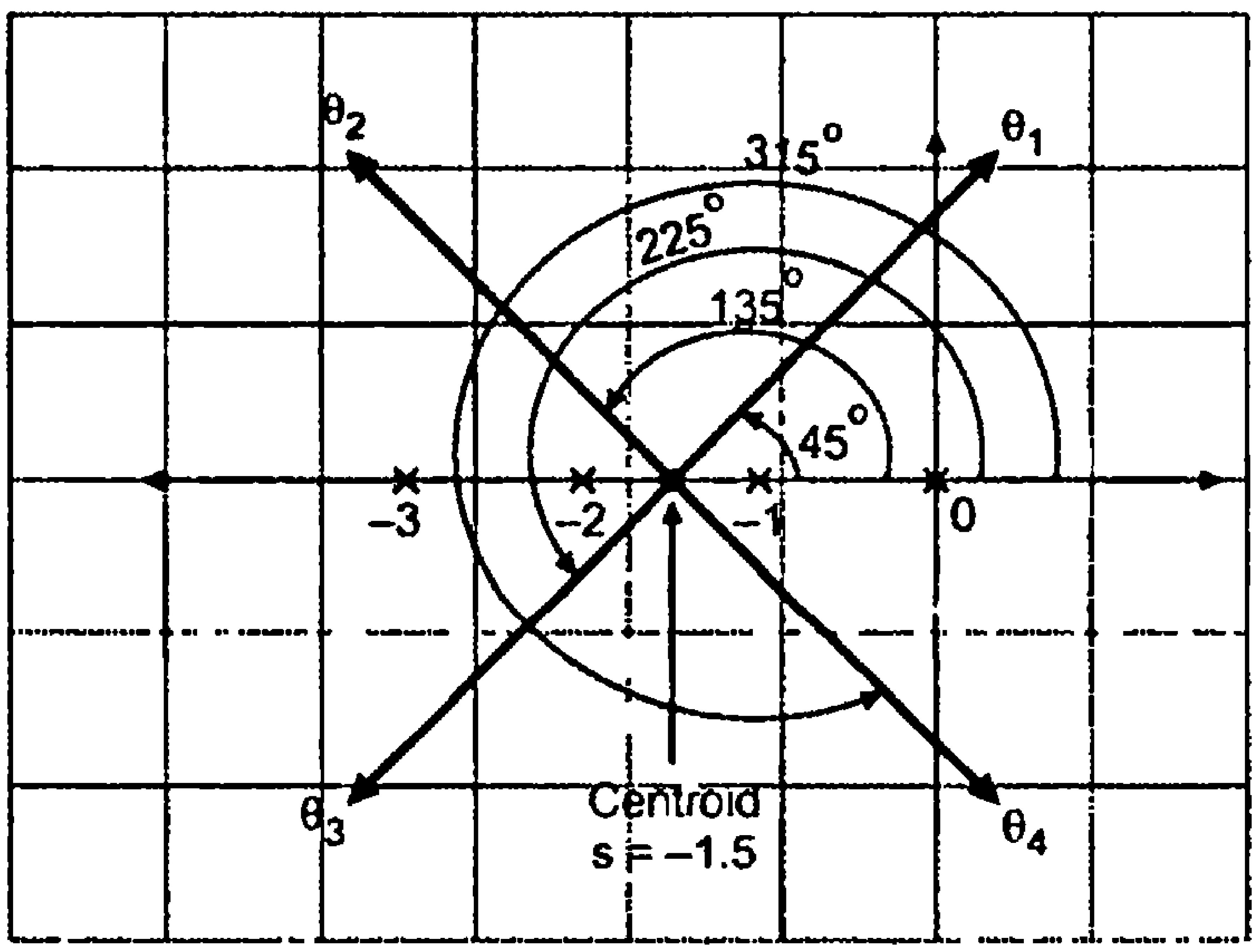
$q = 0, \quad \theta_1 = 45^\circ, \qquad q = 1, \quad \theta_2 = 135^\circ$

$q = 2, \quad \theta_3 = 225^\circ, \qquad q = 3, \quad \theta_4 = 315^\circ$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z}$$

$$= \frac{0-1-2-3}{4} = -1.5.$$



Step 5 : Breakaway point so characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+1)(s+2)(s+3)} = 0$$

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0$$

$$\therefore \quad K = -s^4 - 6s^3 - 11s^2 - 6s$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 22s - 6 = 0$$

i.e. $\quad 4s^3 + 18s^2 + 22s + 6 = 0$

Note : If a section of the real axis is identified for the existence of root locus and a breakaway point is predicted between that section. Try the midpoint of such section for a root of the equation $\frac{dK}{ds} = 0$, first.

Key Point: *If the open loop poles and zeros are distributed symmetrically about a point on the real axis then the point of symmetry is one of the roots of equation* $\frac{dK}{ds} = 0$.

In this problem poles s = 0, – 1 and s = –2, –3 are symmetrically located about a point s = –1.5. So s = –1.5 must be taken as a first trial point for taking trials to find the roots of $\frac{dK}{ds} = 0$.

Let us try s = – 1.5 as a root of above equation. By synthetic division method,

– 1.5	4	18	22	6
		– 6	– 18	– 6
	4	12	4	0

$$\therefore \quad (s+1.5)(4s^2+12s+4) = 0$$

$$\therefore \quad \text{Roots of } \frac{dK}{ds} = 0 \text{ are,}$$

$$s = -1.5 \text{ and } \frac{-12 \pm \sqrt{144-64}}{4 \times 2} \quad \text{i.e. } -0.381, -2.619$$

But as there is no root locus between s = – 1 and s = – 2.

∴ s = – 1.5 cannot be valid breakaway point.

$$\left.\begin{array}{ll} \text{For } s = -0.381, & K = 1 \\ \text{For } s = 2.619, & K = 1 \end{array}\right\} \text{Substituting in expression for K.}$$

∴ Both are valid breakaway points, occurring simultaneously at K = 1.

Step 6 : Intersection with imaginary axis

Characteristic equation $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

Routh's array,

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s^1	$\frac{60-6K}{10}$	0	
s^0	K		

Step 3 : Angles of asymptotes

Number of branches approaching ∞ are 4 so 4 asymptotes are required.

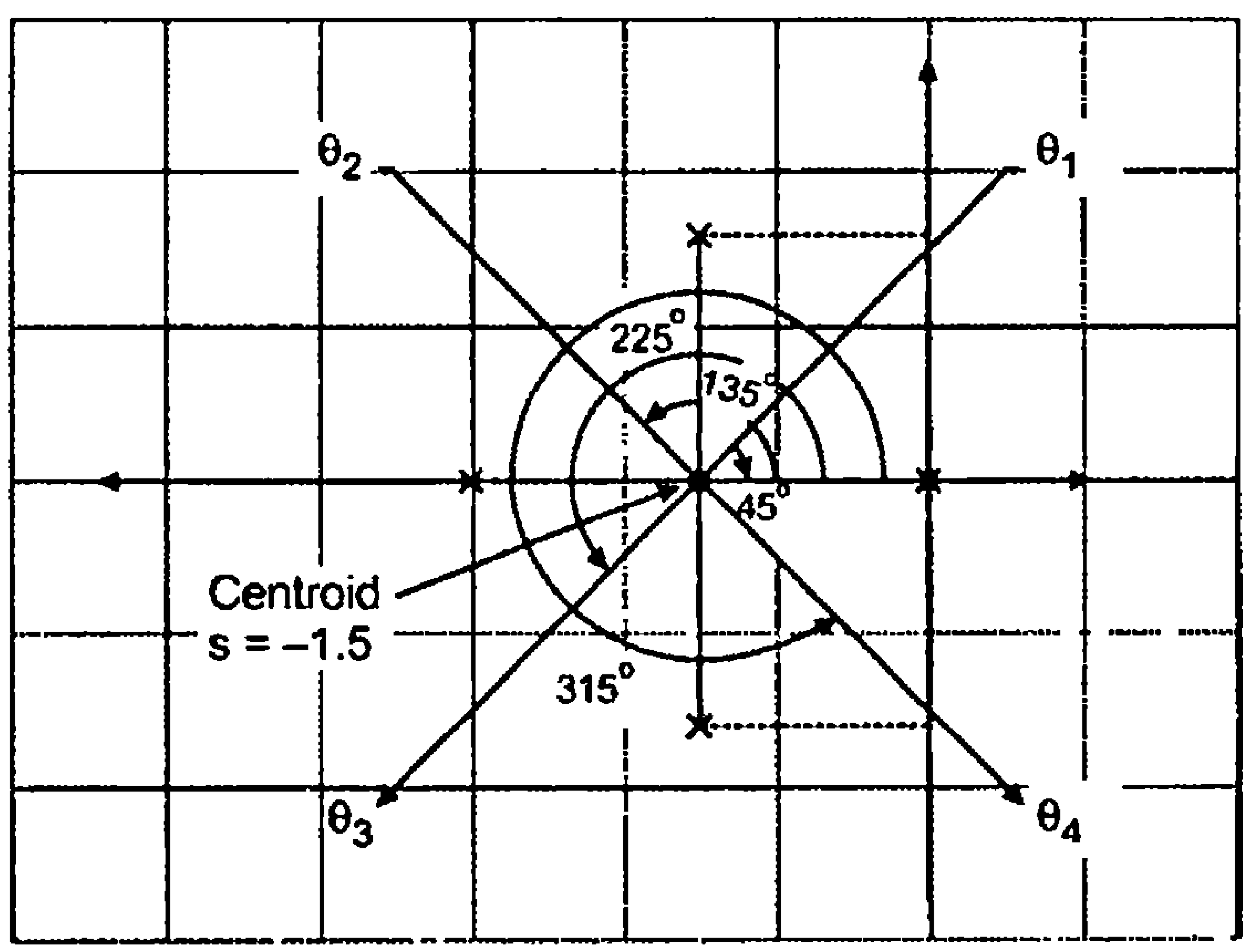
$$\theta = \frac{(2q+1)180°}{P-Z}$$

where
$q = 0,\quad \theta_1 = 45°$
$q = 1,\quad \theta_2 = 135°$
$q = 2,\quad \theta_3 = 225°$
$q = 3,\quad \theta_4 = 315°$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R. P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0-3-1.5-1.5}{4}$$

$$= -1.5$$



Step 5 : Breakaway point

Characteristic equation :

$$1+G(s)H(s) = 0$$

$$1+\frac{K}{s(s+3)(s^2+3s+4.5)} = 0$$

$$\therefore\ s^4+6s^3+13.5s^2+13.5s+K = 0$$

$$\therefore\quad K = -s^4-6s^3-13.5s^2-13.5s$$

$$\therefore\quad \frac{dK}{ds} = -4s^3-18s^2-27s-13.5 = 0$$

$$\therefore \quad 151.87 - 6\,K = 0$$

$$\therefore \quad K_{mar} = \frac{151.87}{6} = 25.3125$$

Auxiliary equation : $A(s) = 11.25s^2 + K = 0$

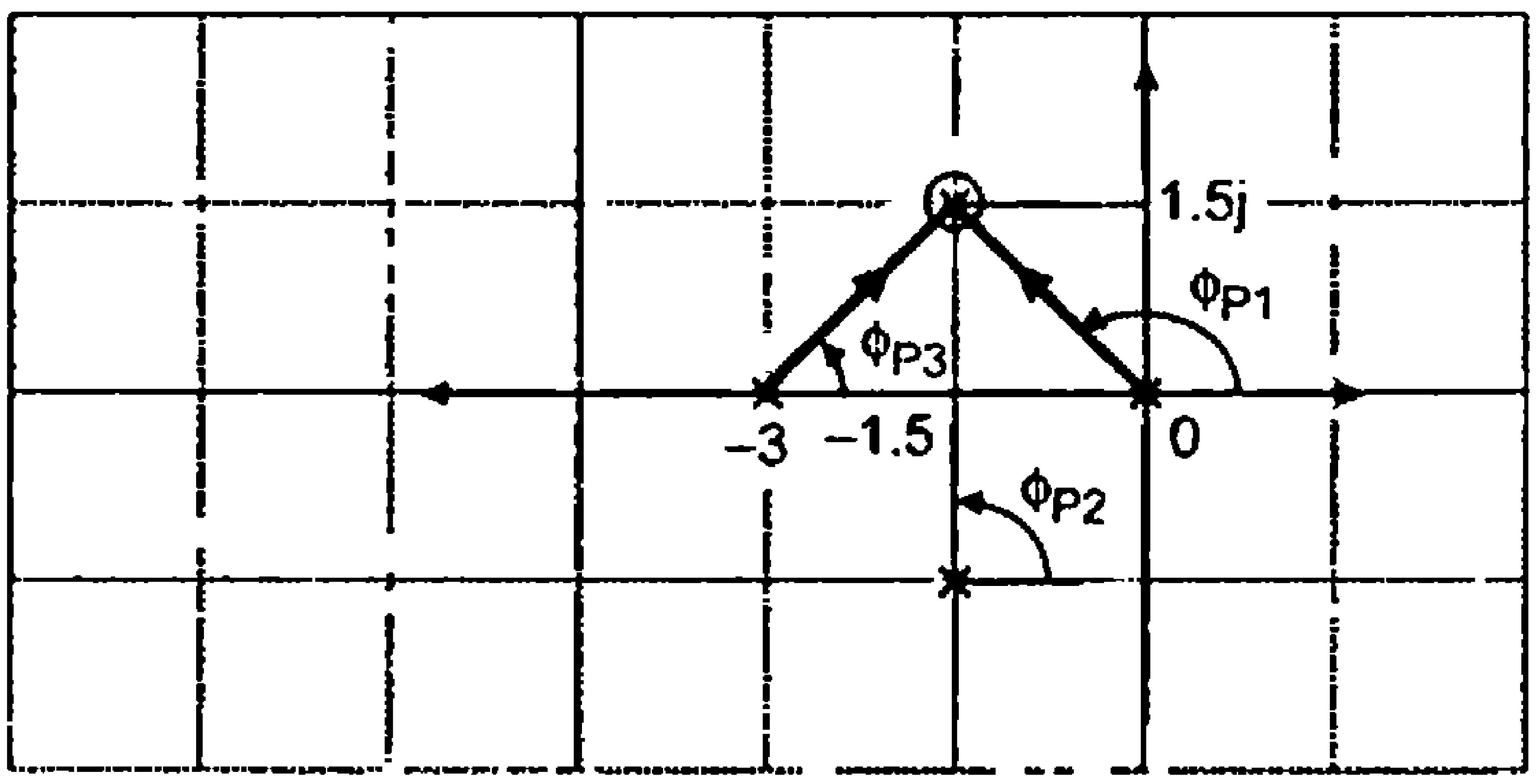
$$\therefore \quad 11.25s^2 + 25.3125 = 0$$

$$\therefore \quad s^2 = -2.25$$

$$\therefore \quad s = \pm j\,1.5$$

Step 7 : Angle of departure

Consider complex pole, $-1.5 + j\,1.5$.



$$\phi_{P1} = 135^\circ,\ \phi_{P2} = 90^\circ,\ \phi_{P3} = 45^\circ$$

$$\sum \phi_P = 135 + 90 + 45 = 270^\circ$$

$$\sum \phi_Z = 0^\circ$$

$$\therefore \quad \phi = \sum \phi_P - \sum \phi_Z = 270^\circ$$

$$\therefore \quad \phi_d = 180 - 270 = -90^\circ \quad \text{at } -1.5 + j\,1.5,$$

$$\therefore \quad \phi_d = +90^\circ \quad \text{at } -1.5 - j\,1.5$$

Step 8 : Complete root locus.

As all branches are behaving identically, this nature of root locus is typical as shown in Fig. 9.29 below.

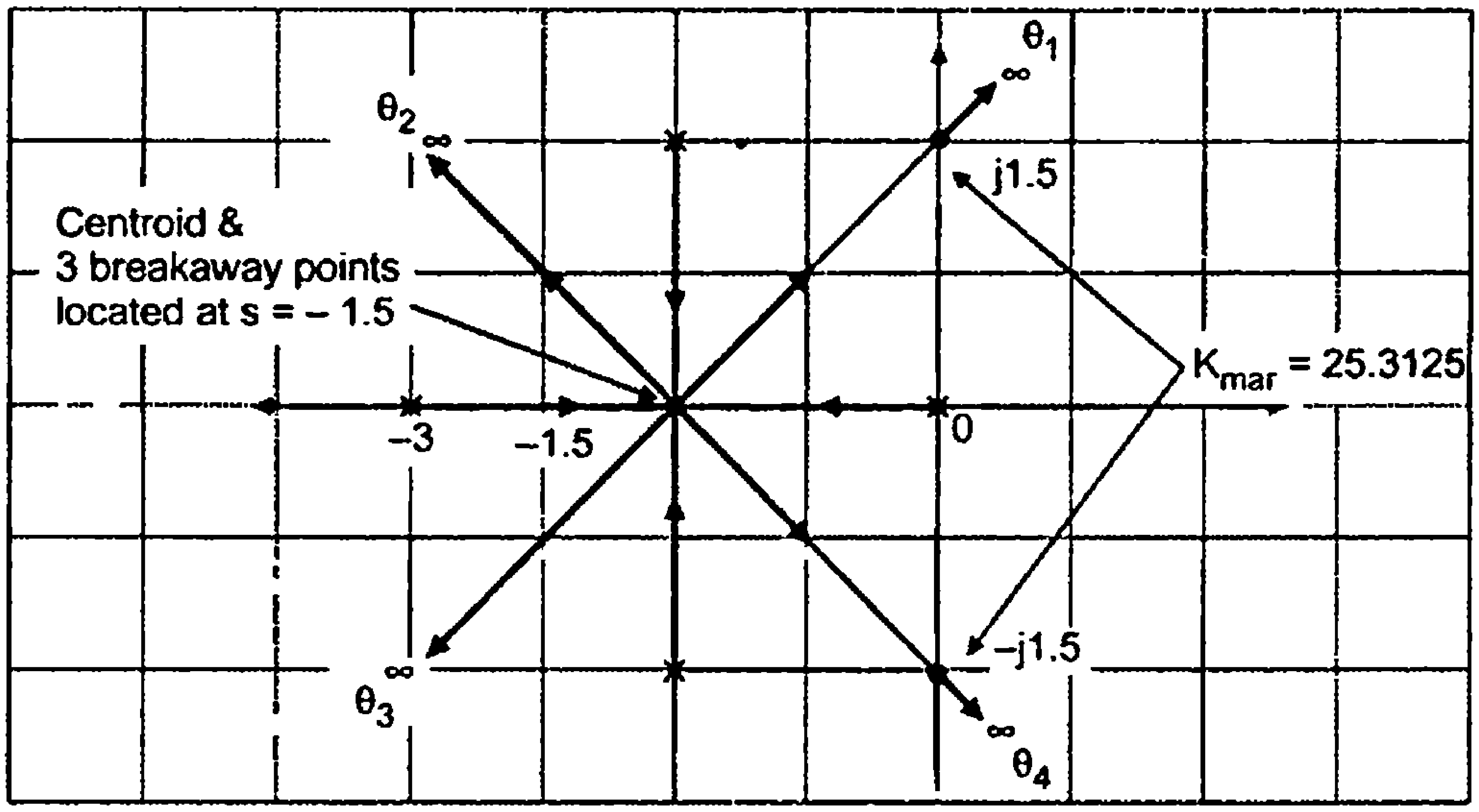


Fig. 9.29

Step 9 : Comment on stability

For $0 < K < 25.3125$, system is absolutely stable.

At $K = 25.3125$, system is marginally stable oscillating with frequency 1.5 rad/sec.

$K > 25.3125$, system is unstable.

➠ **Example 9.21 :** *Now in the example above, if the imaginary parts of complex poles are changed such that their distance from s = –1.5 is more than the distance of real poles s = 0 and – 3 from s = –1.5 , let us see the effect on the root locus. Real part of complex poles is still s = –1.5.*

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+3s+11.25)}$$

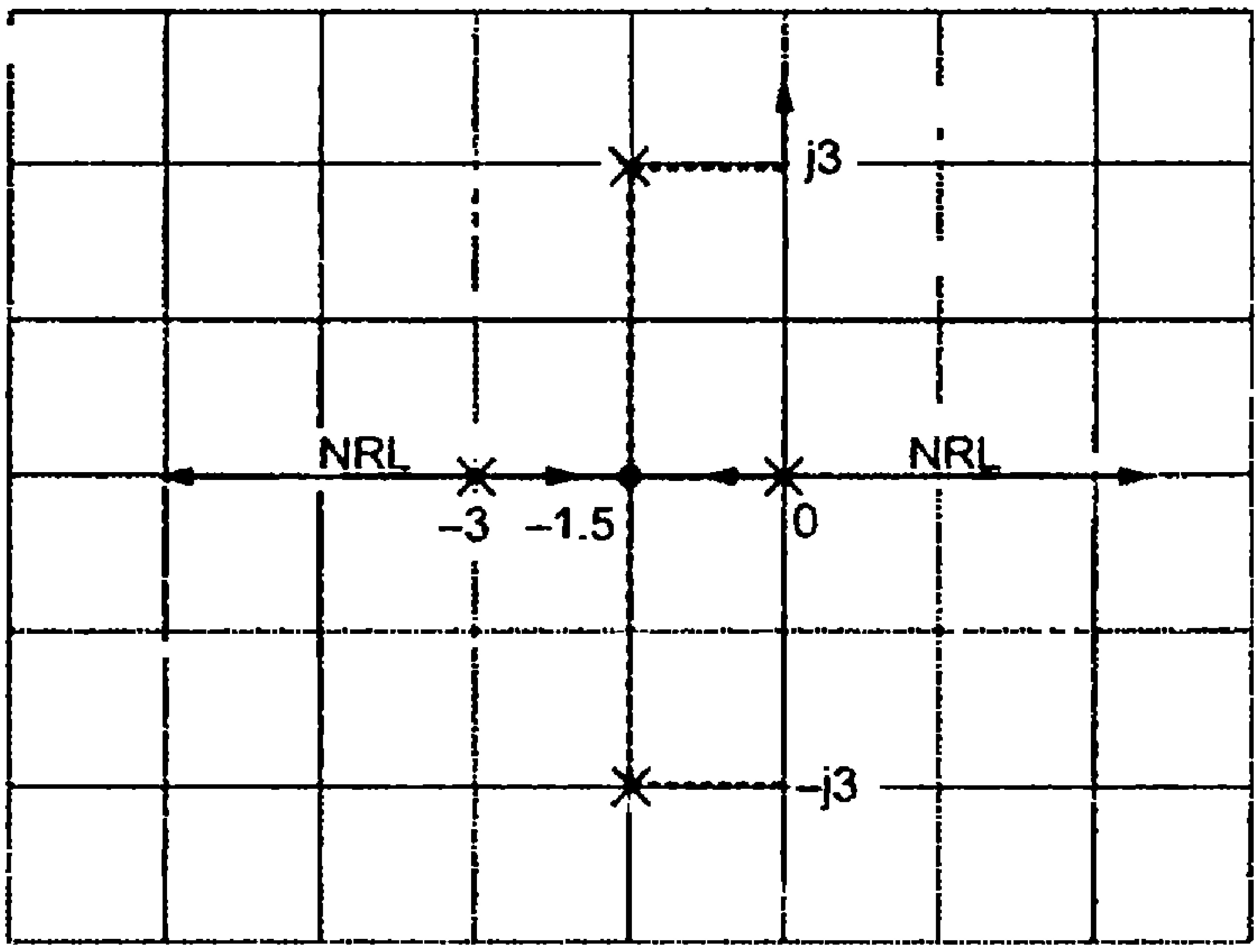
Solution : Step 1 : P = 4 , Z = 0 , N = 4. All branches approaching to ∞. Starting points s = 0, – 3 and –1.5 ± j3.

Step 2 : Pole - Zero plot is as follows.

Section between 0 and –3 is part of root locus.

Overall symmetry is there but imaginary distance of complex poles is more from s = –1.5 point than the distance of s = 0 and – 3 from that point. One breakaway point between s = 0 and –3.

Step 3 : Angles of asymptotes



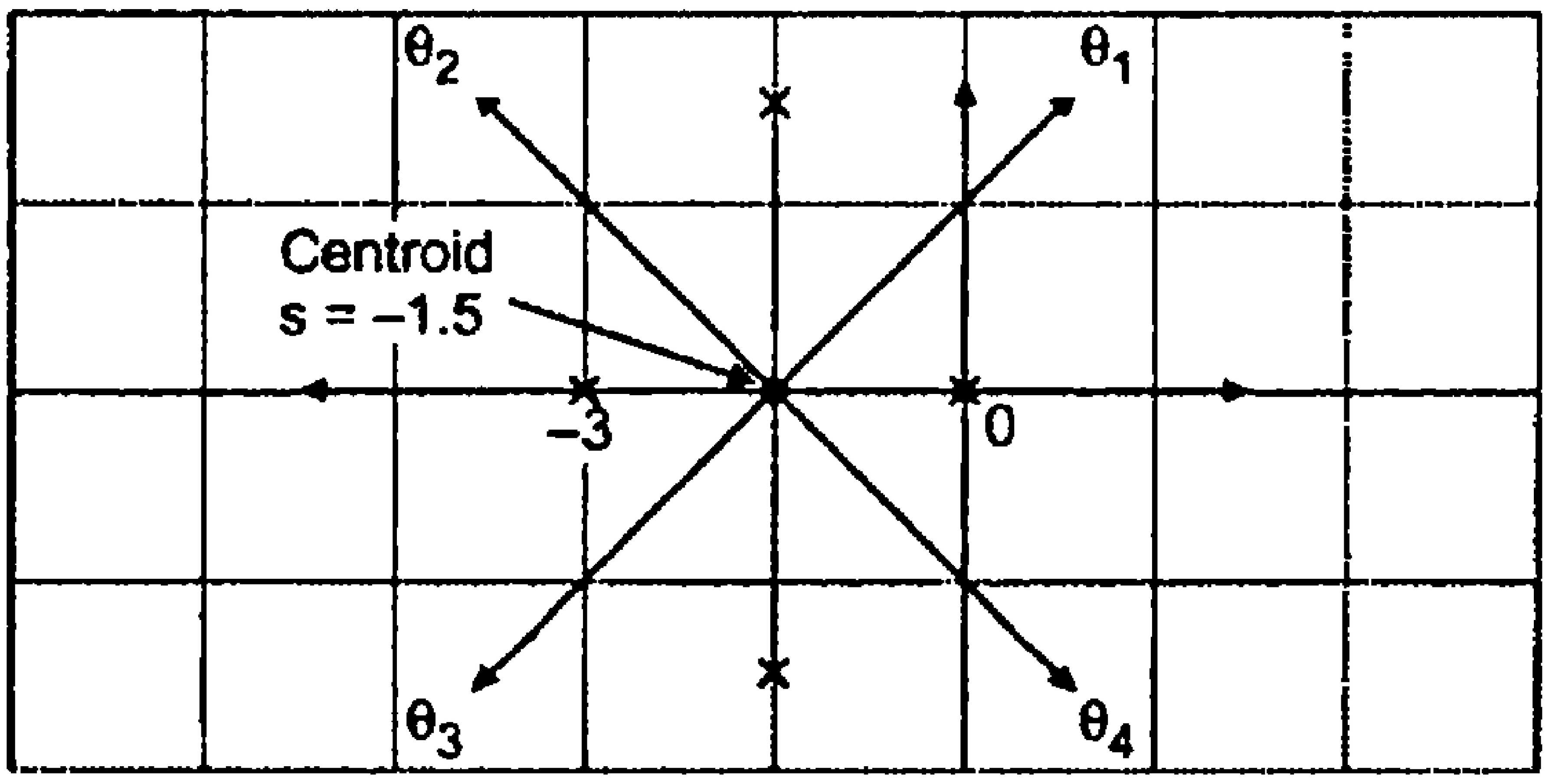
As number of branches approaching to ∞ are 4, angles are $\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$.

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P - Z} = \frac{0 - 3 - 1.5 - 1.5}{4}$$

$$= -1.5$$

Step 5 : Breakaway points



Characteristic equation : $1 + G(s)H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s(s+3)(s^2+3s+11.25)} = 0$$

$$\therefore \quad s^4 + 6s^3 + 20.25s^2 + 33.75s + K = 0$$

$$\therefore \quad K = -s^4 - 6s^3 - 20.25s^2 - 33.75s$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 40.5s - 33.75 = 0$$

$$\therefore \quad 4s^3 + 18s^2 + 40.5s + 33.75 = 0$$

Test s = –1.5,

– 1.5	4	18	40.5	33.75
		– 6	– 18	–33.75
	4	12	22.5	0

$$\therefore \quad \frac{dK}{ds} = (s + 1.5)(4s^2 + 12s + 22.5) = 0$$

$$\therefore \quad s = -1.5 \text{ and } \frac{-12 \pm \sqrt{144 - 4 \times 4 \times 22.5}}{2 \times 4}$$

∴ Three breakaway points are –1.5 , –1.5 ± j 1.8371

At s = –1.5 , Value of K = + 20.25

So s = –1.5 is valid breakaway point.

But to test s = –1.5 ± j 1.8371 , it is difficult to calculate 'K' so we can use angle condition to test their validity.

$$\angle G(s)H(s) = \pm(2q+1)\,180^\circ$$

$$\left.\frac{\angle K + j0}{\angle s \angle s+3 \angle s+1.5+j3 \angle s+1.5-j3}\right|_{at\ s = -1.5+j1.8371}$$

$$= \frac{\angle K + j0}{\angle -1.5 + j1.8371\ \angle -1.5+j1.8371+3 \angle -1.5+j1.8371+1.5+j3 \angle -1.5+j1.8371+1.5-j3}$$

$$\therefore \quad \angle G(s)H(s) = \frac{0^\circ}{\angle -1.5+j1.8371\ \angle 1.5+j1.8371\ \angle j4.8371\ \angle -j1.1629}$$

$$= \frac{0^\circ}{129.23^\circ\ 50.77^\circ\ 90^\circ\ (-90^\circ)}$$

$$= -180^\circ$$

i.e. it satisfies angle condition so both –1.5 ± j1.8371 are valid breakaway points. To find corresponding 'K' use magnitude condition for same point.

$$|G(s)H(s)|_{at\ s = -1.5 \pm j1.8371} = 1$$

Consider a point –1.5 + j 3

$$\phi_{P1} = 180° - \tan^{-1}\frac{3}{1.5} = 116.56°$$

$\therefore$ $$\phi_{P2} = 90°$$

$\therefore$ $$\phi_{P3} = \tan^{-1}\frac{3}{1.5} = 63.43°$$

$$\sum\phi_P = 270°, \quad \sum\phi_Z = 0°$$

$$\phi = \sum\phi_P - \sum\phi_Z = 270°$$

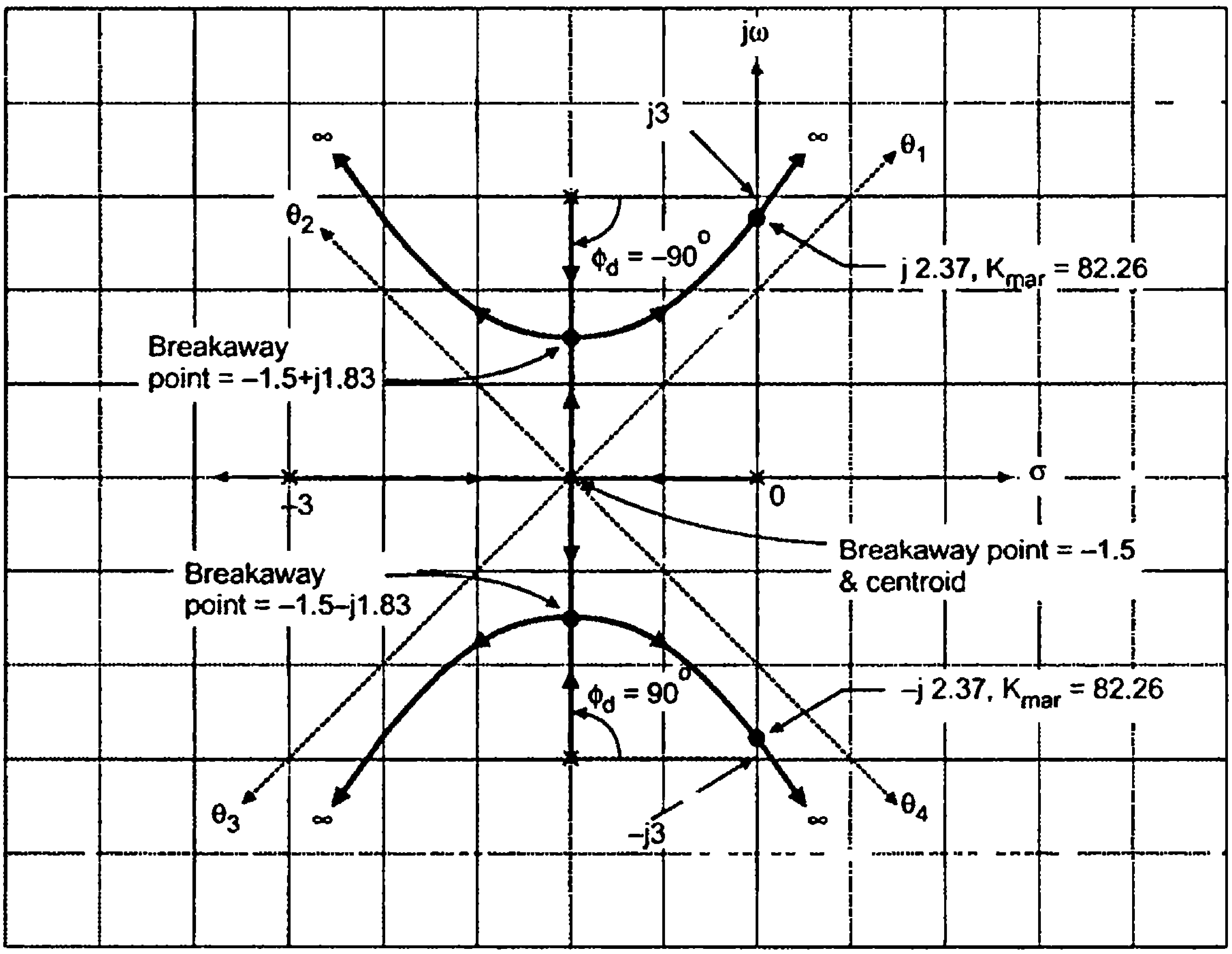
At –1.5 + j 3 $\quad \phi_d = 180° - \phi = -90°$

At –1.5 + j 3 $\quad \phi_d = +90°$

Step 8 : Root Locus

All branches will start from respective starting points. At K = 20.25, branches from s = 0 and – 3 will meet at s = –1.5. Other two will fail to reach to s = –1.5 at K = 20.25 as imaginary distance is more. Then branches will break from s = –1.5 at ± 90° and then at K = 31.63 , remaining two breakaway points will occur. Then branches will further break and will travel to ∞ along the asymptotes as shown. The breakaway point s = – 1.5 will occur first at K = 20.25 while remaining two will occur later at K = 31.63 but simultaneously. The overall root locus is symmetrical about real axis.

Step 9 : Comment on stability



∴ For $0 < K < 82.26$ system is stable.

At $K = 82.26$ system is marginally stable.

$K > 82.26$ system is unstable.

Example 9.22 : *Let us reduce the imaginary part of complex poles such that it is less than distance of s = 0 and − 3 from s = − 1.5.*

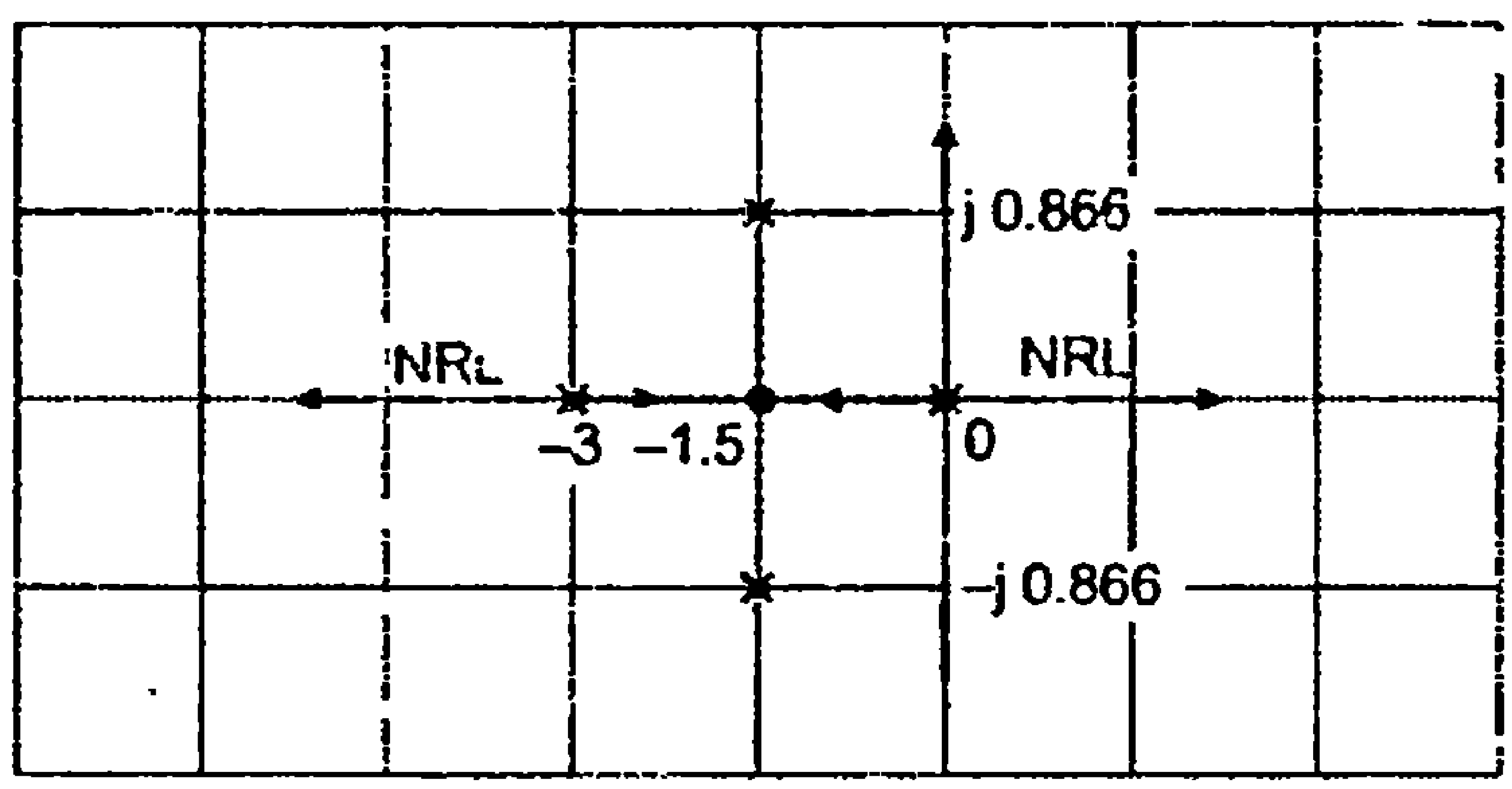
$$G(s)H(s) = \frac{K}{s\,(s+3)\,(s^2+3s+3)}$$

Solution : Step 1 : P = 4, Z = 0, N = 4, All branches approaching to ∞. Starting points $s = 0, -3$ and $\frac{-3 \pm \sqrt{9-12}}{2}$ i.e. $0, -3, -1.5 \pm j\,0.866$.

Step 2 : Pole-Zero plot is as follows.

Minimum one breakaway point exists between 0 and − 3.

Step 3 and 4 : Same as in examples 9.21 and 9.22 as real parts of poles are not changed.



Centroid −1.5 and angles $\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$.

Step 5 : Breakaway point

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s\,(s+3)\,(s^2+3s+3)} = 0$$

$$s^4 + 6s^3 + 12s^2 + 9s + K = 0$$

$$\therefore \quad K = -s^4 - 6s^3 - 12s^2 - 9s \quad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 24s - 9 = 0$$

$$\therefore \; 4s^3 + 18s^2 + 24s + 9 = 0$$

Test s = −1.5

−1.5	4	18	24	9
		− 6	−18	−9
	4	12	6	0

$$\frac{dK}{ds} = (s + 1.5)(4s^2 + 12s + 6) = 0$$

i.e. $$s = -1.5 \text{ and } \frac{-12 \pm \sqrt{144 - 16 \times 6}}{2 \times 4}$$

∴ Breakaway points are = −1.5, − 0.633, −2.366 and all are valid.

For s = −1.5 , K = 1.6875 ... Using equation (1)

s = − 0.633 , K = 2.25
s = − 2.366 , K = 2.25 } These two occur simultaneously.

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^4 + 6s^3 + 12s^2 + 9s + K = 0$

Routh's array,

s^4	1	12	K
s^3	6	9	0
s^2	10.5	K	0
s^1	$\frac{94.5 - 6K}{10.5}$	0	
s^0	K		

$$\therefore \quad 94.5 - 6K = 0$$

$$K_{mar} = 15.75$$

$$A(s) = 10.5\,s^2 + K = 0$$

$$\therefore \quad 10.5s^2 + 15.75 = 0$$

$$s^2 = -1.5$$

$$\therefore \quad s = \pm j\,1.224$$

Step 7 : Angle of departure

ϕ_d at −1.5 + j 0.866 is − 90°. While ϕ_d at −1.5 − j 0.866 is + 90°.

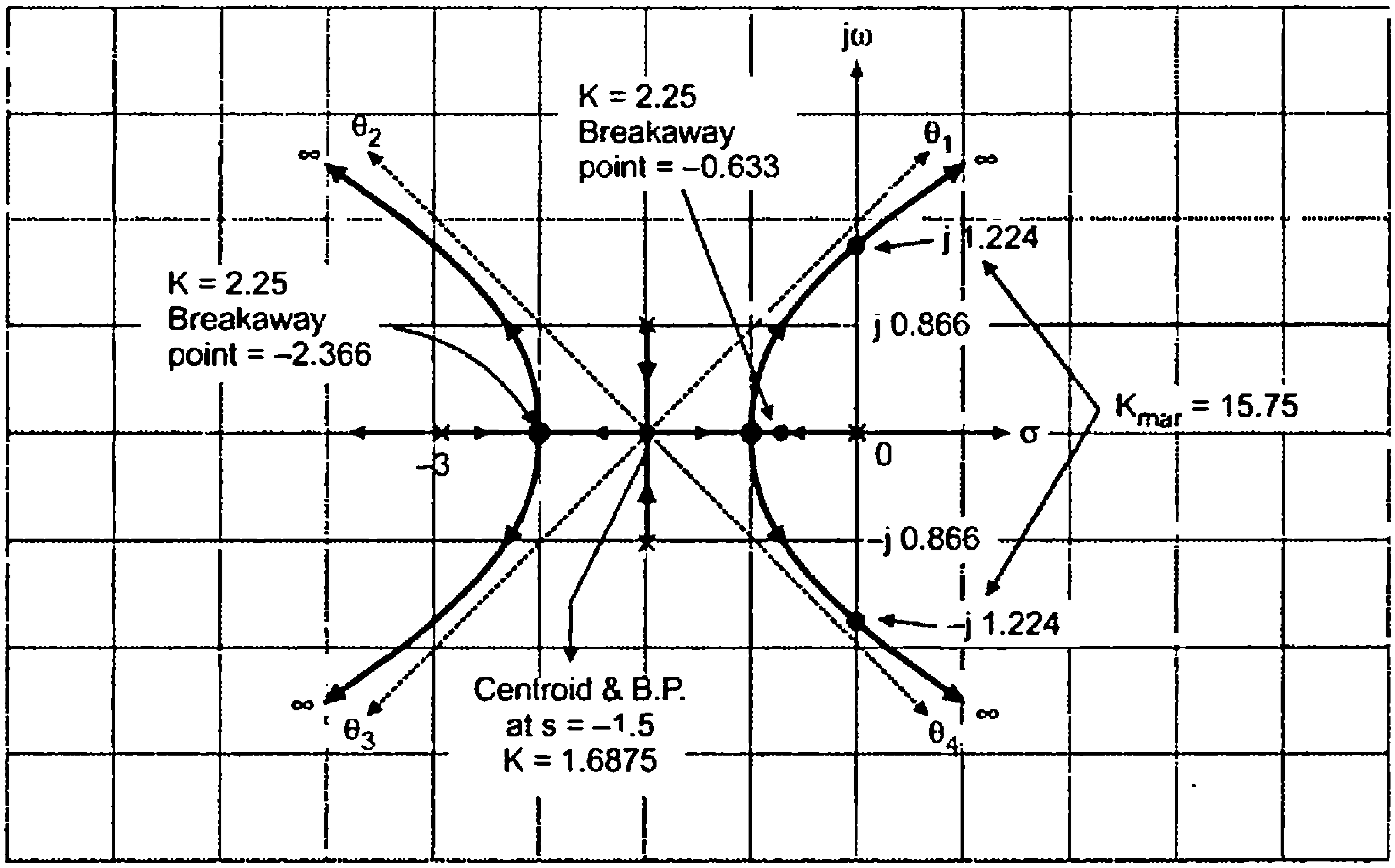
Step 8 : Root locus : In this problem s = −1.5 will occur as a breakaway point first as K = 1.6875 for this breakaway point. But branches from complex poles will reach at s = −1.5 rather than branches from real open loop poles. This is because the complex poles are more closer to s = − 1.5 than the real poles. The remaining breakaway points will occur later simultaneously. These will exist after the complex poles branches break into real branches at s = −1.5.

Step 9 :

For 0 < K < 15.75 system is stable.

At K = 15.75 system is marginally stable.

K > 15.75 system is unstable.



Example 9.23 : *Prove that the breakaway points of the root locus are the solutions of* $\frac{dK}{ds} = 0$ *where K is the open loop gain of the system whose open loop transfer function is G(s).*

Solution :

The characteristic equation 1 + G(s)H(s) = 0 is the combination of s terms and K terms. It can be arranged as,

$$F(s) = P(s) + KQ(s) = 0 \quad ...(1)$$

where P(s) = Polynomial containing s terms

KQ(s) = Polynomial containing K and s terms

Taking K outside from KQ(s), it can be noted that both P(s) and Q(s) are polynomials in s.

Now at **breakaway points, multiple roots occur.** Mathematically this is possible if $\frac{dF(s)}{ds} = 0$.

From (1),
$$\frac{dF(s)}{ds} = \frac{dP(s)}{ds} + K\frac{dQ(s)}{ds} = 0$$

$$\therefore \qquad K = \frac{-\left(\frac{dP(s)}{ds}\right)}{\left(\frac{dQ(s)}{ds}\right)} \qquad ...(2)$$

Substituting in (1), at breakaway point we can write,

$$P(s) - \frac{\left(\frac{dP(s)}{ds}\right)}{\left(\frac{dQ(s)}{ds}\right)} Q(s) = 0$$

i.e. $\frac{dQ(s)}{ds} P(s) - \frac{dP(s)}{ds} Q(s) = 0$ ...(3)

Solving (3) for value of 's', breakaway points can be obtained.

Now from (1), $K = -\frac{P(s)}{Q(s)}$

$$\therefore \qquad \frac{dK}{ds} = -\left\{ \frac{Q(s)\left(+\frac{dP(s)}{ds}\right) - [+P(s)]\left[\frac{dQ(s)}{ds}\right]}{Q(s)^2} \right\} \qquad ...(4)$$

If $\frac{dK}{ds}$ is equated to zero we get,

$$\frac{dK}{ds} = \frac{dQ(s)}{ds} P(s) - \frac{dP(s)}{ds} Q(s) = 0 \qquad ...(5)$$

This is same as equation (3) which yields breakaway points. This proves that **the roots of $\frac{dK}{ds} = 0$ are the actual breakaway points.**

➡ **Example 9.24 :** $G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+5s+20)}$

Sketch the complete root locus with approximate indication of breakaway points. Comment on the stability.

Solution : Step 1 : P = 4, Z = 1, N = P = 4 as P > Z, P – Z = 3 branches approach to ∞.

Starting points are, $s = 0, +1, -2.5 \pm j\,3.708$

Terminating points are, $s = -1$ and ∞, ∞, ∞

By trial and error method, the approximate breakaway points satisfying above equation are,

$s = +0.447$ and $s = -2.45$

Other two roots are complex conjugates and are not the valid breakaway points. It can be confirmed from angle condition. The values of K for valid breakaway points can be obtained from the equation of K and these values must be positive.

For	$s = +0.447$,	$K = +3.833$
For	$s = -2.45$,	$K = +80.16$

Step 6 : Intersection with imaginary axis

Consider the characteristic equation 1 + G(s)H(s) = 0 as,

$$s^4 + 4s^3 + 15s^2 + s(K-20) + K = 0$$

Routh's array,

s^4	1	15	K
s^3	4	K − 20	0
s^2	$\frac{80-K}{4}$	K	0
s^1	$\frac{\left(\frac{80-K}{4}\right)(K-20)-4K}{\left(\frac{80-K}{4}\right)}$	0	
s^0	K		

For marginal value of K, make row of s^1 as row of zeros but at the same time coefficient of s^2 should not be negative.

So $80 - K > 0$

$\therefore \quad 80 > K$ i.e. $K < 80$

Hence $K_{marginal}$ must be less than 80. So if $K_{marginal}$ is less than 80 it is valid.

From s^1 row, $\left(\frac{80-K}{4}\right)(K-20) - 4K = 0$

$$\therefore \quad -K^2 + 100K - 1600 - 16K = 0$$

$$\therefore \quad K^2 - 84K + 1600 = 0$$

$$K = \frac{84 \pm \sqrt{(84)^2 - 4 \times 1600}}{2} = 54.8,\ 29.19$$

Both are less than 80 and are valid. So this **root locus intersects twice with the imaginary axis i.e. at K_{mar} = 29.19 and again at K_{mar} = 54.8.**

The intersection points can be obtained from auxiliary equation.

$$A(s) = \left(\frac{80-K}{4}\right)s^2 + K = 0$$

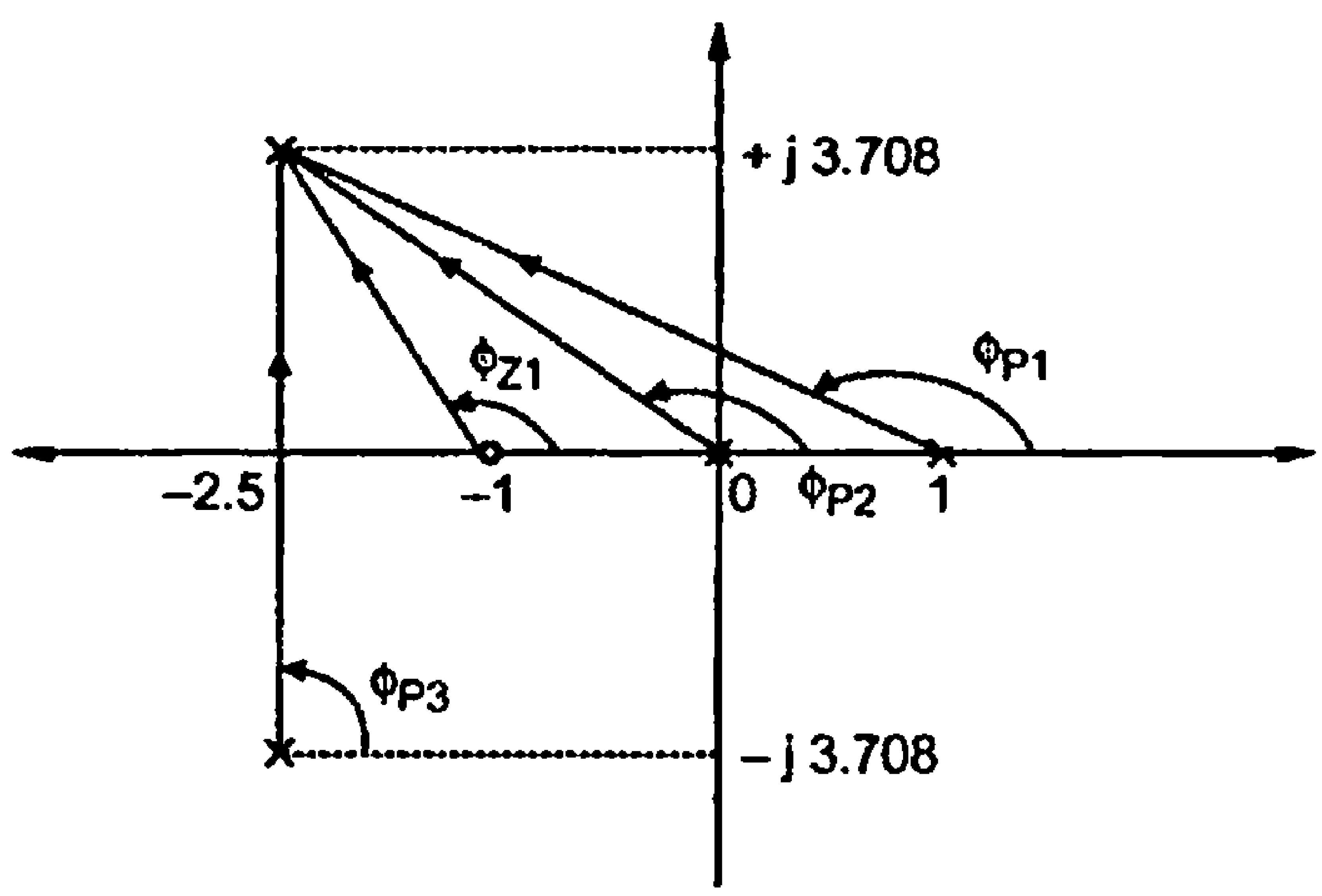
$$\therefore \quad s^2 = -\frac{K}{\left(\frac{80-K}{4}\right)}$$

At K_{mar} = 29.19,	s^2 = – 2.298,	s = ± j 1.515
At K_{mar} = 54.8,	s^2 = – 8.698,	s = ± j 2.94

Step 7 : Angles of departure at complex poles

Consider a pole – 2.5 + j 3.708 and calculate $\sum \phi_P$ and $\sum \phi_Z$ as shown.

From the geometry of the Pole-Zero plot,



$$\phi_{P1} = 180° - \tan^{-1}\frac{3.708}{3.5} = 130.79°$$

$$\phi_{P2} = 180° - \tan^{-1}\frac{3.708}{2.5} = 123.98°$$

$$\phi_{P3} = +90°$$

$$\phi_{Z1} = 180° - \tan^{-1}\frac{3.708}{1.5} = 112.02°$$

$$\sum \phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3} = +344.77°, \qquad \sum \phi_Z = 112.02°$$

Step 3 : Angles of asymptotes

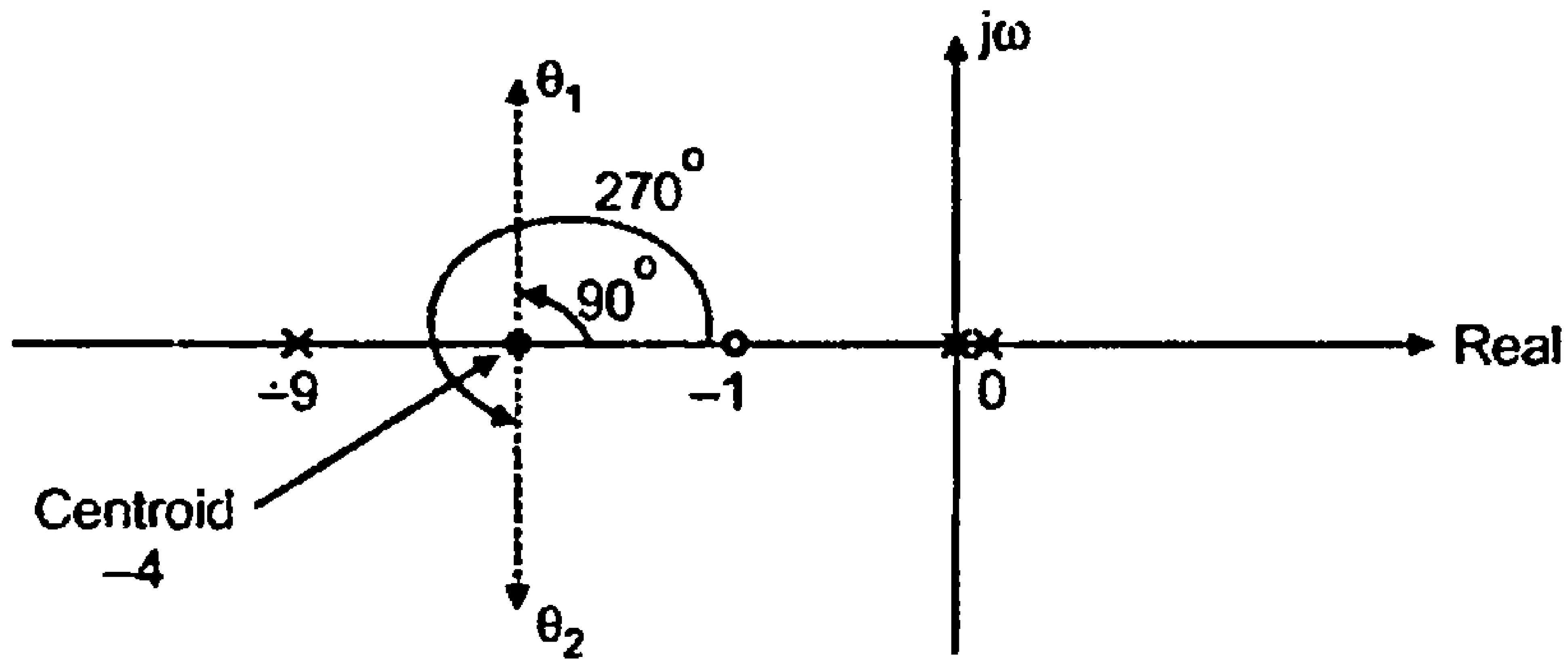
$$\theta = \frac{(2q+1)180^\circ}{P-Z}$$

$$\theta_1 = \frac{180^\circ}{2} = 90^\circ \quad \text{and} \quad \theta_2 = \frac{3\times 180^\circ}{2} = 270^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0+0-9-(-1)}{2}$$

$$= -4$$



Step 5 : As per normal predictions there are no breakaway points but let us find out roots of $\frac{dK}{ds} = 0$.

The characteristic equation is given as

$$s^3 + 9s^2 + Ks + K = 0$$

$$\therefore \quad K = \frac{-s^3 - 9s^2}{(s+1)}$$

$$\therefore \quad \frac{dK}{ds} = \frac{(s+1)(-3s^2 - 18s) - (-s^3 - 9s^2)(1)}{(s+1)^2} = 0$$

$$\therefore \quad -3s^3 - 3s^2 - 18s^2 - 18s + s^3 + 9s^2 = 0$$

$$\therefore \quad -2s^3 - 12s^2 - 18s = 0$$

$$\therefore \quad s(2s^2 + 12s - 18) = 0$$

$$\therefore \quad s = 0, \quad s = \frac{-12 \pm \sqrt{144-144}}{2\times 2}$$

$$\therefore \quad s = 0, -3, -3$$

As s = – 3 is lying on real axis which is the part of root locus as identified earlier, it is valid breakaway point.

Note : This is the rare case where breakaway point actually exists though according to general predictions there is no breakaway points. Such cases are rare.

Substituting s = –3 in expression for K,

$$K = \frac{(-3)^3 - 9(-3)^2}{(-3+1)} = \frac{27-81}{-2} = +27$$

As K is positive, s = –3 is valid breakaway point where all roots are going to occur simultaneously.

Step 6 : Intersection with imaginary axis :

The characteristic equation is $s^3 + 9s^2 + Ks + K = 0$

∴ Routh's array is

s^3	1	K
s^2	9	K
s^1	$\frac{8K}{9}$	0
s^0	K	

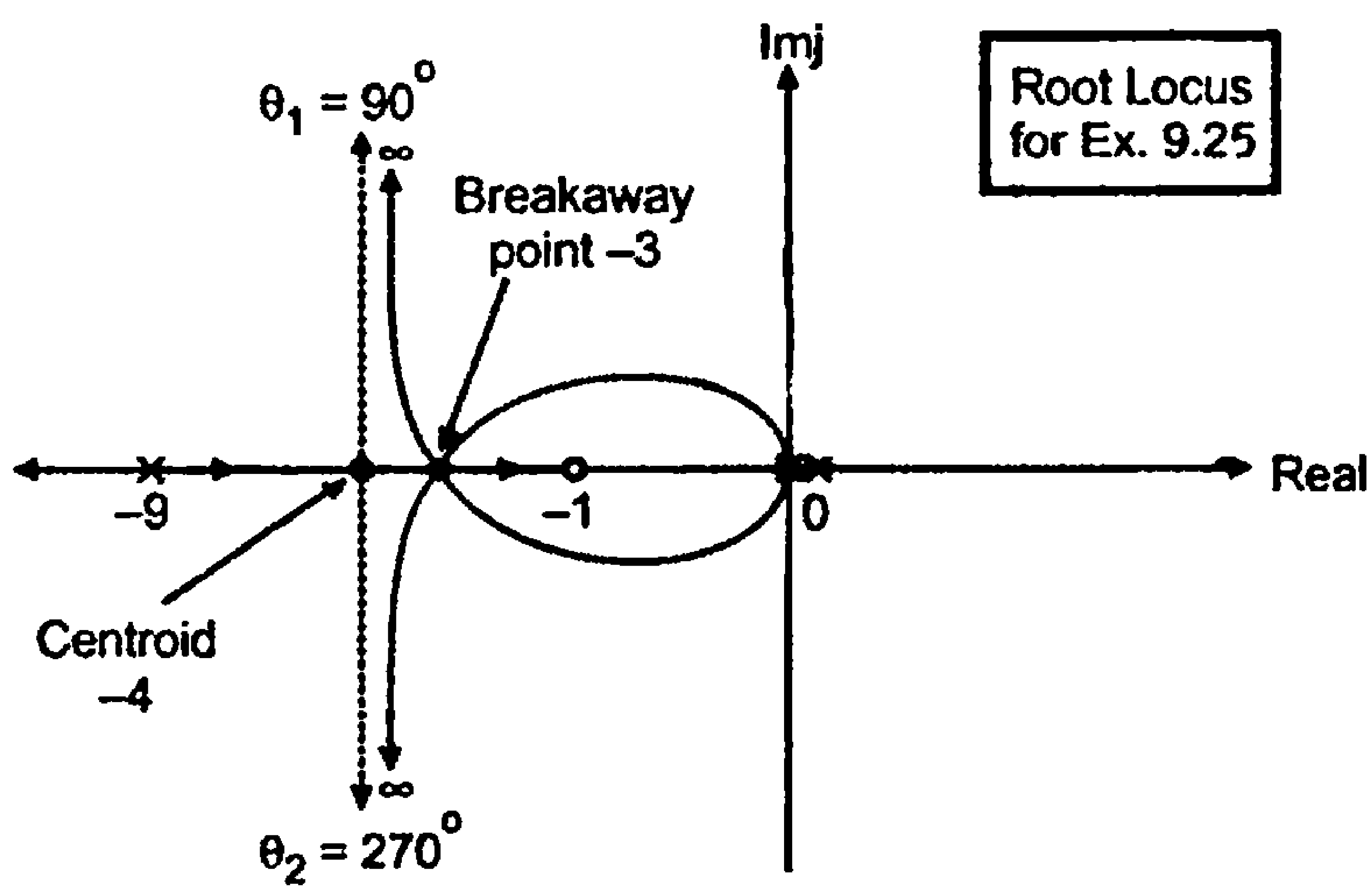
So marginal value of K which will make row of s^1 as row of zeros, is K = 0. But this is invalid. System cannot work with K = 0. This indicates that there is no intersection of root locus with the imaginary axis.

Step 7 : Angle of departure

As there are no complex conjugate poles, there is no need to calculate angles of departures. Root locus will depart at angles ± 90° after meeting at a breakaway point s = –3.

Step 8 : Sketch of Root Locus

'The system is absolutely stable as for all values of K from 0 to ∞, root locus lies in the left half of s-plane.'



Root Locus for Ex. 9.25

➡ Example 9.26 : *Prove that a combination of two poles s = – a_1 and s = – a_2 one zero s = – b to the left of both of them on the real axis, results in a root locus whose complex root branches form a circle centred at the zero s = b, with radius given by $\sqrt{(b-a_1)(b-a_2)}$: the root locus gain varying from 0 to ∞.*

Solution : There are two poles at $-a_1, -a_2$ and one zero at $-b$.

$$G(s)H(s) = \frac{K(s+b)}{(s+a_1)(s+a_2)}$$

For a complex point on the root locus,

$$s = \sigma + j\omega$$

$$\therefore \quad G(s)H(s) = \frac{K(\sigma+j\omega+b)}{(\sigma+j\omega+a_1)(\sigma+j\omega+a_2)} \quad \text{at that point}$$

$$\therefore \quad \angle G(s)H(s) = \frac{\tan^{-1}\left[\frac{\omega}{\sigma+b}\right]}{\tan^{-1}\left[\frac{\omega}{\sigma+a_1}\right]\tan^{-1}\left[\frac{\omega}{\sigma+a_2}\right]} \quad \text{at that point}$$

$$\therefore \quad \angle G(s)H(s) = \tan^{-1}\left[\frac{\omega}{\sigma+b}\right]-\left\{\tan^{-1}\left[\frac{\omega}{\sigma+a_1}\right]+\tan^{-1}\left[\frac{\omega}{\sigma+a_2}\right]\right\}$$

$$= \tan^{-1}\left[\frac{\omega}{\sigma+b}\right]-\tan^{-1}\left[\frac{\frac{\omega}{\sigma+a_1}+\frac{\omega}{\sigma+a_2}}{1-\left(\frac{\omega}{\sigma+a_1}\right)\times\left(\frac{\omega}{\sigma+a_2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\omega}{\sigma+b}\right]-\tan^{-1}\left[\frac{\omega(2\sigma+a_1+a_2)}{(\sigma+a_1)(\sigma+a_2)-\omega^2}\right]$$

$$= \tan^{-1}\left[\frac{\frac{\omega}{\sigma+b}-\frac{\omega(2\sigma+a_1+a_2)}{(\sigma+a_1)(\sigma+a_2)-\omega^2}}{1+\left(\frac{\omega}{\sigma+b}\right)\left(\frac{\omega[2\sigma+a_1+a_2]}{(\sigma+a_1)(\sigma+a_2)-\omega^2}\right)}\right]$$

Now for a point on the root locus, $\angle G(s)H(s)$ is 180°.

$$180° = \tan^{-1}\left[\frac{\frac{\omega}{\sigma+b}-\frac{\omega(2\sigma+a_1+a_2)}{(\sigma+a_1)(\sigma+a_2)-\omega^2}}{1+\left(\frac{\omega}{\sigma+b}\right)\left(\frac{\omega[2\sigma+a_1+a_2]}{(\sigma+a_1)(\sigma+a_2)-\omega^2}\right)}\right]$$

Taking tan of both sides, and substituting tan 180° = 0 we get,

$$(\sigma+a_1)(\sigma+a_2)-\omega^2 = (\sigma+b)(2\sigma+a_1+a_2)$$

$$\therefore \quad -b(a_1+a_2)+a_1a_2 = \sigma^2+2b\sigma+\omega^2$$

$$\therefore \quad \sigma^2+2b\sigma+b^2+\omega^2-b^2 = -b(a_1+a_2)+a_1a_2$$

$$\therefore \quad (\sigma+b)^2+\omega^2 = b^2-b(a_1+a_2)+a_1a_2$$

$$\therefore \quad (\sigma+b)^2+(\omega-0)^2 = \left(\sqrt{(b-a_1)(b-a_2)}\right)^2$$

This is the equation of circle with centre as (– b, 0) which is location of the zero and the radius $\sqrt{(b-a_1)(b-a_2)}$. Thus complex root branches form a circle is proved.

Example 9.27 : *A system has* $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)(s+8)}$

Draw the root locus plot and determine the position of closed loop poles for damping factor of 0.707 for the dominant poles. **(M.U. : Jan.-1993, Dec.-1996)**

Solution : $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)(s+8)}$

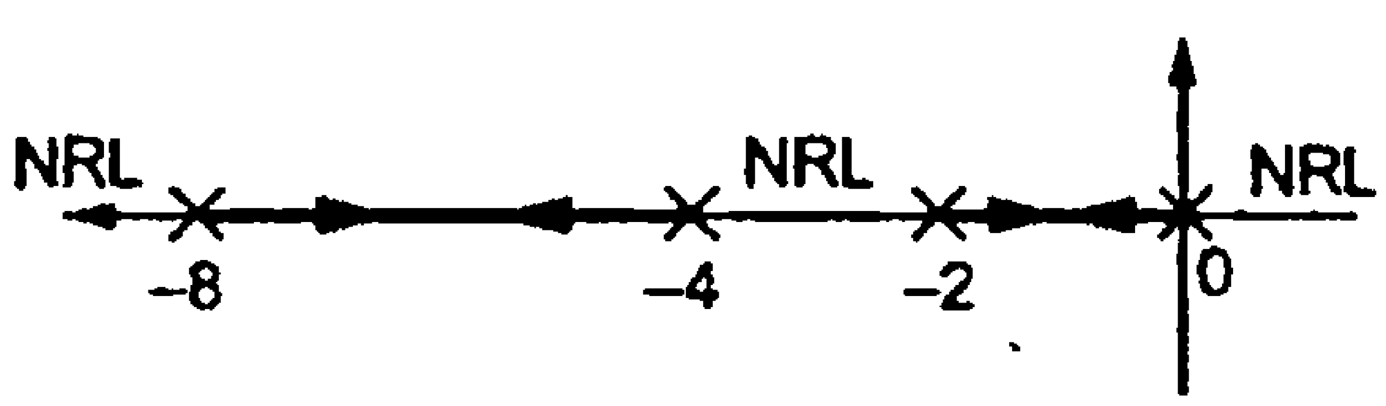
Step 1 : Number of poles = P = 4 and Number of zeros = Z = 0, N = P = 4,

N = P – Z = 4 branches of root locus will terminate at ∞.

Starting points : s = 0, – 2, – 4, – 8

Terminating points : ∞, ∞, ∞, ∞

Step 2 : Sections of real axis

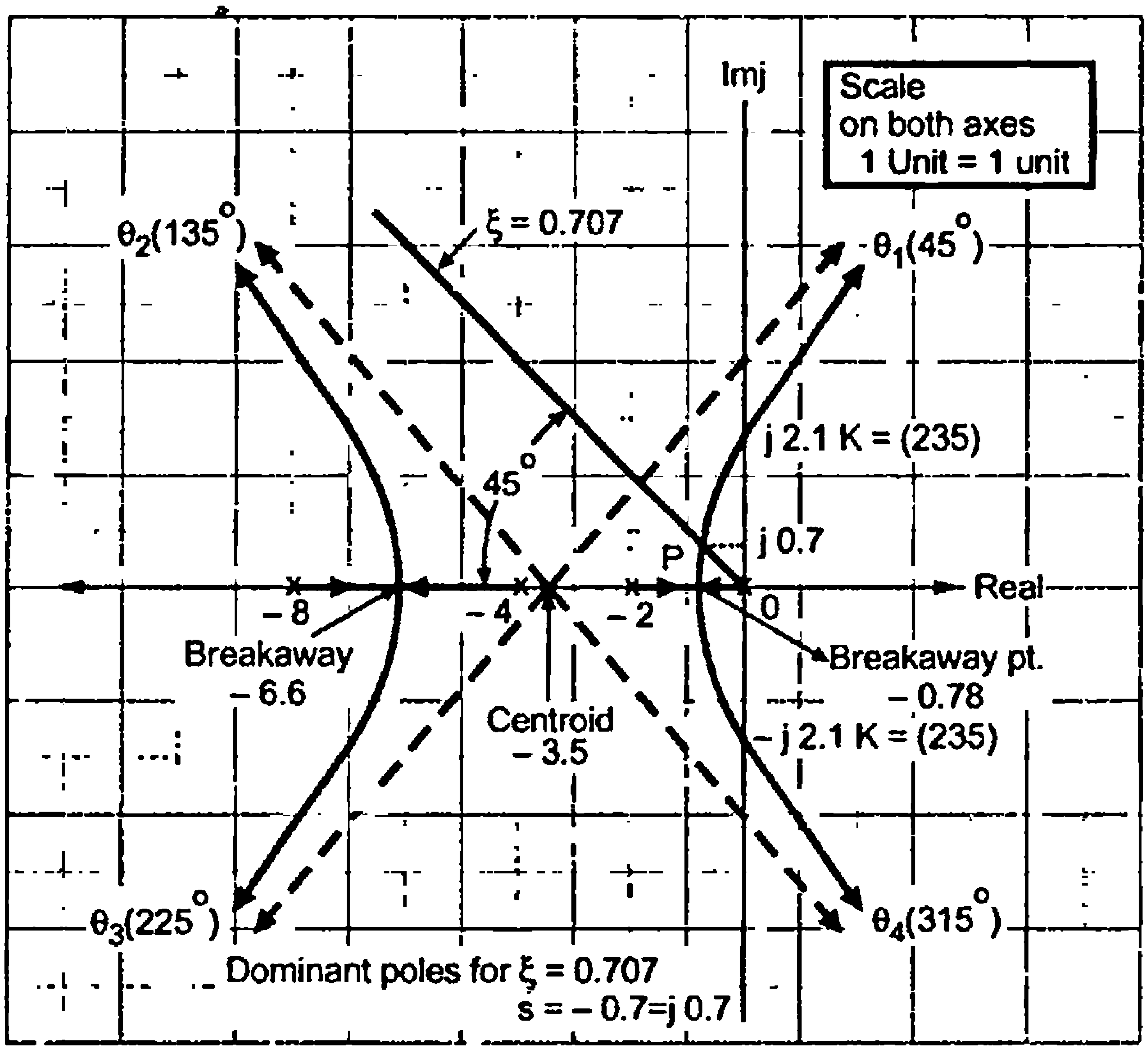


Two breakaway points possible between 0 and – 2 and – 4 and – 8.

Step 3 : Angle of asymptote = $\theta = \dfrac{(2q+1)180}{P-Z}$, q = 0, 1, 2, 3.

$\theta_1 = 45°, \quad \theta_2 = 135°, \quad \theta_3 = 225°, \quad \theta_4 = 315°$

Step 4 : Centroid $= \dfrac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \dfrac{-2-4-8}{4} = -3.5$

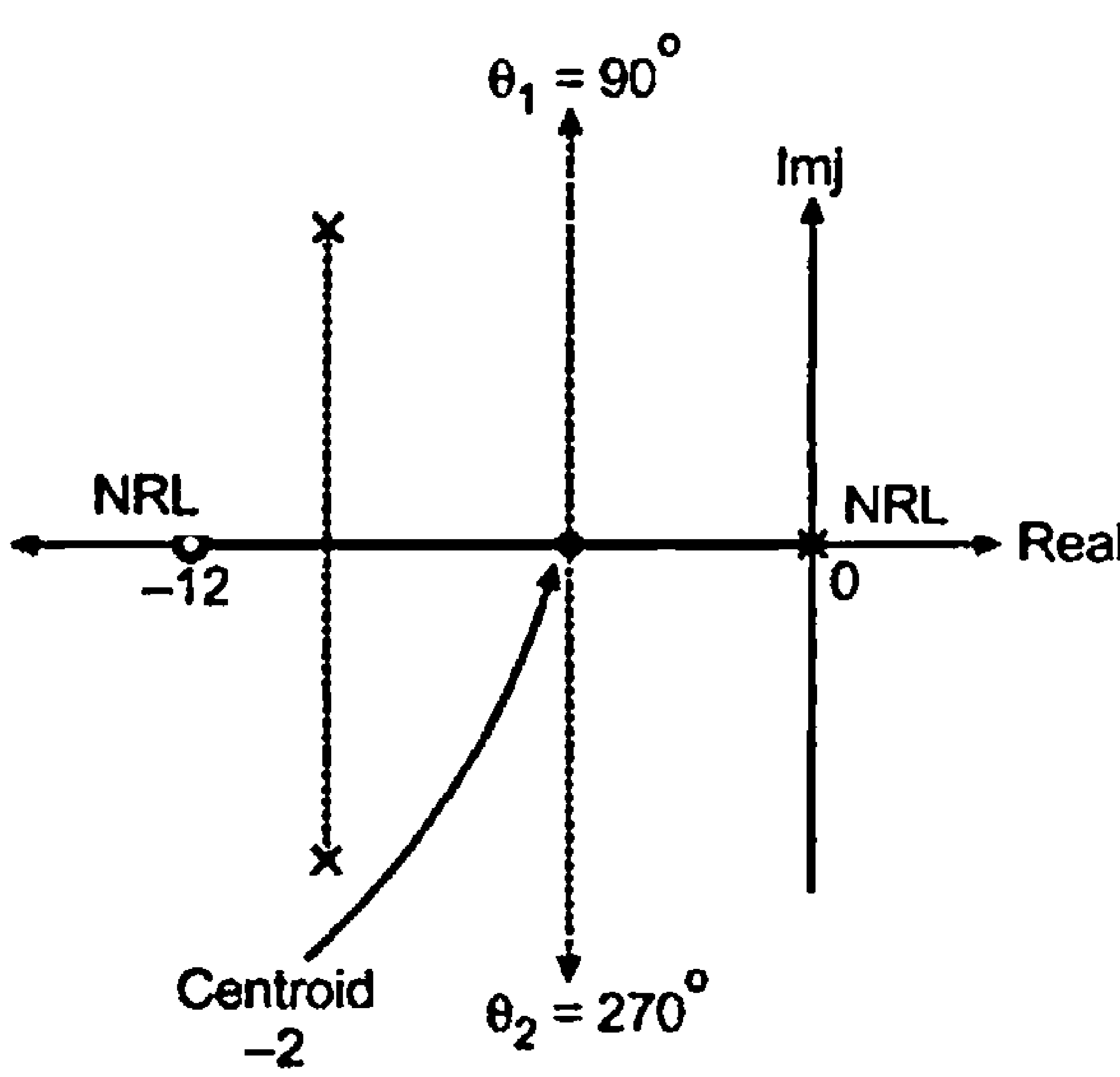


Example 9.28 : *For the transfer function*

$$G(s)H(s) = \frac{K(s+12)}{s(s^2+16s+100)}$$

Sketch the root locus and determine all pertinent data. **(M.U. : July-1991)**

Solution : $G(s)H(s) = \frac{K(s+12)}{s(s^2+16s+100)}$



Step 1 : Number of poles = P = 3 and Number of zeros = Z = 1

$$P_1 = 0, \quad P_{2,3} = \frac{-16 \pm \sqrt{256-400}}{2} = -8 \pm j\,6 \qquad Z_1 = -12$$

Step 2 : N = P − Z = 2 branches of root locus will terminate at ∞

Step 3 : Angle of asymptotes $= \theta = \frac{(2q+1)180}{P-Z}$; $\theta = 0, 1.$

$$\theta_1 = 90°, \; \theta_2 = 270°.$$

Step 4 : Centroid $= \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z}$

$$= \frac{-8-8+12}{2} = -2$$

Step 5 : Breakaway point does not exist.

Step 6 : Angle of departure

Consider $s = -8 + j6$

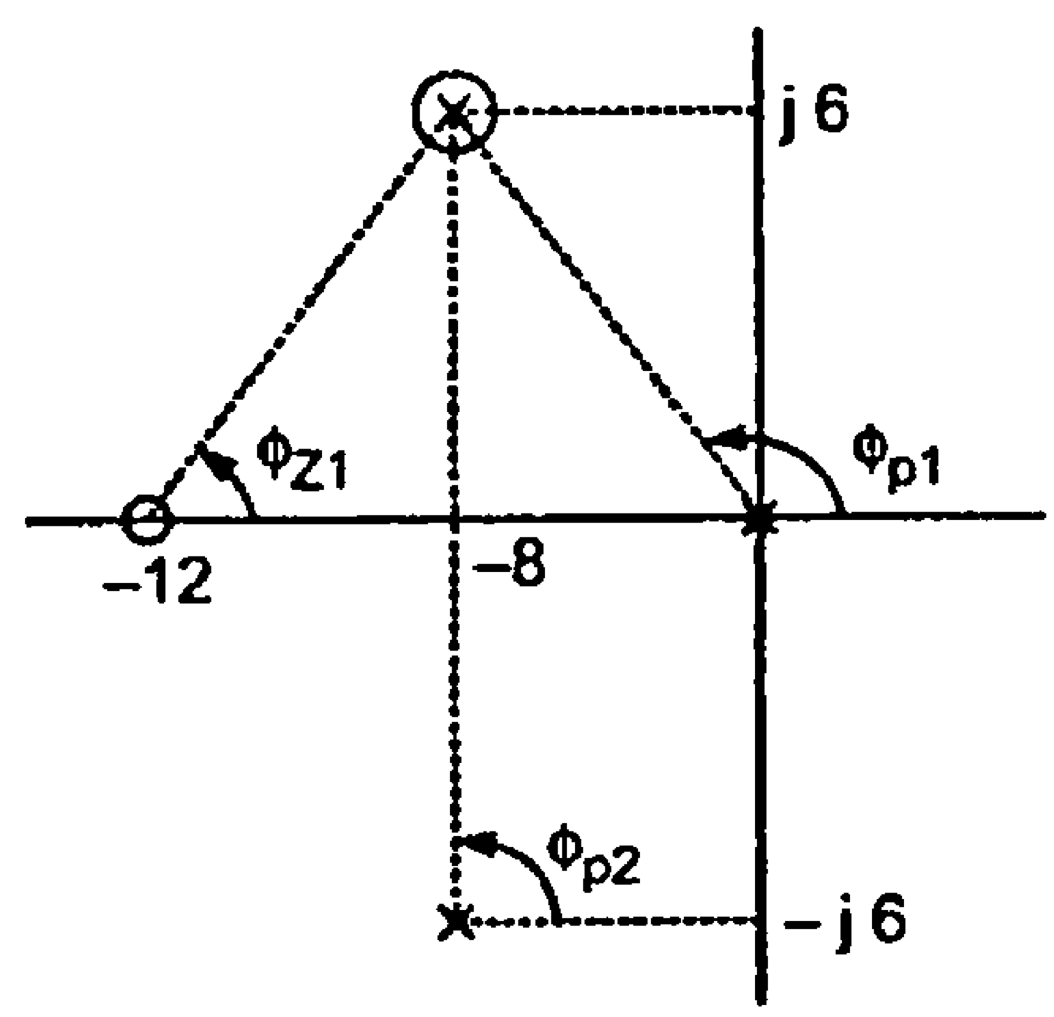
$$\phi_d = 180 - \phi \qquad \text{where} \quad \phi = \Sigma\phi_P - \Sigma\phi_Z$$

$$\therefore \phi = \left[180° - \tan^{-1}\frac{6}{8}\right] + 90° - \tan^{-1}\frac{6}{4}$$

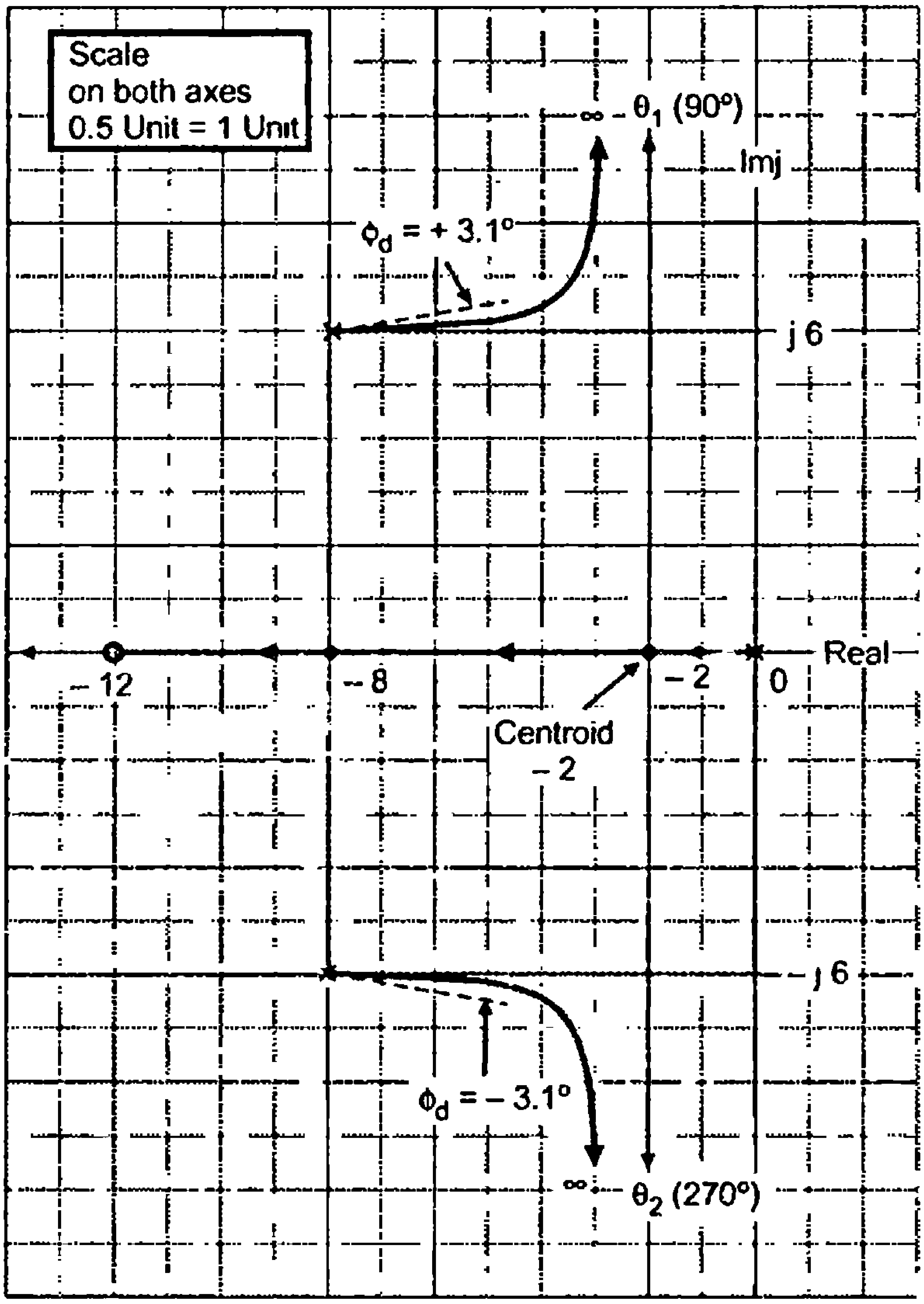
$$= 176.82°$$

$$\phi_d = 180° - 176.82°$$

$$\phi_d = +3.179° \quad \text{... at } -8 + j6$$



Similarly ϕ_d at $-8 - j6$ is $-3.179°$



Example 9.29 : *Construct root locus for*

$$G(s)H(s) = \frac{K}{(s+3)(s+5)(s^2+2s+5)}; \quad K > 0$$

Also find the gain margin of the system, if the design value of K = 16.

(M.U. : May-1993)

Solution : $G(s)H(s) = \dfrac{K}{(s+3)(s+5)(s^2+2s+5)}$

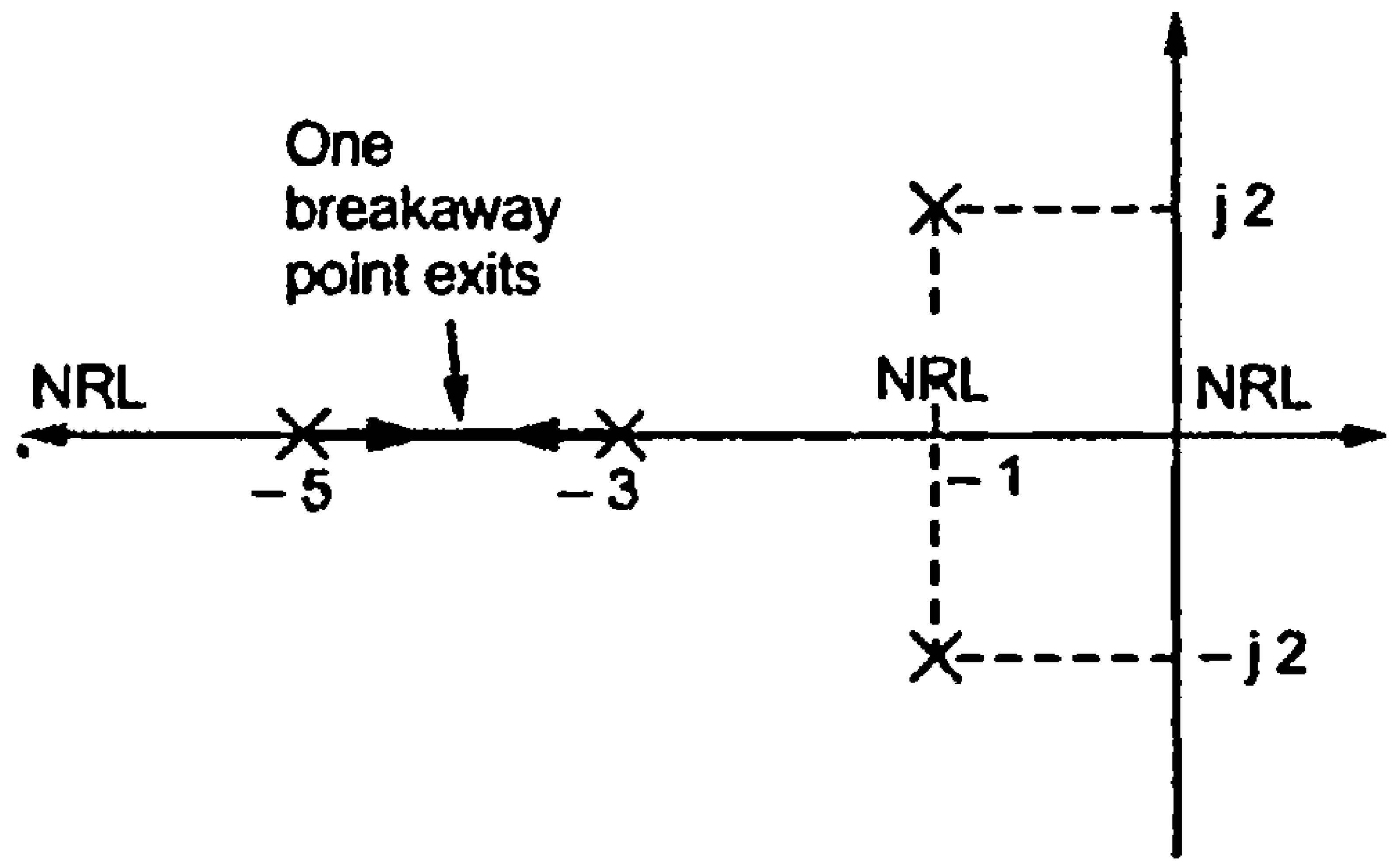
Step 1 : Number of poles = P = 4 and Number of zeros = Z = 0

N = P – Z = 4 branches of root locus will terminate at ∞

Starting points : – 3, – 5, – 1 ± j2

Terminating points : ∞, ∞, ∞, ∞.

Step 2 : Sections of real axis



Step 3 : Angle of asymptotes $= \theta = \frac{(2q+1)180}{P-Z}$; q = 0, 1, 2, 3.

$$\theta_1 = 45°, \quad \theta_2 = 135°, \quad \theta_3 = 225°, \quad \theta_4 = 315°$$

Step 4 : Centroid $= \sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z}$

$$= \frac{-3-5-1-1}{4} = -2.5$$

Step 5 : Breakaway point . Characteristic equation is

$$s^4 + 10s^3 + 36s^2 + 70s + 75 + K = 0$$

$$\therefore \quad K = -[s^4 + 10s^3 + 36s^2 + 70s + 75]$$

$$\frac{dK}{ds} = 4s^3 + 30s^2 + 72s + 70 = 0$$

Solving, s = − 4.21, − 1.645 ± j 1.201

where s = − **4.21** is valid breakaway point.

Step 6 : Intersection with imaginary axis.

$$s^4 + 10s^3 + 36s^2 + 70s + 75 + K = 0$$

The Routh array is

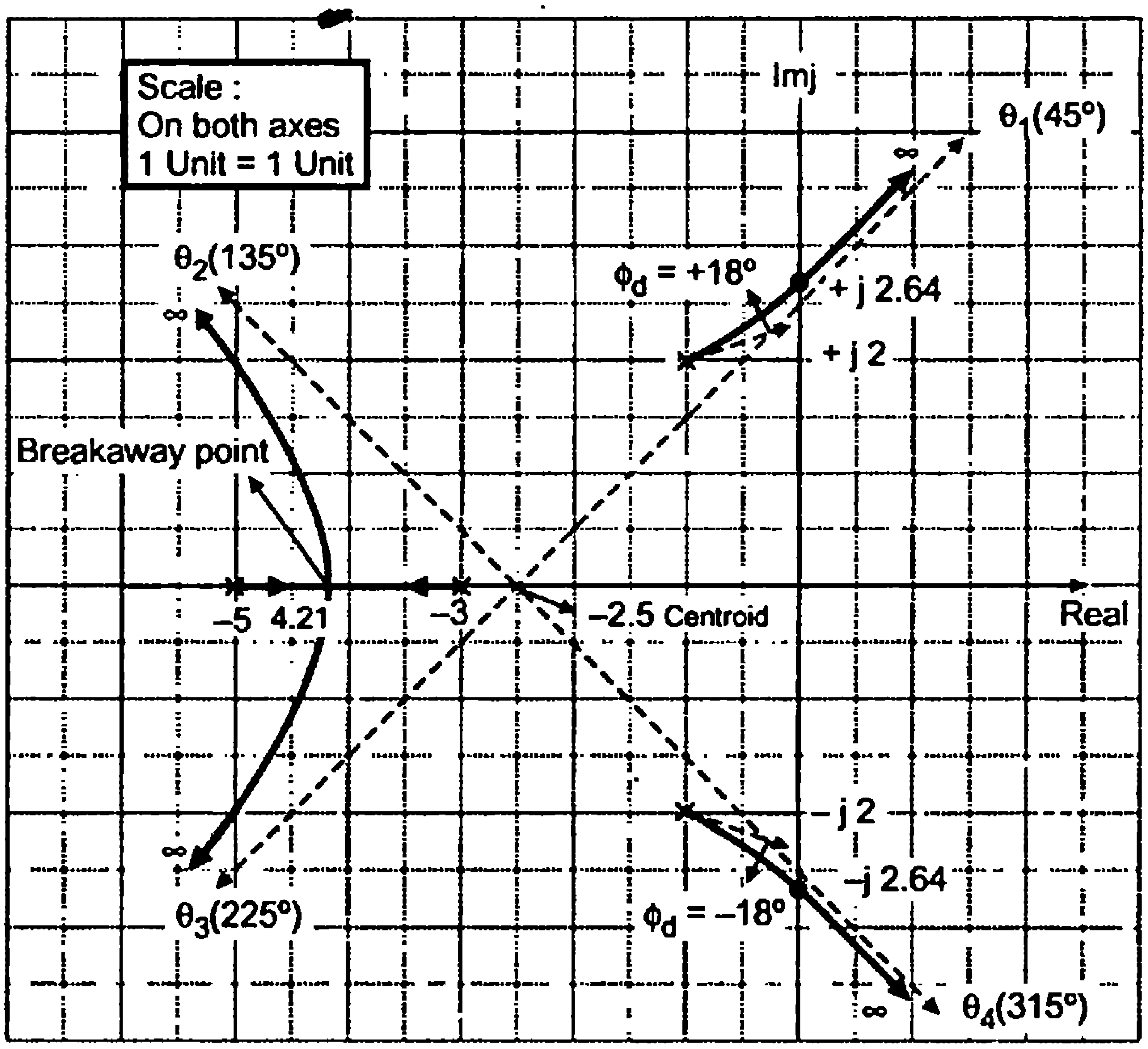
s^4	1	36	75 + K
s^3	1	7	0
s^2	29	75 + K	0
s^1	$\frac{203-75-K}{29}$	0	
s^0	75 + K		

$\therefore K_{mar} = 128$

The Auxiliary equation

$$A(s) = 29s^2 + 203 = 0$$

$$s = \pm j2.64$$



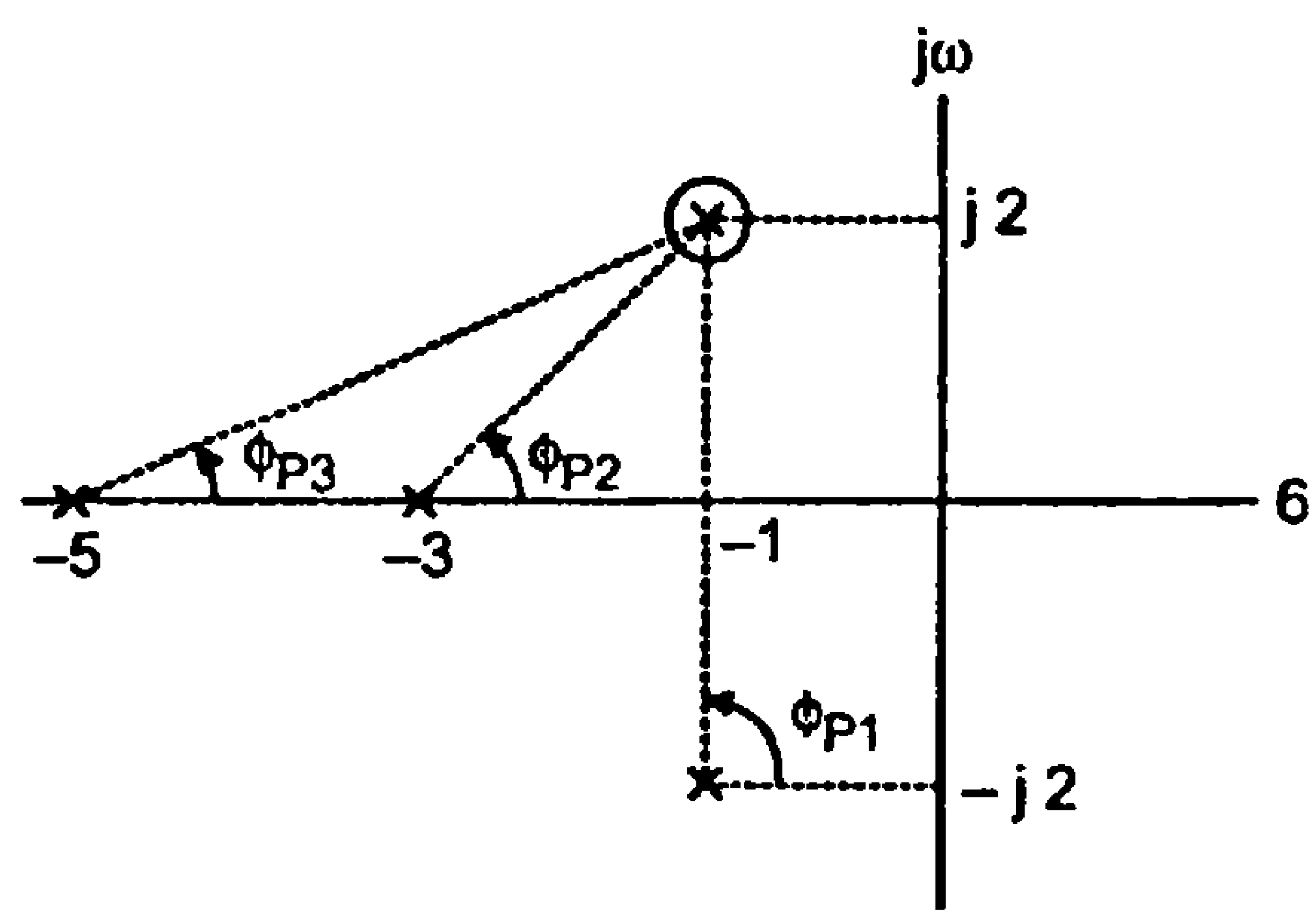
Step 7 : Angle of departure

Consider, $s = -1 + j2$

$$\phi_d = 180° - \phi \quad \text{where} \quad \phi = \Sigma\phi_P - \Sigma\phi_Z$$

$$\Sigma\phi_P = 90° + \tan^{-1}\frac{2}{2} + \tan^{-1}\frac{2}{4} = 90° + 45° + 26.56° = 161.56°$$

$$\phi_d = 180° - 161.56° = 18.4° \quad \text{... at } -1 + j2 \text{ and } -18.4° \text{ ... at } -1 - j2$$



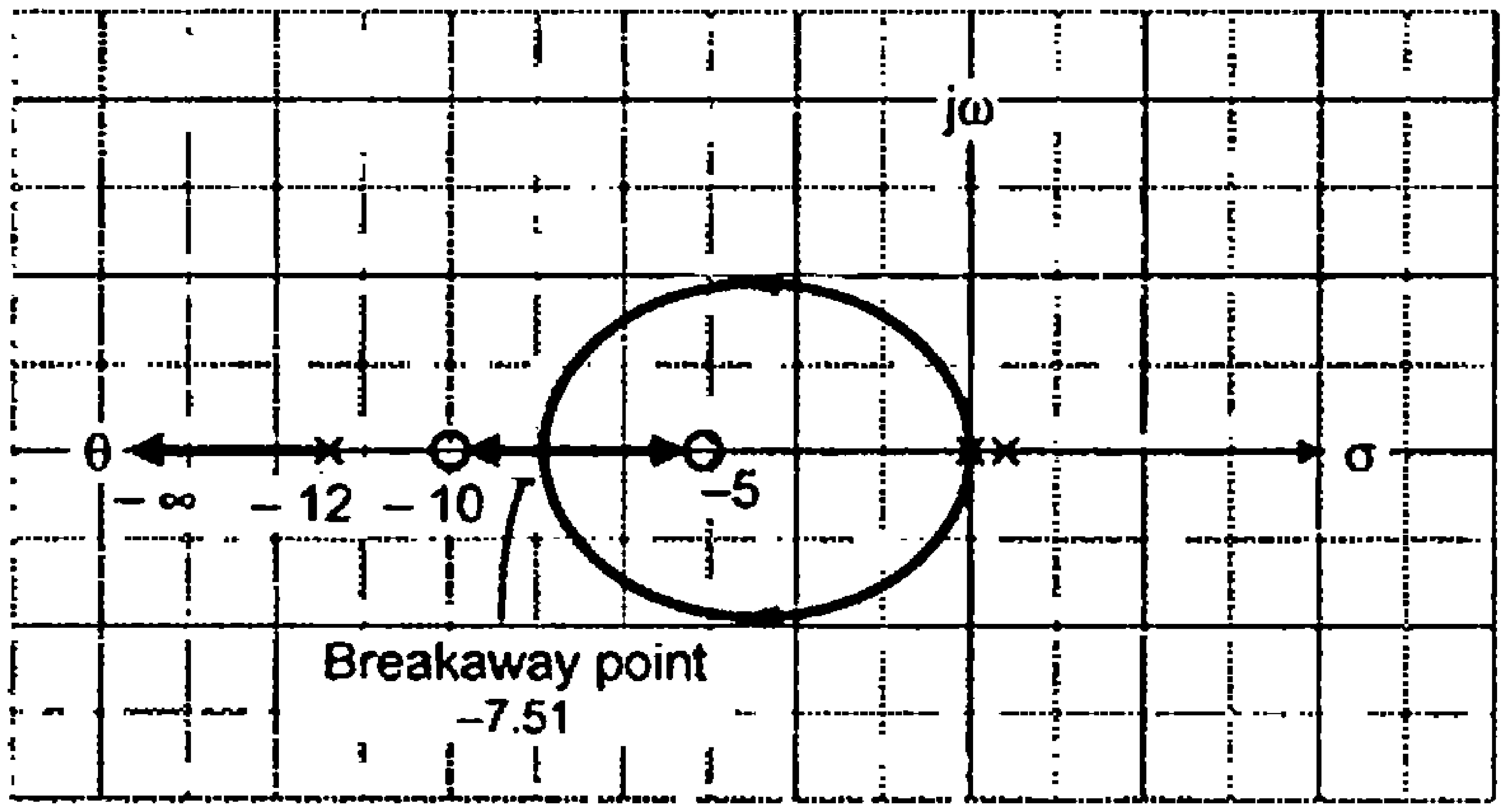
$$\therefore \quad K_2 = \frac{-[s^3 + 12s^2]}{0.24(s+10)\ (s+5)}$$

$$\frac{dK}{ds} = 2s^4 + 39s^3 + 178s^2 - 9s + 50 = 0$$

Approximate breakaway point = –7.51

Step 6 : No intersection with imaginary axis.

Step 7 : The complete root locus is as shown.



Example 9.31 : *Sketch the root locus of a system having*

$$G(s)H(s) = \frac{K}{s(s^2 + 4s + 1)}$$

i) What is the range of damping factor for the dominant poles?

ii) What is the angle of departure from complex open loop poles?

iii) For what values of K the system crosses the imaginary axis. **(M.U. : Nov.-1995)**

Solution : $G(s)H(s) = \dfrac{K}{s(s^2 + 4s + 1)}$

Step 1 : Number of poles = P = 3 and Number of zeros Z = 0

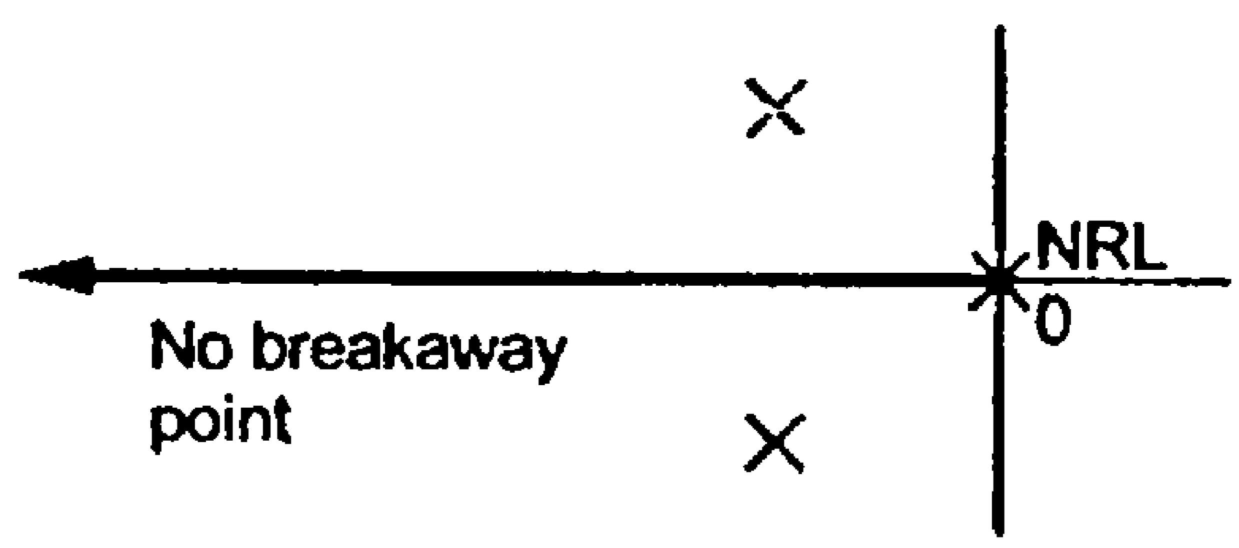
$s_1 = 0; \quad s_2 = -2 \pm j1.73$

N = P – Z = 3 branches of root locus will terminate at ∞

Starting points : 0, – 2 + j1.73, – 2 – j1.73

Terminating points : ∞, ∞ , ∞

Step 2 : Sections of real axis,

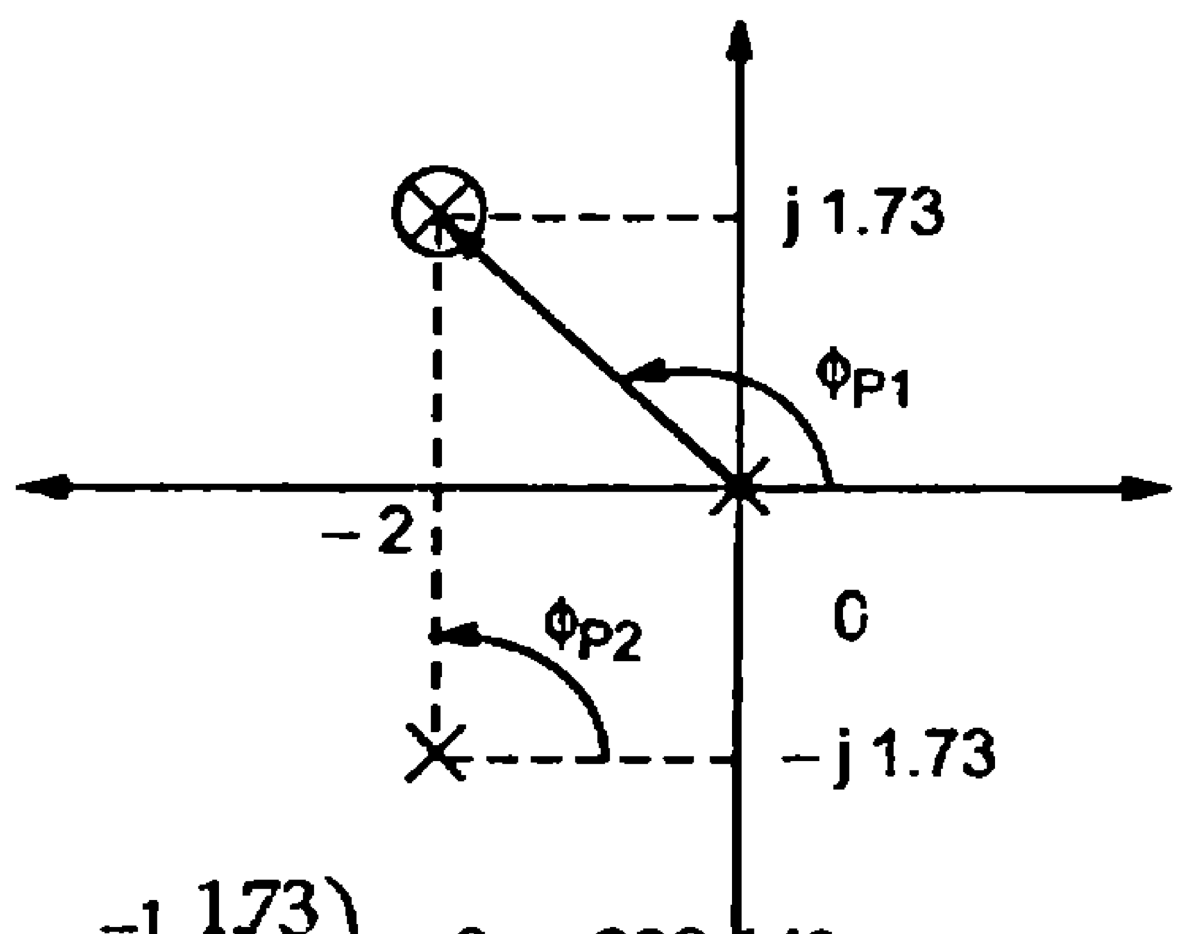


Step 3 : Angle of asymptotes $= \theta = \dfrac{(2q+1)180}{P-Z}$; q = 0, 1, 2.

$$\theta_1 = 60°;\quad \theta_2 = 180°;\quad \theta_3 = 300°$$

Step 4 : Centroid $= \dfrac{\Sigma \text{ R.P. of poles} - \Sigma \text{ R.P. of zeros}}{P-Z}$

$$= \frac{-2-2}{3} = -1.33$$



Step 5 : Breakaway point does not exist.

Step 6 : Angle of departure

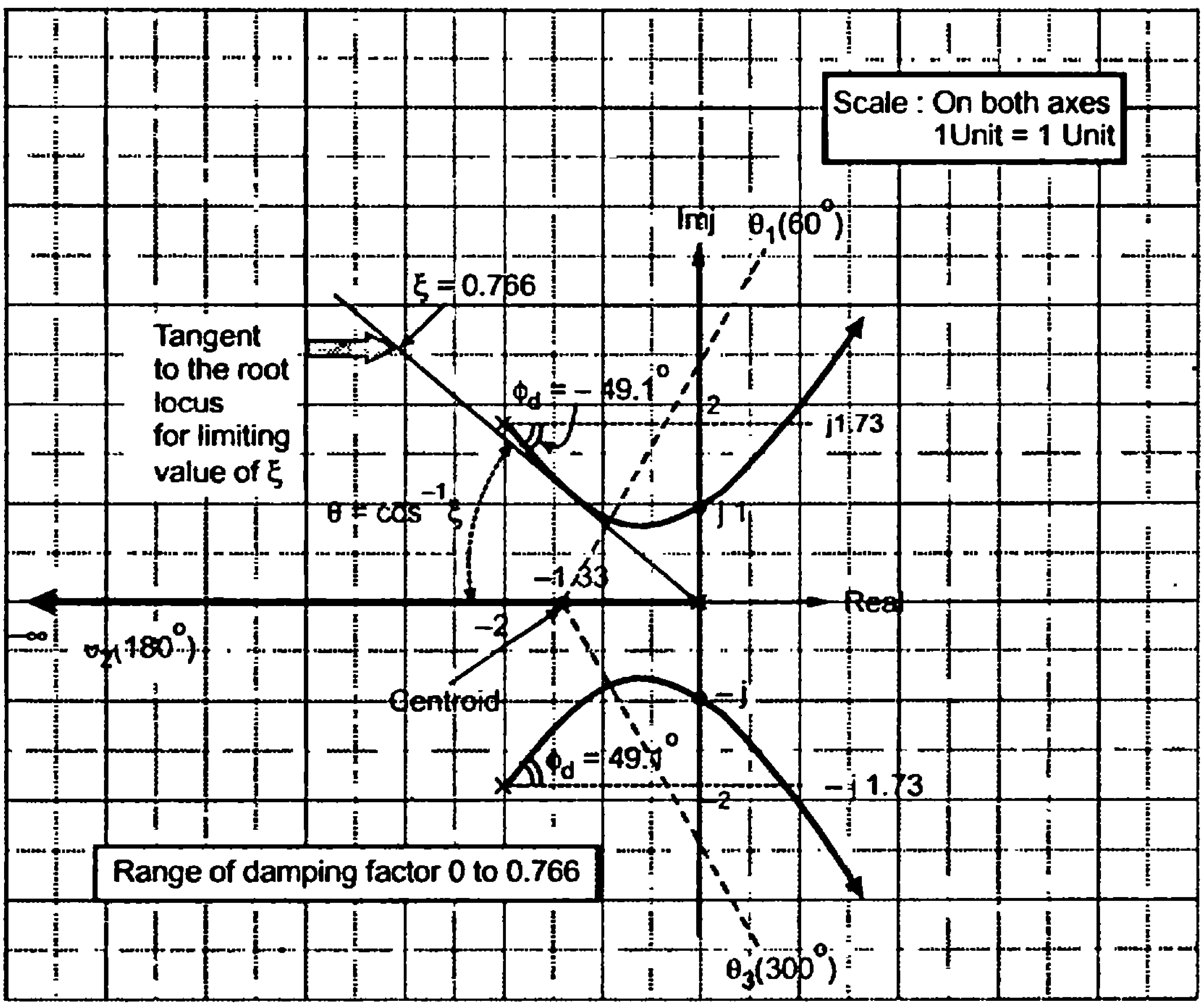
Consider $s = -1 + j\,1.73$

$$\phi_d = 180 - \phi$$

$$\phi = \Sigma\phi_P - \Sigma\phi_Z = 90° + \left(180° - \tan^{-1}\frac{1.73}{2}\right) - 0 = 229.14°$$

$$\phi_d = 180° - 229.14° = -49.14° \text{ at } -2 + j1.73$$

Similarly $\phi_d = +49.14°$ at $-2 - j1.73$



Step 7 : Intersection with imaginary axis. The characteristic equation is

$$s^3 + 4s^2 + s + K = 0$$

The Routh array is

s^3	1	1
s^2	4	K
s^1	$\frac{4-K}{4}$	0
s^0	K	0

$\therefore \quad K_{mar} = \mathbf{4}$

$$A(s) = 4s^2 + 4 = 0 \qquad s = \pm j1$$

To find range of ξ, draw from origin a line tangential to the root locus and measure angle made by that line with negative real axis.

The range of damping factor for dominant poles is,

$$\xi = \cos 90° \text{ to } \cos 40°$$

$\therefore$ Damping factor range = **0 to 0.766.**

Example 9.32 : *Sketch the root locus for a negative feedback control system having*

$$G(s) = \frac{K(s+1)}{s^2(s+2)\ (s+4)}.\quad H(s) = 1$$

Comment on the salient features of this system. **(M.U. : May-95)**

Solution : $G(s) = \dfrac{K(s+1)}{s^2(s+2)\ (s+4)}$

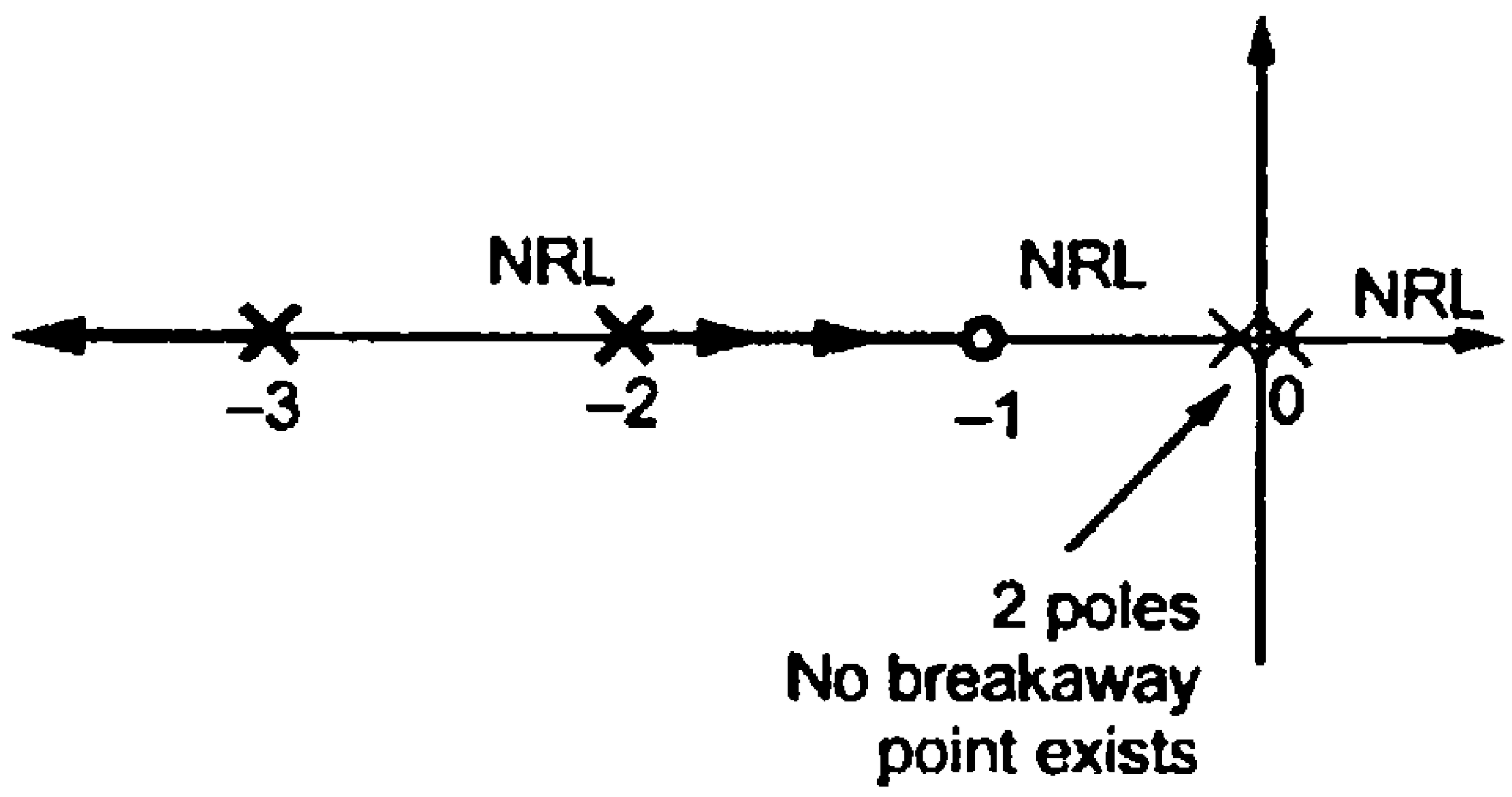
Step 1 : Number of poles = P = 4; Number of zeros = Z = 0

N = P − Z = 4 − 1 = 3 branches of root locus will terminate at ∞

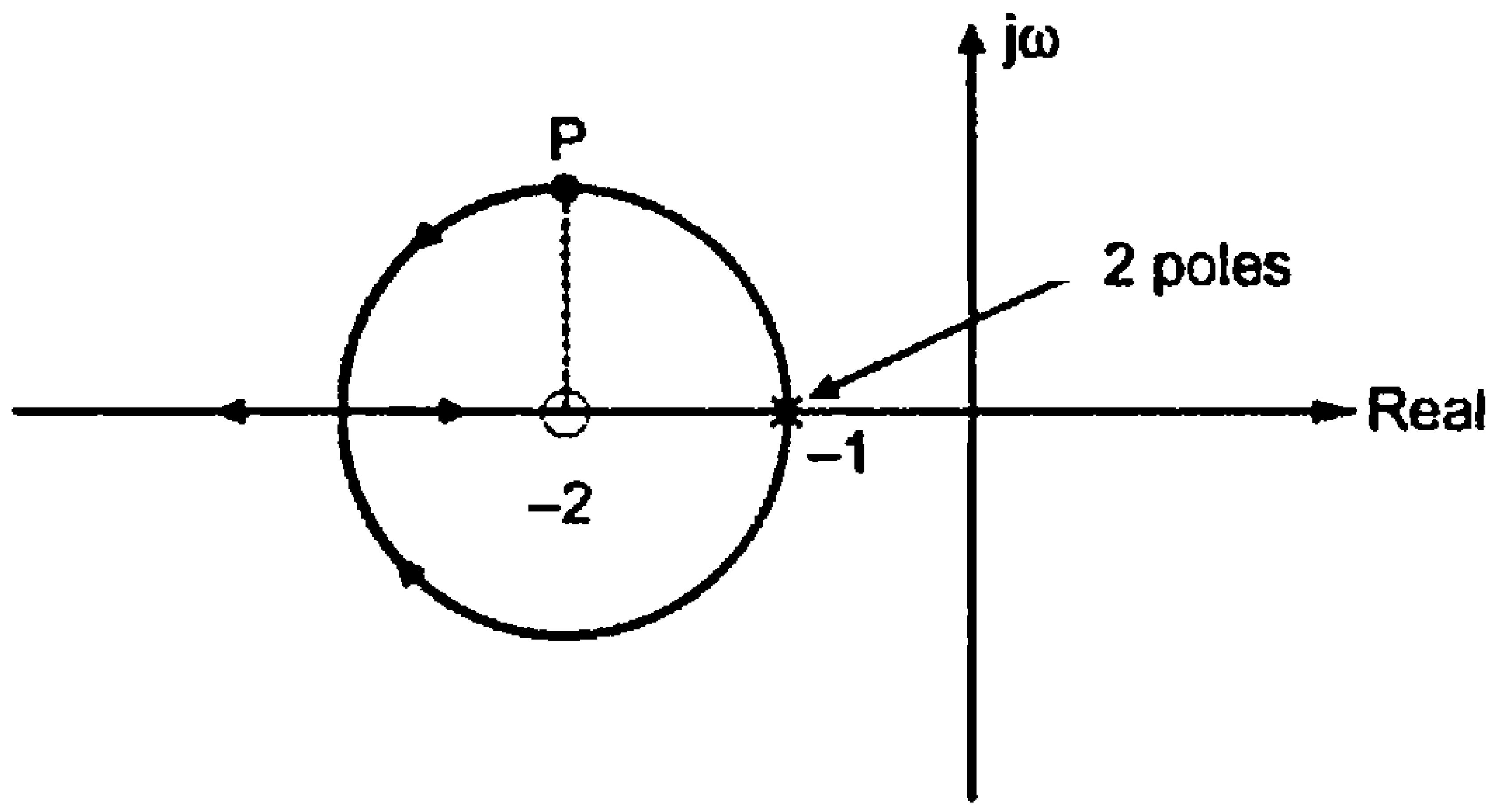
Starting points : 0, 0, − 2, − 4

Terminating points : ∞, ∞, ∞, − 1

Step 2 : Sections of real axis.



Example 9.33 : *Calculate the value of K at point P for root loci in figure.* **(M.U. : May-98)**



Solution : From the root locus shown we can write G(s)H(s) value as

$$G(s)H(s) = \frac{K(s+2)}{(s+1)^2}$$

and Point P $= -2 + j$

Now use the magnitude condition to get K at P.

$$|\,G(s)H(s)\,| \text{ at point P} = 1$$

$$\therefore \frac{K \times |-2+j+2|}{|-2+j+1|\;|-2+j+1|}$$

$$\therefore \quad \frac{K \times 1}{\sqrt{2} \times \sqrt{2}} = 1$$

$$\therefore \quad K = 2$$ **at point P.**

Example 9.34 : *The characteristics equation for the control system for an a.c. induction motor is (s + 2) (s + 4) (s + a) + K = 0. To achieve good dynamic behaviour, it is desired that the damping ratio $\xi = 0.5$ and that the natural frequency $\omega_n = 4$, determine a and K.* **(M.U. : Dec.-97)**

Solution : As $\xi = 0.5$ and $\omega_n = 4$ and system is third order hence the dominant roots of characteristic equation are the roots of $s^2 + 2\xi\,\omega_n s + \omega_n^2 = 0$

i.e. $$s^2 + 2 \times 0.5 \times 4\,s + (4)^2 = 0$$

i.e. $$s^2 + 4s + 16 = 0$$

i.e. $$s = -2 \pm j3.464$$

Now the G(s)H(s) can obtained from the given characteristic equation by comparing it with 1 + G(s)H(s) = 0

$$\therefore \quad 1+\frac{K}{(s+2)(s+4)(s+a)} = 0$$

$$\therefore \quad G(s)H(s) = \frac{K}{(s+2)(s+4)(s+a)}$$

Now at point s = –2 + j3.464, as it is on the root locus

$$|\ G(s)H(s)\ | = 1$$

This is according to the magnitude condition.

$$\therefore \quad \frac{|K|}{|-2+j3.464+2|\ |-2+j3.464+4|\ |-2+j3.464+a|} = 1$$

$$\therefore \quad \frac{K}{3.464\times 4\times\sqrt{(a-2)^2+(3.464)^2}} = 1$$

Squaring both sides

$$5.1984\times 10^{-3}\ K^2 = (a-2)^2+11.8336$$

$$\therefore \quad K^2 = 192.367\ [a^2-4a+4+11.8336]$$

$$\therefore \quad K^2 = 192.367\,a^2-769.467\,a+3045.86 \qquad \text{... (1)}$$

Now use the angle condition

$$\angle G(j\omega)H(j\omega) \text{ at } s = -2+j\,3.464 = -180^\circ$$

$$\therefore \frac{\angle K+j0}{\angle(-2+j3.464+2)\ \angle-2+j3.464+\ \angle 4-2+j3.464+a} = -180^\circ$$

$$\therefore \quad \frac{0^\circ}{90^\circ+60^\circ+\tan^{-1}\left(\frac{3.464}{a-2}\right)} = -180^\circ$$

$$\therefore \quad -90^\circ-60^\circ-\tan^{-1}\left(\frac{3.464}{a-2}\right) = -180^\circ$$

$$\therefore \quad -\tan^{-1}\left(\frac{3.464}{a-2}\right) = -30^\circ$$

$$\therefore \quad \frac{3.464}{a-2} = \tan 30$$

$$\therefore \quad \frac{3.464}{a-2} = 0.5773$$

$\therefore \quad a - 2 = 6$

$\therefore \quad a = 8$

Substituting in (1) we get

$$K^2 = 192.367 \times 64 - 769.467 \times 8 + 3045.86$$

$$\therefore \quad K^2 = 9201.612$$

$$\therefore \quad K = 95.92$$

$\therefore$ For $\xi = 0.5$ and $\omega_n = 4$ the values of constants are

$$K = 95.92 \text{ and } a = 8$$

Example 9.35 : *Given* $G(s) = \frac{K(s+2)}{s(s+1)}$. *Plot root locus for the above transfer function.*

i) Find breakaway and entry points on real axis.

ii) Find gain and roots when the real part of complex roots is located at – 2.

(M.U. : May - 2003)

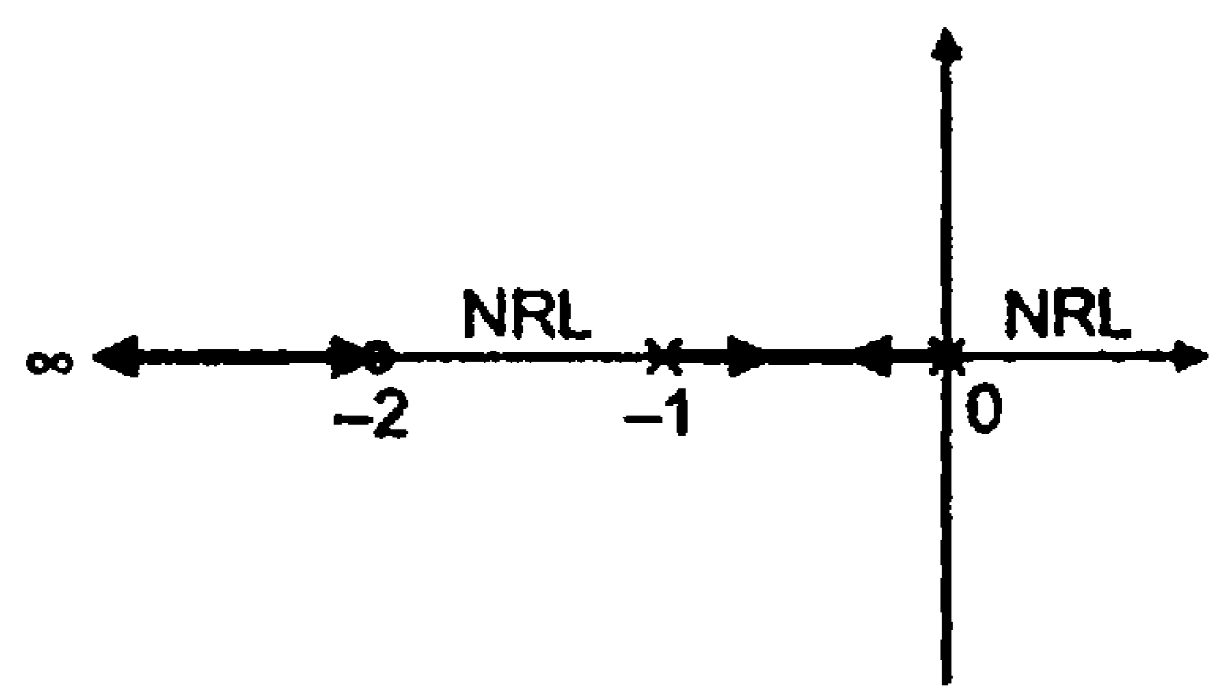
Solution : Step 1 : P = 2, Z = 1, N = P = 2, Branches approaching to ∞ = 1

Starting points = 0, –1

Terminating points = –2, ∞

Step 2 : The section of real axis on which root locus exists is shown in the figure

Two breakaway points possible, one between 0 and –1 while other to the left of s = –2.



Step 3, 4 : Only one branch approaching to ∞ along negative real axis so no asymptote and centroid is required.

Step 5 : Breakaway point,

$$1 + G(s)H(s) = 0$$

$$\therefore \quad 1 + \frac{K(s+2)}{s(s+1)} = 0$$

$$\therefore \quad K = \frac{-s^2 - s}{(s+2)} \quad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = \frac{(s+2)(-2s-1) - (-s^2 - s)(1)}{(s+2)^2} = 0$$

$$\therefore \quad \frac{K \times 1.375}{2.427 \times 1.7} = 1$$

$$\therefore \quad K = 3$$

So gain is 3 at these roots.

Example 9.36 : *Plot root locus of a unity feedback control system.*

$$G(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

Also find K marginal and K critical for damping $\xi = 1$ at breakaway point.

((M.U. : May-2004)

Solution : Step 1 : P = 5, Z = 1, P − Z = 4 branches approach to ∞

N = P = 5

Starting points : s = 0, − 5, − 6, − 1 + j, − 1 − j

Terminating points : s = − 3, ∞, ∞, ∞, ∞

Step 2 : Sections of real axis

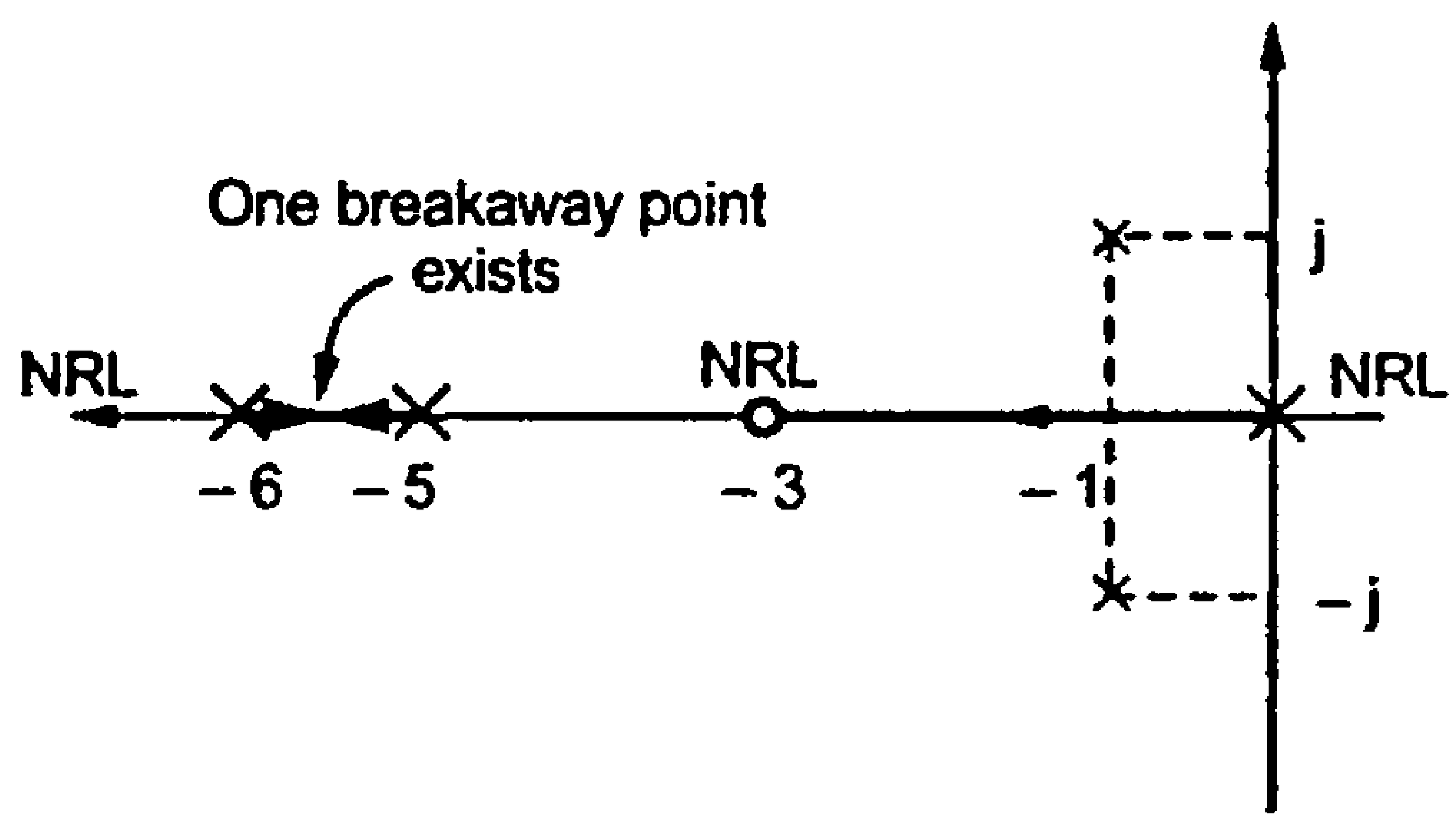


Fig. 9.30

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)\,180°}{P-Z}, \quad q = 0, 1, 2, 3.$$

$\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$

Step 4 :

$$\text{Centroid} = \frac{\sum \text{R.P. of open loop poles} - \sum \text{R.P. of open loop zeros}}{P-Z} = \frac{0-5-6-1-1-(-3)}{4}$$

$$= -2.5$$

Step 5 : Breakaway point

$$1+G(s)H(s) = 1+\frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} = 0$$

$$\therefore \quad s(s+5)(s+6)(s^2+2s+2)+K(s+3) = 0$$

$$\therefore \quad K = \frac{-s^5-13s^4-54s^3-82s^2-60s}{(s+3)} \quad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = \frac{(s+3)(-5s^4-52s^3-162s^2-164s-60)-(-s^5-13s^4-54s^3-82s^2-60s)(1)}{(s+3)^2}$$

$$\therefore \quad \frac{dK}{ds} = 0 \text{ gives } -4s^5-54s^4-264s^3-568s^2-492s-180 = 0$$

By trial and error method the breakaway point is **s = –5.525.**

From (1), K = **+11.718** at the breakaway point.

Step 6 : Intersection with imaginary axis

The characteristic equation is,

$s^5+13s^4+54s^3+82s^2+s(60+K)+3K = 0$

s^5	1	54	60 + K
s^4	13	82	3K
s^3	47.6923	$\frac{780+10K}{13}$	0
s^2	$\frac{3130.7686-10K}{47.6923}$ → X	3K	0
s^1	$\frac{\left(\frac{780+10K}{13}\right)X-143.0769K}{X}$	0	
s^0	3K		

For K_{mar}, equate coefficient of s^1 to zero.

$$\therefore \quad \left(\frac{780+10K}{13}\right)\left(\frac{3130.7686-10K}{47.6923}\right)-143.0769\,K = 0$$

$$\therefore \quad -100K^2-65199.977\,K+2441999.508 = 0$$

i.e. $$K^2+625\,K-24420 = 0$$

Solution : The given G(s)H(s) is,

$$G(s)H(s) = \frac{K}{s(1+0.02s)(1+0.1s)} = \frac{K}{s\times 0.02(s+50)\times 0.1\times(s+10)}$$

$$= \frac{500K}{s(s+50)(s+10)} = \frac{K_1}{s(s+50)(s+10)} \text{ with } K_1 = 500\text{ K}$$

Step 1 : P = 3, Z = 0, N = P = 3, P – Z = 3 branches towards ∞

Starting points . s = 0, – 10, – 50

Terminating points : s = ∞, ∞, ∞

Step 2 : Sections of real axis shown in the Fig. 9.33.

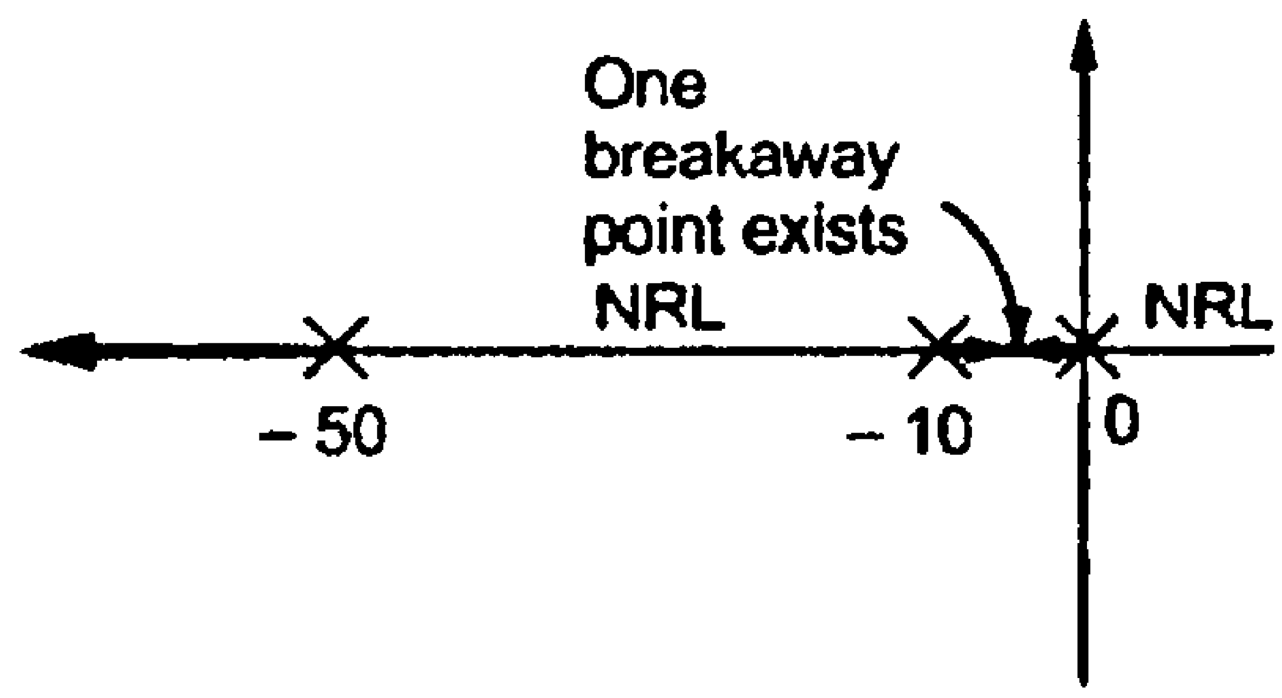


Fig. 9.33

Step 3 : Angles of asymptotes = $\frac{180°(2q+1)}{P-Z}$,

q = 0, 1, 2

$\therefore\ \theta_1 = 60°,\ \ \theta_2 = 180°,\ \theta_3 = 300°$

Step 4 : Centroid = $\frac{\sum \text{R.P. of open loop poles} - \sum \text{R.P. of open loop zeros}}{P-Z}$

$$= \frac{0-10-50}{3} = -20$$

Step 5 : Breakaway point

$$1+G(s)H(s) = 1 + \frac{K_1}{s(s+50)(s+10)} = 0$$

$$\therefore\quad s^3+60s^2+500s+K_1 = 0 \qquad \text{i.e. } K_1 = -s^3-60s^2-500s \qquad \text{... (1)}$$

$$\therefore\quad \frac{dK_1}{ds} = -3s^2-120s-500 = 0 \text{ i.e. } s^2+40s+166.667 = 0$$

Solving, s = – 4.7247, 35.2752

Thus s = – 4.7247 is valid breakaway point at which the value of K_1= + 1128.4510, using (1) . But K_1 = 500 K hence,

K = +2.2569 at breakaway point s = – 4.7247.

The co-ordinates of Q (– 4 + j 9.25). Using magnitude condition,

$$\left.\frac{K_1}{|s||s+10||s+50|}\right|_{At\ Q} = 1 \qquad \text{i.e.} \qquad \frac{K_1}{|-4+j\,9.25||6+j\,9.25||46+j9.25|} = 1$$

$$\therefore \quad K_1 = 10.0778 \times 11.0255 \times 46.9208 = 5213.5019$$

$$\therefore \quad K = \frac{K_1}{500} = \mathbf{10.427} \text{ for } \xi = 0.4$$

Example 9.38 : *Plot root locus :*

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

Hence obtain :

i) K for ξ = 0.707, ii) K for ξ = 0.866, iii) K for ξ = 1. **(M.U. : May-2005, Dec.-2006)**

Solution : The open loop poles are located at,

$$s = 0, -4 \text{ and } \frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j4$$

Step 1 : P = 4, Z = 0, N = P = 4, P – Z = 4 branches approaching to ∞.

Starting points, s = 0, – 4, –2 + j4 , –2 – j4

Terminating points : s = ∞, ∞, ∞, ∞

Step 2 : Sections of real axis shown in the Fig. 9.35.

One breakaway point between 0 and –4 exists.

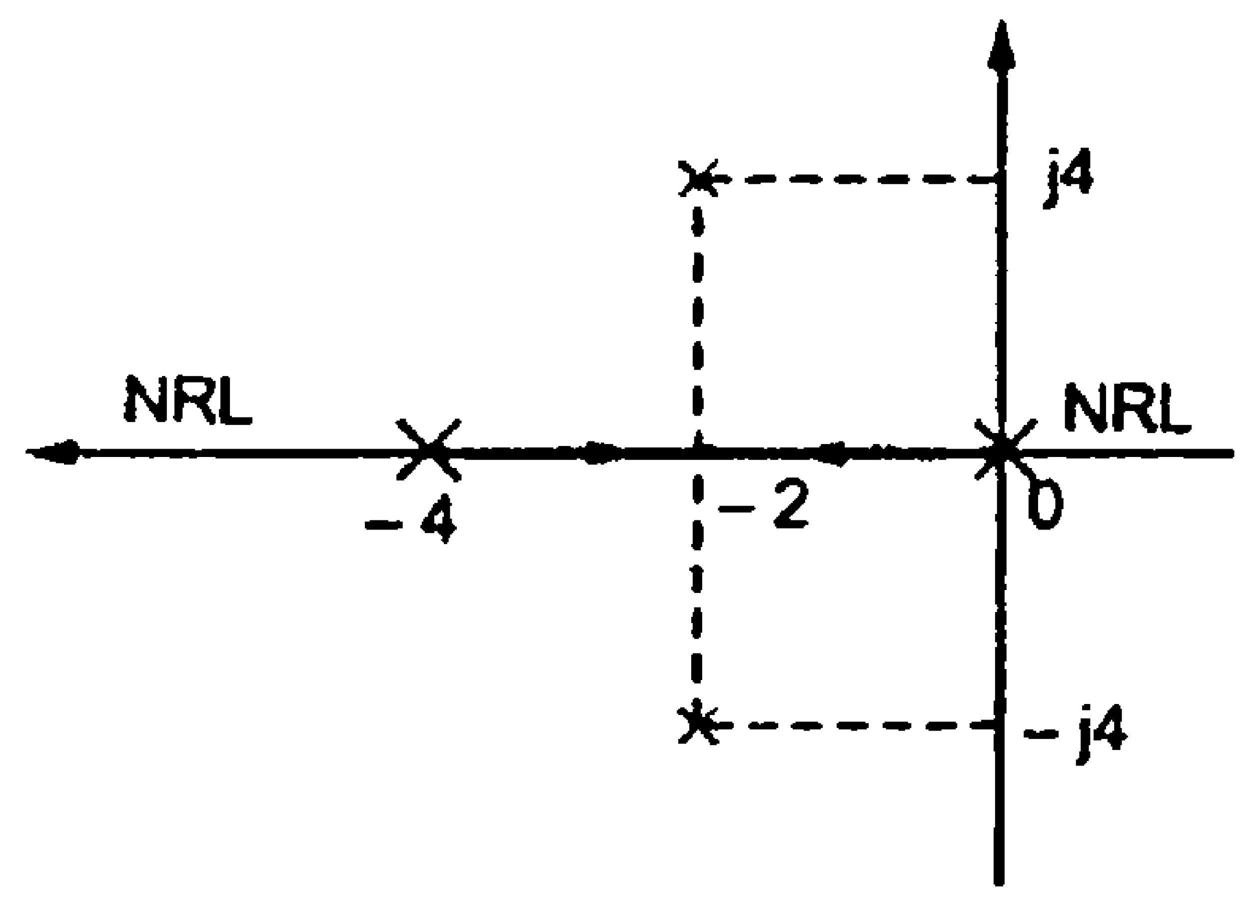


Fig. 9.35

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)\,180°}{P-Z}, \; q = 0, 1, 2, 3$$

$$\therefore \theta_1 = 45°, \; \theta_2 = 135°, \theta_3 = 225°, \; \theta_4 = 315°$$

Step 4 : Centroid = $\dfrac{\sum \text{R.P. of open loop poles} - \sum \text{R.P. of open loop zeros}}{P-Z}$

$$= \frac{0-4-2-2}{4} = \mathbf{-2}$$

Step 5 : Breakaway point

The characteristic equation is 1 + G(s)H(s) = 0

$$\therefore \quad 1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

$$\therefore \quad s^4+8s^3+36s^2+80s+K = 0$$

$$\therefore \quad K = -s^4-8s^3-36s^2-80s \qquad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = -4s^3-24s^2-72s-80 = 0 \text{ i.e. } s^3+6s^2+18s+20 = 0$$

Solving, $s = -2,\ -2 \pm j\,2.45$

Key Point : *All three are valid breakaway points. The validity of $-2 \pm j\,2.45$ can be confirmed using angle condition. (Refer Ex. 9.21)*

While the value of K at $-2 \pm j\,2.45$ can be obtained using magnitude condition.

$$|G(s)H(s)|_{s=-2+j2.45} = 1$$

$$\therefore \quad \left|\frac{K}{s(s+4)(s+2+j4)(s+2-j4)}\right|_{s=-2+j2.45} = 1$$

$$\therefore \quad \frac{|K|}{|-2+j2.45|\,|-2+j2.45+4|\,|-2+j2.45+2+j4|\,|-2+j2.45+2-j4|} = 1$$

$$\therefore \quad K = 3.1626\times 3.1626\times 6.45\times 1.55 = 100$$

As $-2-j2.45$ is complex conjugate, K = **100** for $s = -2-j2.45$ also.

For $s = -2$, $K = -(-2)^4-8(-2)^3-36(-2)^2-80(-2) = +\,\mathbf{64}$ ... from (1)

Step 6 : Intersection with imaginary axis

$$s^4+8s^3+36s^2+80s+K = 0 \qquad \text{... Characteristic equation}$$

s^4	1	36	K
s^3	8	80	0
s^2	26	K	0
s^1	$\frac{2080-8K}{26}$	0	
s^0	K		

From the row of s^1, $2080-8K = 0$

$\therefore K_{mar} = \frac{2080}{8} = \mathbf{260}$

For this K_{mar}, $A(s) = 0$ is,

$26s^2 + K_{mar} = 0$

$\therefore\ s^2 = \frac{-K_{mar}}{26} = -10$

$\therefore\ s = \pm\, j3.162$

These are intersection points with imaginary axis.

Step 7 : Angle of departure

Consider complex pole at $-2+j4$.

From the geometry,

$\phi_{P1} = 180° - \tan^{-1}\frac{4}{2} = +116.56°,\ \phi_{P2} = 90°$

$\phi_{P3} = \tan^{-1}\frac{4}{2} = +63.43°$

$\sum\phi_P = 270°,\ \sum\phi_Z = 0°$

$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 270°$

$\therefore \quad \phi_d = 180° - \phi = 180° - 270°$

$= -90°$ ... at $-2 + j4$

$\phi_d = +90°$ ... at $-2 - j4$

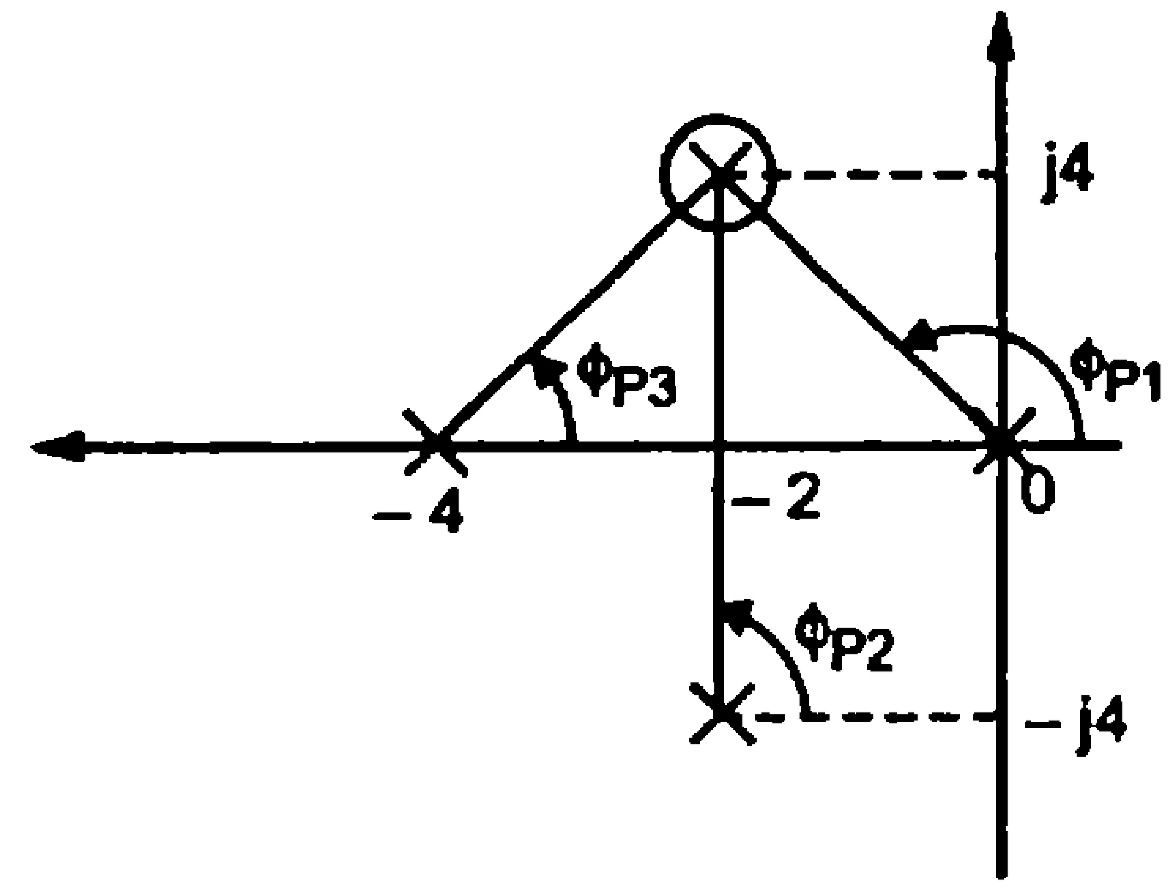


Fig. 9.36

Step 8 : Draw the root locus to the scale on the graph paper as shown.

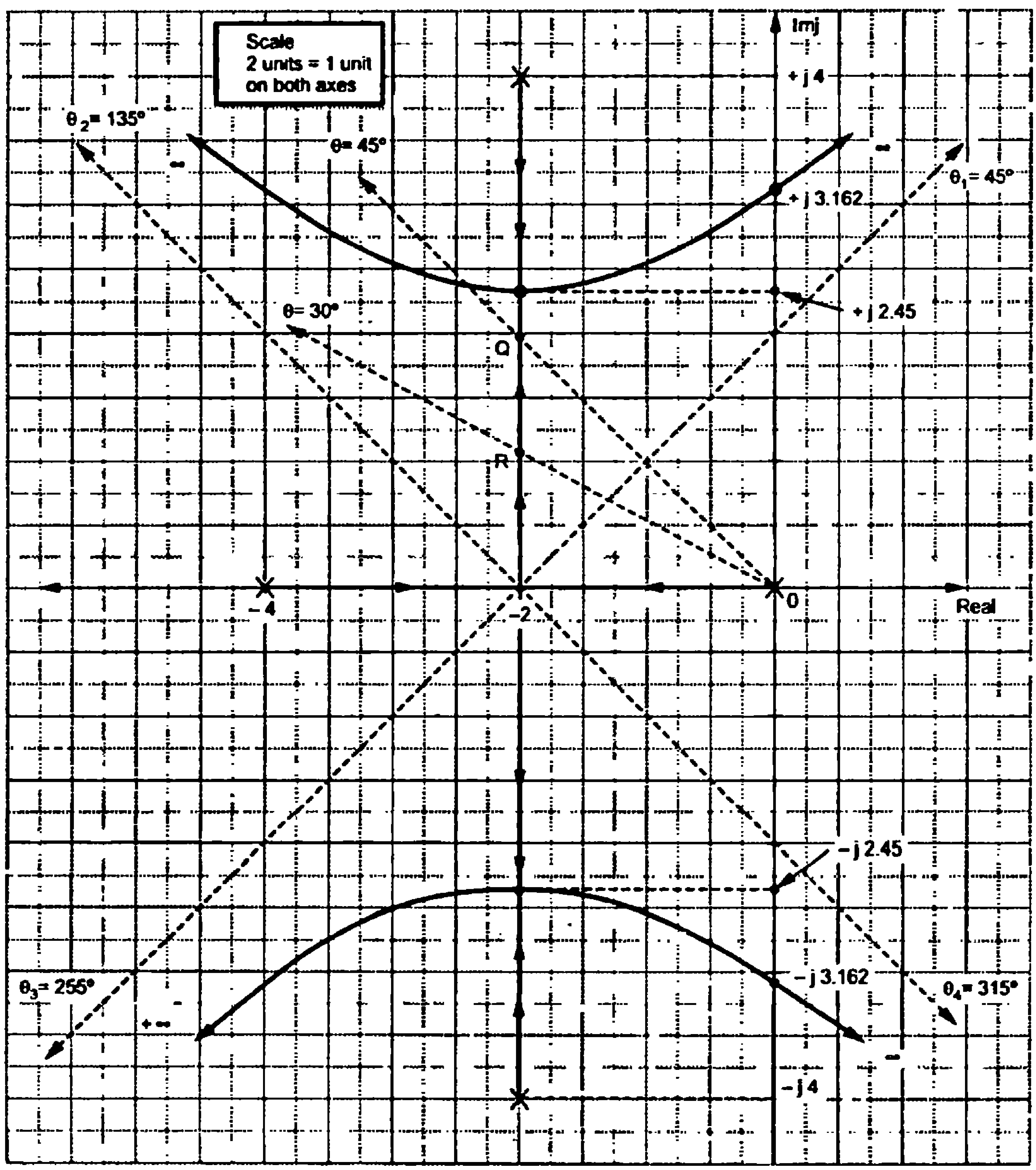


Fig. 9.37

$$\therefore \qquad \cos^{-1}\xi = 45° \text{ i.e. } \xi = 0.7071 \text{ for } K = 1.33$$

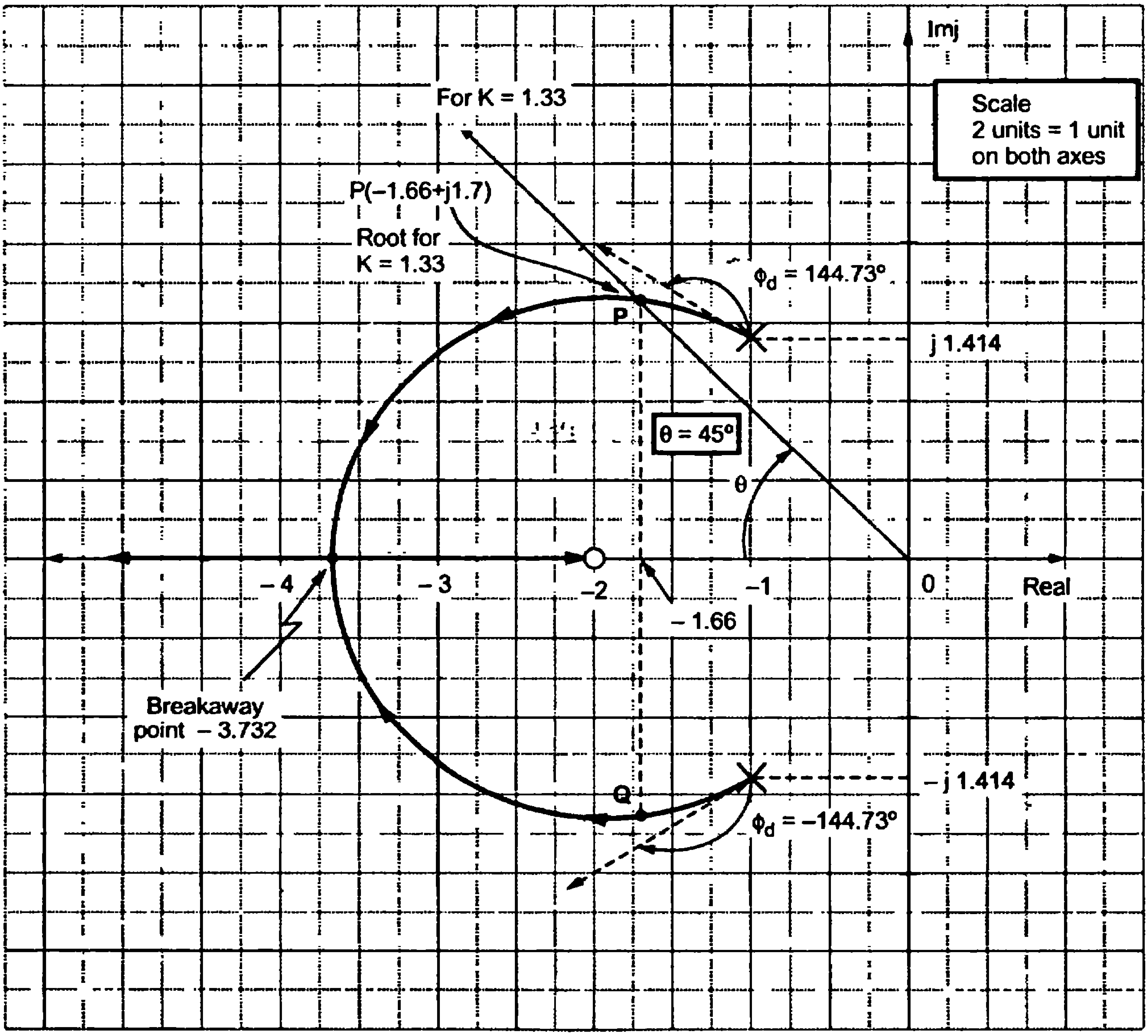


Fig. 9.40

➡ **Example 9.40 :** *Sketch the root locus of the system shown in the Fig. 9.41 and determine range of K for stability.* **(M.U. : May - 2006)**

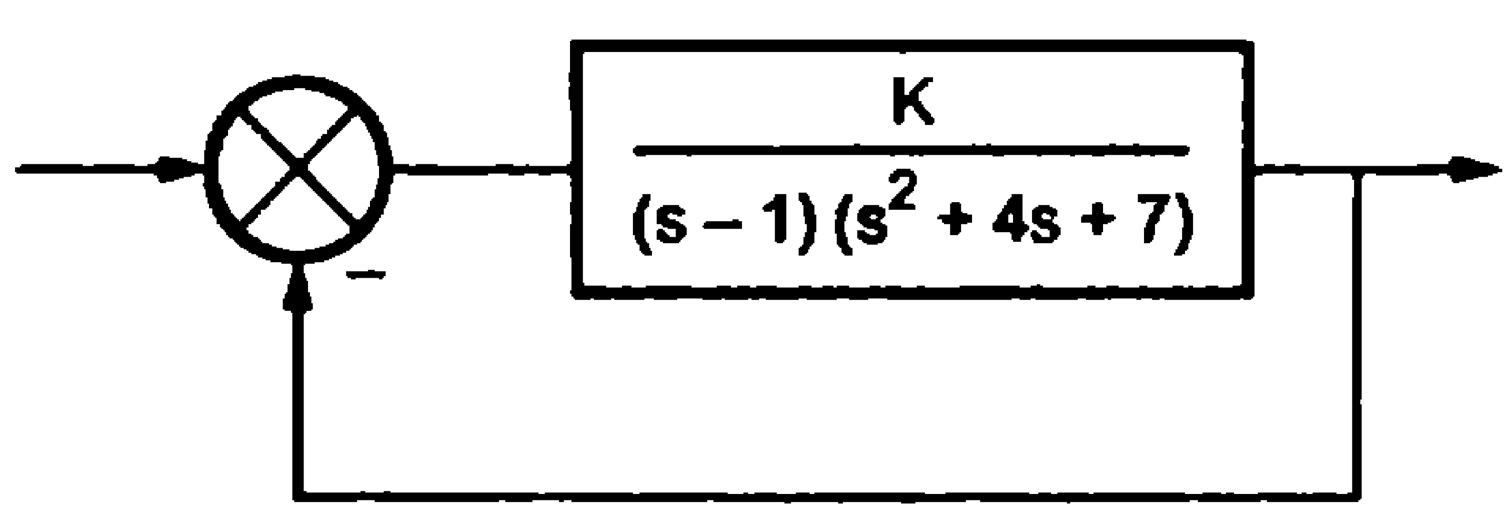


Fig. 9.41

Solution : $G(s) = \dfrac{K}{(s-1)(s^2+4s+7)}$

Step 1 : P = 3, Z = 0, N = P = 3, P − Z = 3 branches approaching to ∞ .

Starting points : 1, −2 + j 1.732, −2 − j1.732

Terminating points : ∞, ∞, ∞

Step 2 : Sections of real axis shown in the Fig. 9.42.

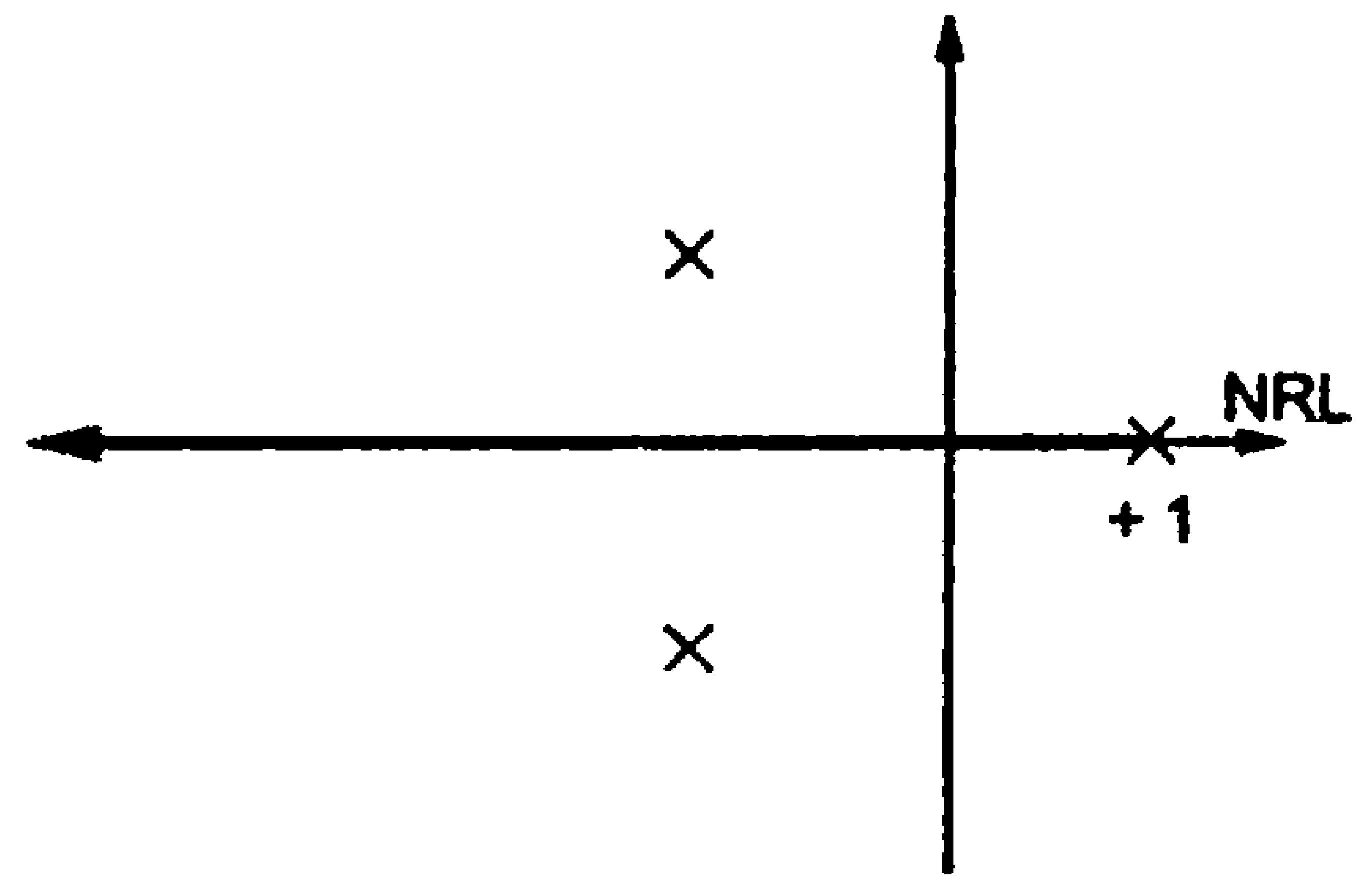


Fig. 9.42

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z},\ q = 0, 1, 2$$

$$\theta_1 = 60^\circ,\ \theta_2 = 180^\circ, \theta_3 = 300^\circ$$

Step 4 : Centroid $= \dfrac{\sum \text{R.P. of open loops poles} - \sum \text{R.P. of open loops zeros}}{P-Z}$

$$= \frac{+1-2-2}{3} = -1$$

Step 5 : Breakaway point

The characteristic equation is 1+ G(s) = 0

$$\therefore \quad 1 + \frac{K}{(s-1)(s^2+4s+7)} = 0 \text{ i.e. } s^3+3s^2+3s-7+K = 0$$

$$\therefore \quad K = -s^3-3s^2-3s+7 \qquad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = -3s^2-6s-3 = 0 \text{ i.e. } s^2+2s+1 = 0 \text{ i.e. } (s+1)^2 = 0$$

$$\therefore \quad s = -1, -1$$

From (1), $K = -(-1)^3 - 3(-1)^2 - 3(-1) + 7 = +8$ at $s = -1$

As K is positive , s = − 1 is valid breakaway point. As there are two breakaway points, all three branches meet at s = − 1 and then breakaway to ∞ along the asymptotes.

Step 6 : Intersection with imaginary axis

s^3	1	3
s^2	3	K − 7
s^1	$\frac{16-K}{3}$	0
s^0	K − 7	

From s^1, $16 - K = 0$

$\therefore \quad K_{mar} = +16$

$A(s) = 3s^2 + (K-7) = 0$

$\therefore \quad s^2 = -\frac{9}{3} = -3$

$\therefore \quad s = \pm j\sqrt{3} = \pm j\,1.732$

These are intersection with imaginary axis.

From the Routh's array in step 6, it can be seen that $K - 7 > 0$ to keep the coefficient of s^0 positive.

$\therefore \quad K > 7$ ... from row of s^0

While, $\quad K < 16$ ... from row of s^1

Hence $7 < K < 16$ is the range of K for stability.

Example 9.41 : *Sketch the root locus :* $G(s)H(s) = \dfrac{K}{s(s+6)(s^2+2s+2)}$.

Find K marginal, ω_{pc} *and comment on stability.* **(M.U. : May - 2007)**

Solution : The locations of poles are $s = 0, -6, -1 \pm j$

Step 1 : $P = 4$, $Z = 0$, $N = P = 4$, $P - Z = 4$ branches approaching to ∞.

Starting points : $s = 0, -6, -1+j, -1-j$

Terminating points : $s = \infty, \infty, \infty, \infty$

Step 2 : Sections of real axis

One breakaway point exists between $s = 0$ and $s = -6$.

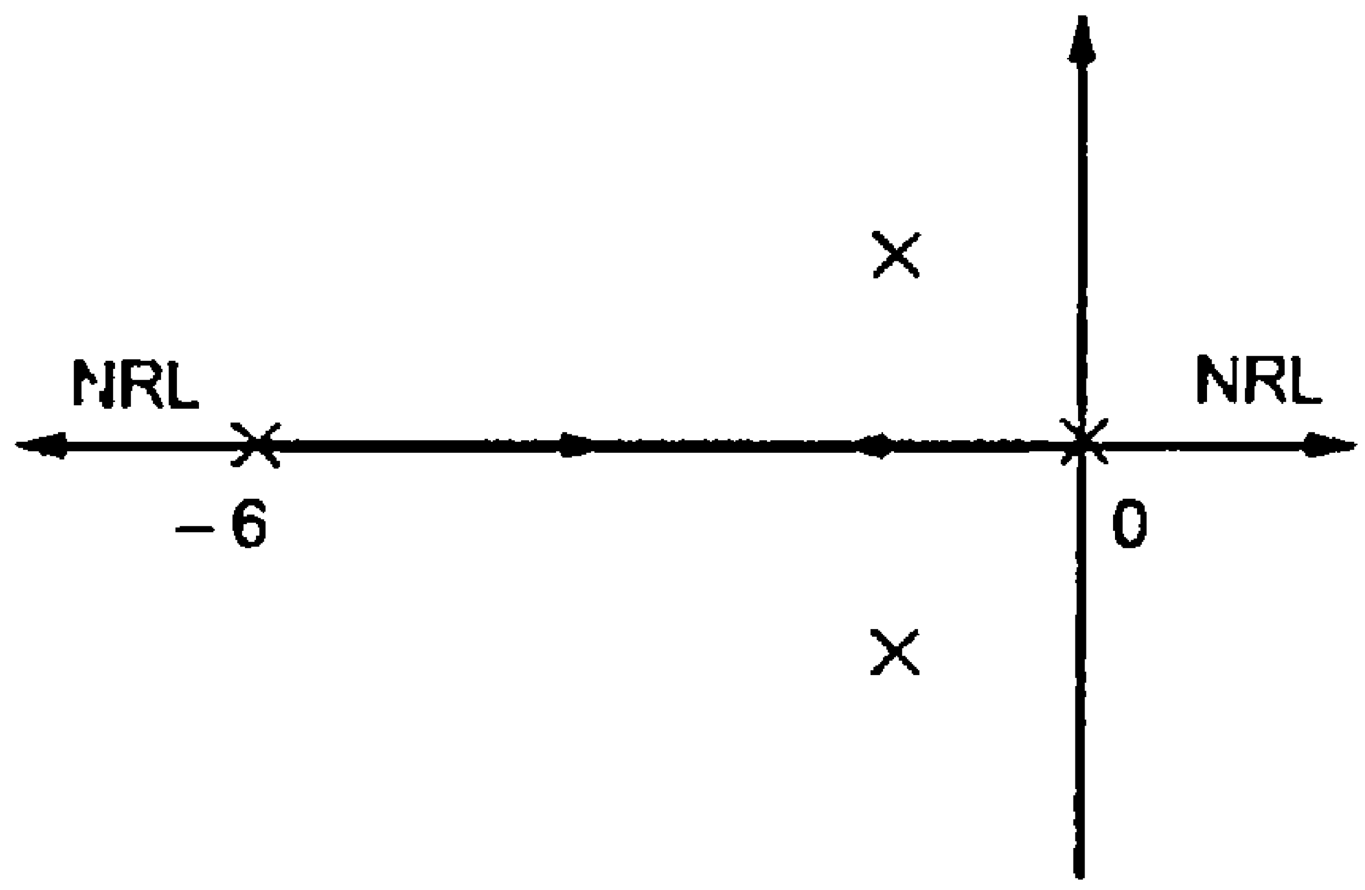


Fig. 9.45

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)\,180°}{P-Z}, \quad q = 0, 1, 2, 3$$

$$\theta_1 = 45°, \theta_2 = 135°, \theta_3 = 225°, \theta_4 = 315°$$

Step 4 : Centroid $= \dfrac{\sum \text{R.P. of open loop poles} - \sum \text{R.P. of open loop Zeros}}{P-Z}$

$$= \frac{0-6-1-1}{4} = -2$$

Step 5 : Breakaway point

The characteristic equation is $1 + G(s)H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s(s+6)(s^2+2s+2)} = 0$$

$$\therefore \quad s^4 + 8s^3 + 14s^2 + 12s + K = 0 \text{ i.e. } K = -s^4 - 8s^3 - 14s^2 - 12s \qquad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 24s^2 - 28s - 12 = 0 \text{ i.e. } s^3 + 6s^2 + 7s + 3 = 0$$

Solving, s = – 4.627 is a valid breakaway point.

From (1), K = + 89.9257 at the breakaway point.

Step 6 : Intersection with imaginary axis

From the characteristic equation,

s^4	1	14	K
s^3	8	12	0
s^2	12.5	K	
s^1	$\frac{150-8K}{12.5}$	0	
s^0	K		

From row of s^1, 150 – 8K = 0

For marginal K,

$$\therefore \quad K_{mar} = \frac{150}{8} = 18.75$$

$$A(s) = 12.5s^2 + K_{mar} = 0$$

$$\therefore \quad s^2 = \frac{-18.75}{12.5} = -1.5$$

$$\therefore \quad s = \pm j1.2247$$

These are intersection points with imaginary axis.

Step 7 : Angle of departure

consider the pole – 1 + j

From the geometry,

$$\phi_{P1} = 180° - \tan^{-1}\frac{1}{1} = 135°$$

$$\phi_{P2} = 90°$$

$$\phi_{P3} = \tan^{-1}\frac{1}{5} = 11.31°$$

$$\therefore \quad \sum\phi_P = 236.31°, \sum\phi_Z = 0°$$

$$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 236.31°$$

$$\therefore \quad \phi_d = 180° - \phi = -56.31° \quad \text{... for } -1+j$$

$$\phi_d = +56.31° \quad \text{... for } -1-j$$

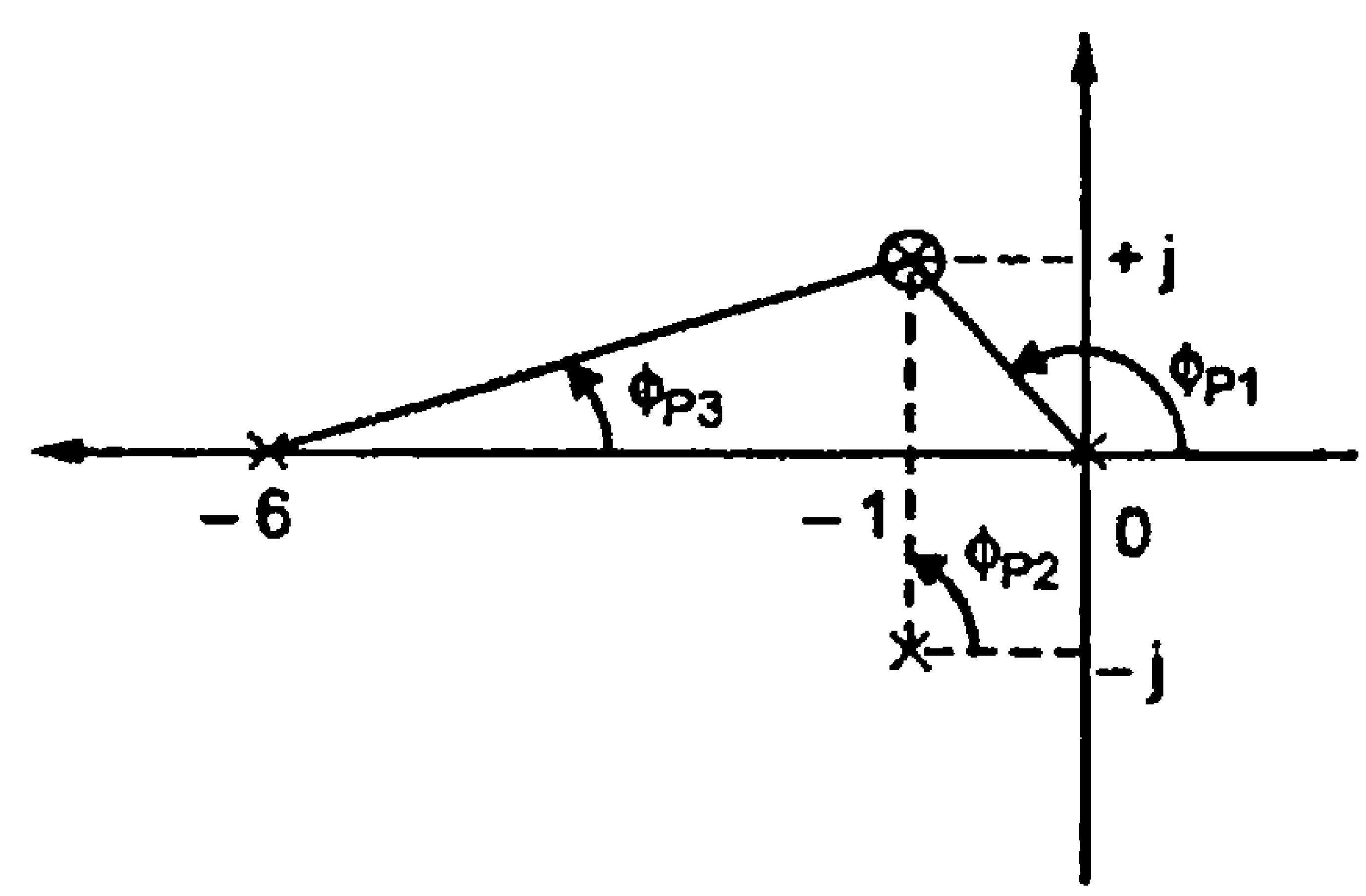


Fig. 9.46

Step 8 : The root locus is shown in the Fig. 9.47.

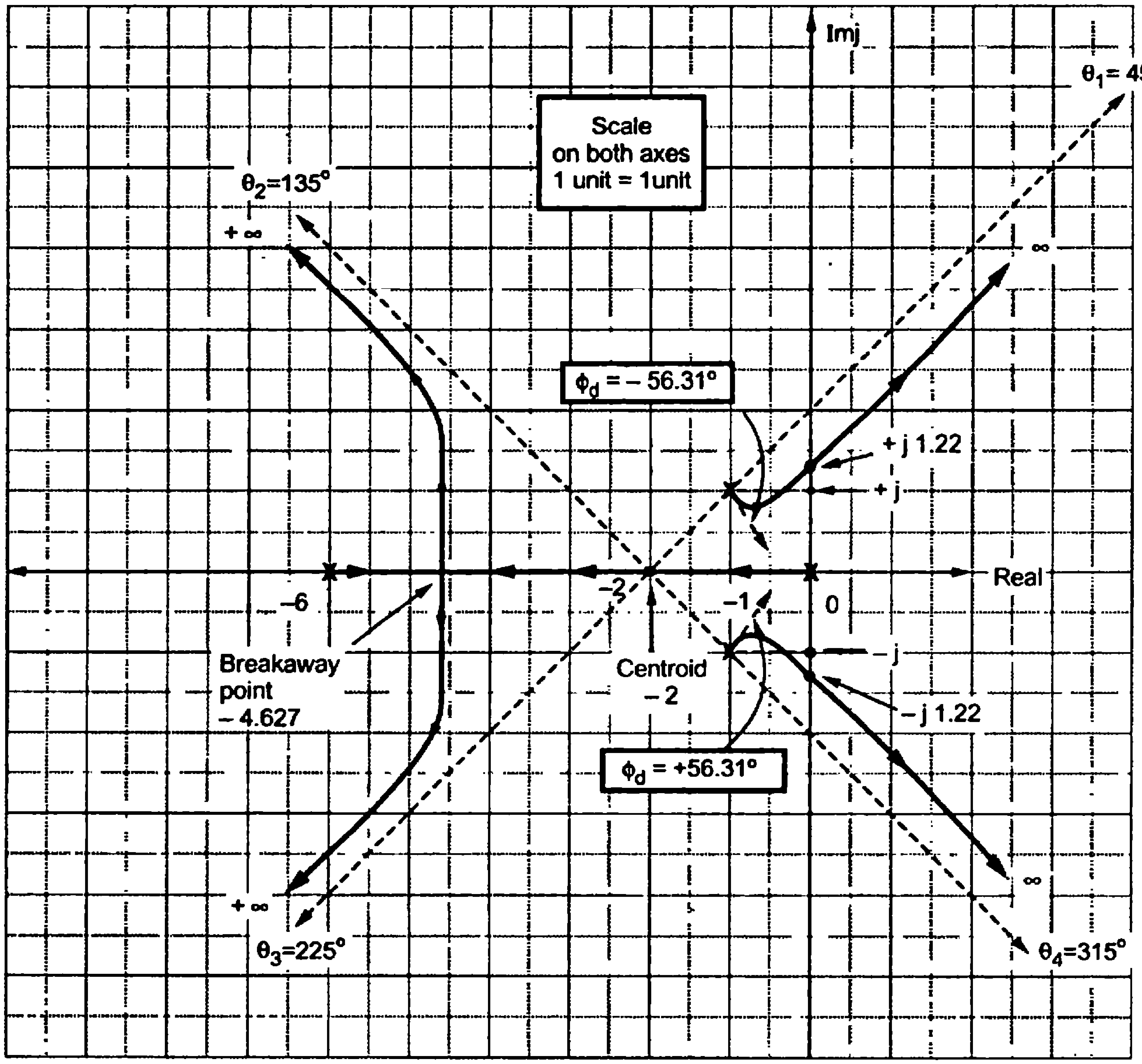


Fig. 9.47

K_{mar} = **18.75** and ω_{pc} = **1.2247** rad/sec

The system is stable for 0 < K < 18.75, marginally stable at K = 18.75 and unstable for K > 18.75.

15. *For* $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$, *sketch the root locus and obtain the value of K for* $\xi = 0.5$

(M.U. : Jan.-93)

(Ans. : K = 0.987)

16. *For* $G(s)H(s) = \dfrac{K(s^2+8s+3)}{s(s+2)(s^2+2s+101)}$, *calculate angles of departures at complex poles.*

(M.U. : May-98)

(Ans. : $\phi_d = 60°$**)**

17. *Show that the part of root locus of a system with*

$G(s)H(s) = \dfrac{K(s+3)}{s(s+2)}$ *is a circle having centre (– 3, 0) and radius at* $\sqrt{3}$.

18. *For a feedback control system*

$G(s)H(s) = \dfrac{K(s+4)}{s(s+2)(s+6)(s+8)}$.

Draw complete root locus. Mark salient points on it.

19. *Plot the root loci for the system having the loop transfer function*

$G(s)H(s) = \dfrac{K}{s(s+2)(s^2+4s+8)}$.

Mark salient points on the root loci.

10 Basics of Frequency Domain Analysis

10.1 Background

There are many time-domain methods to analyse control systems yet even after several decades, frequency response methods continue to be used for analysis and design. It should be remembered that during 1930 to 1940 when the design of several military systems were first undertaken in U.S.A. and Europe, the only methods available to analyse stability, were those given by Bode and Nyquist. These methods were used for analysis and design of feedback amplifiers. It was therefore natural to use these methods for control system analysis.

The Root Locus methods were developed during 1940-1950 and time response methods gained importance particularly due to simulation on digital computers, after 1960. The analog computer simulation was however used extensively during 1930-1950.

Thus, we see that historically, frequency response methods were used earlier than other methods for stability analysis and design, as compared to Root Locus methods and digital computer simulation. Literature on the frequency response methods is easily available.

These methods are simple to use and offer many advantages compared to time-domain methods. Some of the advantages are stated below.

10.2 Advantages of Frequency Domain Approach

The various advantages of frequency domain analysis are,

1) Without the knowledge of the transfer function, the frequency response of stable open loop system can be obtained experimentally.
2) These methods are easy to use for design of control systems and for finding absolute as well as relative stability of the system. Calculations are simple and methods of design are well tested.
3) When it is difficult to find transfer function of a given system by writing differential equations, the transfer function of the system can be determined practically in the laboratory by obtaining the frequency response of the system.
4) Frequency response tests are simple and can be made accurately by use of readily available signal generators and the precise measuring instruments.

5) Even if the system has moderate degree of non-linearity it can be depicted by an approximate transfer function. Stability analysis can be carried out and step-response can be predicted. Ofcourse it does not give exact results but then it does give the fairly good idea about stability without using costly, complicated and time consuming procedures.

6) Frequency response can be precisely applied to the system those do not have rational T.F. i.e. e^{-Ts} etc. Time domain methods can be applied to such elements only after making such elements rational (using certain approximation).

7) There is a close relation between frequency response of a system and its step response. If we know the frequency response, we can have a rough idea about the step response. Thus if we want a particular step response (peak overshoot, rise time etc.), we can translate the given requirements in the frequency domain. We may then design the system almost completely.

8) The apparatus required for obtaining frequency response is simple and inexpensive, and easy to use.

9) For difficult cases, such as conditionally stable systems, Nyquist Plot is probably the only method to analyse stability.

There are however certain limitations of frequency response methods. These are stated below.

10.3 Limitations of Frequency Response Methods

The various limitations of frequency response methods are,

1) Basically the methods can be applied only to linear systems. When these methods are applied to systems possessing some degree of non linearity, the result of analysis and design are not exact. Highly mathematically minded persons are not satisfied by such methods since the evaluation of results require experience and judgement regarding the degree of non-linearity.

2) The methods are considered some what 'old' and 'outdated' in view of extensive methods developed for digital computer simulation and modelling.

3) Even for linear systems, the estimation of the step response (or impulse response) from frequency response is usually carried out using the fact that a higher order system often behaves approximately as a second order system, when under damped. Often the step response is estimated using the formulae for second order system (even if the system is of third order or fourth order). Naturally the estimation is only approximate. This should not give an impression that there is no exact relation between frequency response and step (or impulse) response. There is exact relation, but the relationship is via Fourier transform methods, which involve extensive calculations and hence difficult to apply.

4) For an existing system, obtaining frequency response is possible only if the time constants are upto few minutes. However if the time constants are few hours or days, then the methods are not convenient to apply for practical testing.

5) Obtaining frequency response practically is fairly time consuming.

10.4 Conceptual Approach to Frequency Response

Let us define the term frequency response.

The steady state response of a system to a purely sinusoidal input is defined as frequency response of a system. In such method frequency of the input signal is to be varied over a certain range and the resulting response of system is to be studied. Such response is called frequency response.

10.4.1 Steady State Response to Sinusoidal Input : (Frequency Response)

Consider a linear time invariant system with its transfer function G(s) as shown.

Consider input r(t) to be purely sinusoidal.

$$r(t) = A \sin \omega t$$

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

Fig. 10.1

While $$G(s) = \frac{N(s)}{(s+s_1)(s+s_2)....(s+s_n)}$$

$\therefore$ Output $$C(s) = R(s) \cdot G(s) = \frac{A\omega}{(s^2+\omega^2)} \cdot \frac{N(s)}{(s+s_1)(s+s_2)....(s+s_n)}$$

Assuming system to be stable, we can write,

$$C(s) = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + ... + \frac{b_n}{s+s_n}$$

where a is constant while $\bar{a}$ is complex conjugate of a. Taking inverse Laplace,

$$\therefore \quad c(t) = ae^{-j\omega t} + \bar{a}\, e^{+j\omega t} + b_1 e^{-s_1 t} + ... + b_n e^{-s_n t}$$

Steady state of the output is $\lim_{t \to \infty} c(t)$. Then all the terms from c(t) like $b_1 e^{-s_1 t}, ... b_n e^{-s_n t}$ will vanish.

$$\therefore \quad C_{ss}(t) = a\, e^{-j\omega t} + \bar{a}\, e^{+j\omega t}$$

Now a and $\bar{a}$ can be determined as follows.

$$a = G(s) \cdot \frac{A\omega}{s^2+\omega^2} \cdot (s+j\omega)\Big|_{s=-j\omega} = -\frac{A\, G(-j\omega)}{2j}$$

$$\bar{a} = G(s)\frac{A\omega}{s^2+\omega^2}\cdot(s-j\omega)\Big|_{s=+j\omega} = \frac{A\,G(+j\omega)}{2j}$$

Now as G(jω) is a complex quantity having its own magnitude and phase angle ϕ. It can be written as,

$$G(j\omega) = |G(j\omega)|\,e^{j\phi}$$

where $|G(j\omega)| = \sqrt{(\text{Real part})^2 + (\text{Imj}\cdot\text{part})^2}$

and $\phi = \angle G(j\omega) = \tan^{-1}\left[\dfrac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)}\right]$

Similarly $G(-j\omega) = |G(-j\omega)|\,e^{-j\phi} = |G(j\omega)|\,e^{-j\phi}$

$$\therefore \quad C_{ss}(t) = \frac{-A\,|G(-j\omega)|}{2j}\,e^{-j\phi}\cdot e^{-j\omega t} + \frac{A\,|G(j\omega)|}{2j}\,e^{j\phi}\cdot e^{j\omega t}$$

$$= A\,|G(j\omega)|\,\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}$$

Now $e^{j\theta} = \cos\theta + j\sin\theta \quad \therefore \cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2}$

$e^{-j\theta} = \cos\theta - j\sin\theta \quad \therefore \sin\theta = \dfrac{e^{j\theta}-e^{-j\theta}}{2j}$

$$\therefore \quad \boxed{C_{ss}(t) = A\,|G(j\omega)|\sin(\omega t+\phi)}$$

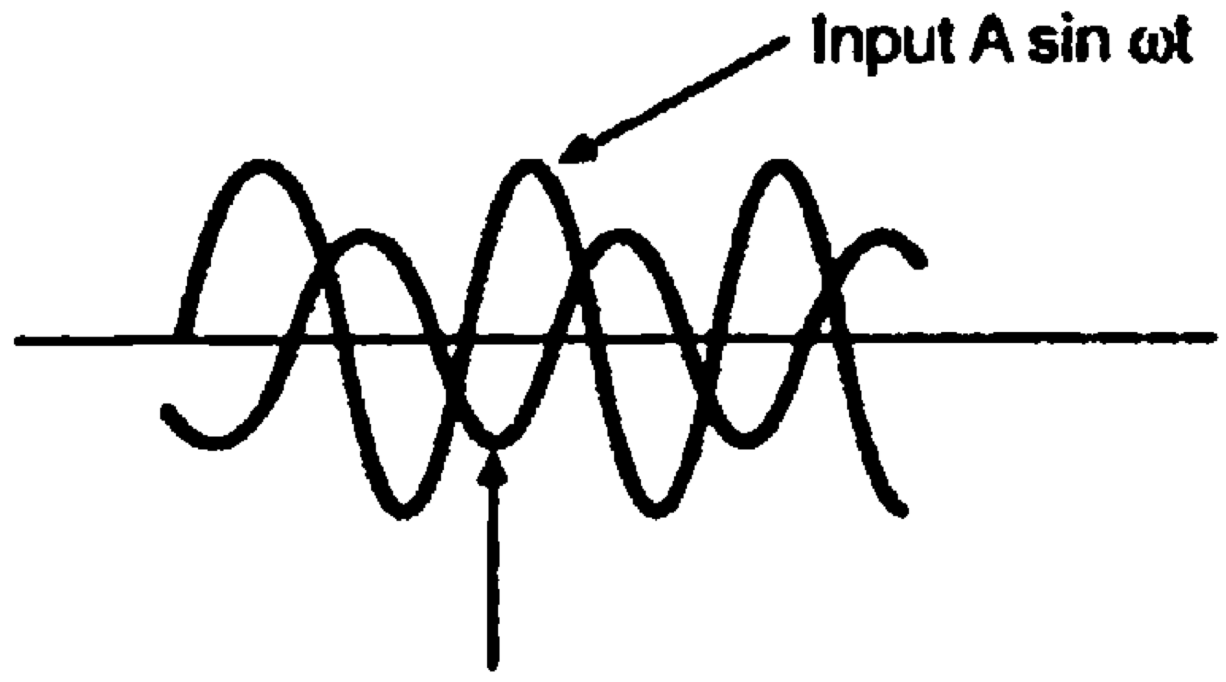


Fig. 10.2

Key Point: *From above we can conclude that when linear time invariant system is subjected to a sinusoidal input it will produce sinusoidal output at steady state having same frequency as that of input. But the amplitude i.e. magnitude and phase angle of the output is different from that of input.*

Magnitude of output = Product of magnitude of input and |G(jω)|

While phase angle difference ϕ = ∠ G(jω).

The sinusoidal transfer function G(jω) is a complex quantity and can be represented as magnitude and phase angle with frequency ω as a variable parameter.

Such frequency domain transfer function can be obtained by substituting jω for 's' in the transfer function G(s) of the system.

$$G(j\omega) = G(s)|_{s=j\omega} = \text{Frequency domain transfer function}$$

So to get frequency response means to sketch the variation in magnitude and phase angle of $G(j\omega)$, when ω is varied from 0 to ∞.

$$G(j\omega) = M \angle \phi$$

$$M = \text{Magnitude} \rightarrow f(\omega)$$

$$\phi = \text{Phase angle} \rightarrow f(\omega)$$

where ω = Input frequency

Key Point : *Frequency response means to sketch variation in M and ϕ against ω. The stability of system can be decided from such frequency responses.*

10.5 Apparatus Required for Frequency Response

Obviously we need a variable frequency oscillator (O), an amplitude measuring device (M) and a phase measuring device (P) to obtain the ratio of output amplitude, to the input amplitude and the phase angle between the input and the output amplitude.

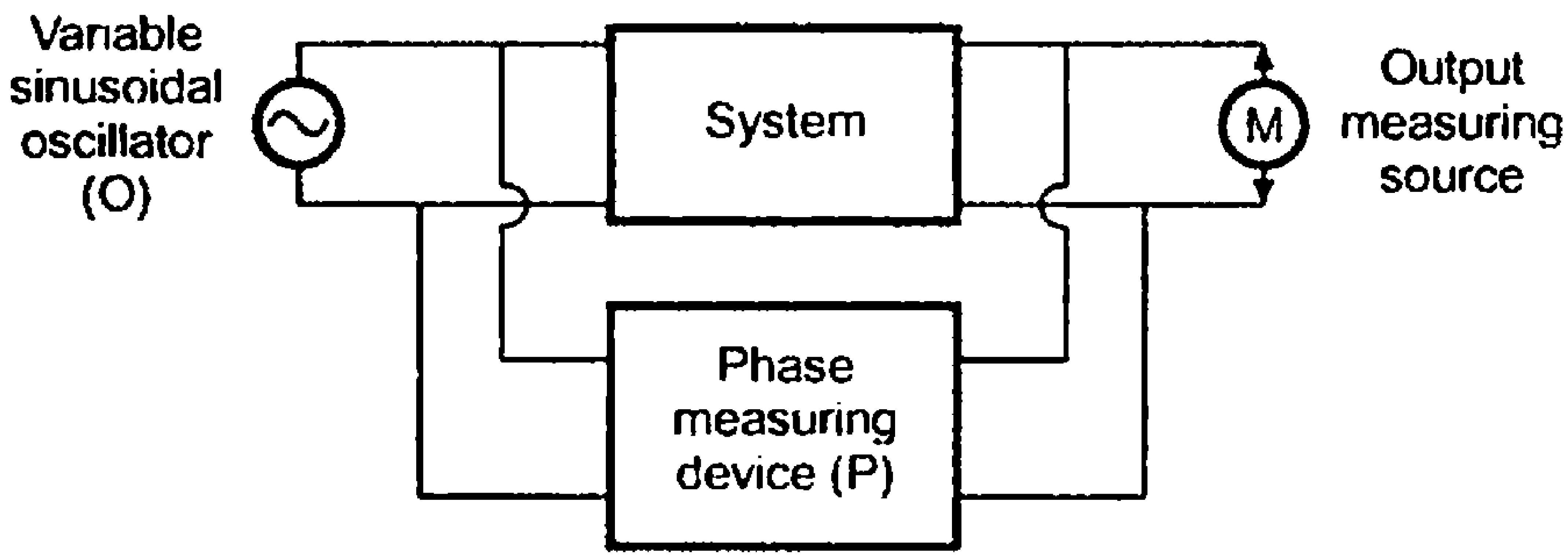


Fig. 10.3 Transfer function analyser

For example for an electronic circuit the sinusoidal oscillator may have frequency range from 10 Hz to 100 kHz and voltage may be measured using CRO (or electronic voltmeter), the phase may be measured using CRO (or phase meter).

However for servomechanisms, the useful frequency range may be very low say 0.001 Hz to 10 Hz. A special low frequency oscillator, a special CRO having long persistance screen and a low frequency phase measuring device is necessary. This system is supplied by many manufacturers and is generally known as a 'Transfer Function Analyser'.

For non electronic systems such as pneumatic systems the transfer function analyser should supply air pressure whose variation is sinusoidal. Such systems have also been built and used.

At each input frequency, one must wait for the transients to die out and the steady state sinusoidal output to result. Then using the amplitude measuring device (such as electronic voltmeter), the input and output are measured and 'Gain' is determined.

Similarly using the phase measuring device, the 'Phase' is determined. This procedure is repeated at other frequencies. The results are tabulated giving at each frequency, the gain and the phase. By this procedure we generate the data for the frequency response.

10.6 Relation between Transfer Function and Frequency Response

Consider that the transfer function of a system is T(s).

Then $$T(s) = \frac{\tau(s)}{R(s)}$$

The frequency response function or 'system function' is then obtained by simply replacing s by 'jω'.

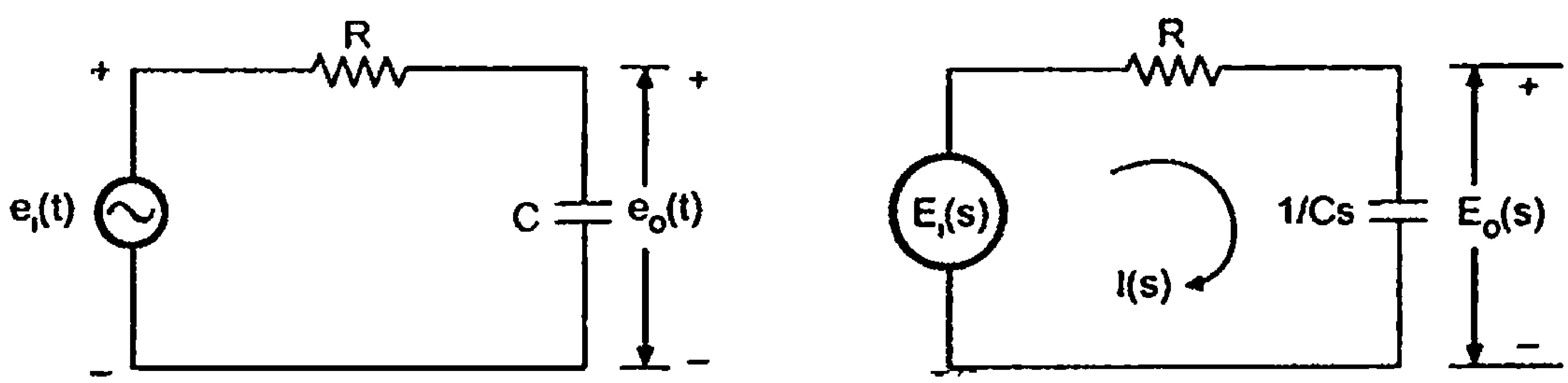
Then $$T(j\omega) = \frac{\tau(j\omega)}{R(j\omega)}$$

At a given frequency $\tau(j\omega)$ is a complex number. The magnitude of $\tau(j\omega)$ is Gain and the angle of $\tau(j\omega)$ = Phase.

Gain = $|\tau(j\omega)|$ and Phase = $\angle\tau(j\omega)$

10.7 Transfer Function and Frequency Response of a Series R-C Circuit

We can find the transfer function of a series R-C circuit by drawing its transform diagram, if initial conditions are zero.



(a) A series R-C circuit **(b) Laplace transform of circuit**

Fig. 10.4

Then $$T(s) = \frac{E_0(s)}{E_i(s)} = \frac{1/Cs}{R+1/Cs} = \frac{1}{RCs+1}$$

Also for sinusoidal input if we write input as E_i and output as E_0 then

$$\frac{E_0}{E_i} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{RCj\omega+1}$$

We can see in this simple case that for sinusoidal input the gain and phase can be obtained by writing s as jω.

Thus $$T(j\omega) = \frac{E_0(j\omega)}{E_i(j\omega)} = \frac{1}{RCj\omega+1}$$

10.9 Co-relation between Time Domain and Frequency Domain for Second Order System

Consider a standard second order system with open loop, T.F.

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}, \qquad H(s) = 1$$

T.F. $$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)}, \qquad H(j\omega) = 1.$$

Now the closed loop transfer function in time domain is,

T.F. $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

10.9.1 Derivations of M_r and ω_r

We have seen in the time domain that second order underdamped system shows overshoot M_p at $t = T_p$. Similarly in frequency response, the second order system shows a peak. This is called resonant peak M_r and corresponding frequency is called resonant frequency ω_r.

Now to find resonant peak M_r, similar to M_p we must find input frequency ω which will maximise closed loop T.F. magnitude, which will be ω_r.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2+2\xi\omega_n j\omega+\omega_n^2} = \frac{\omega_n^2}{-\omega^2+2\xi\omega_n j\omega+\omega_n^2}$$

Dividing by ω_n^2,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+2\xi j\frac{\omega}{\omega_n}}$$

Replacing $\frac{\omega}{\omega_n} = x$,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{[1-x^2]+2\xi j x}$$

Now $$\text{Magnitude} = \frac{1}{\sqrt{(1-x^2)^2+4\xi^2x^2}}$$

At 'x' which will maximise magnitude, we can write ,

$$\frac{dM}{dx} = 0$$

This can be shown in the graphs as below.

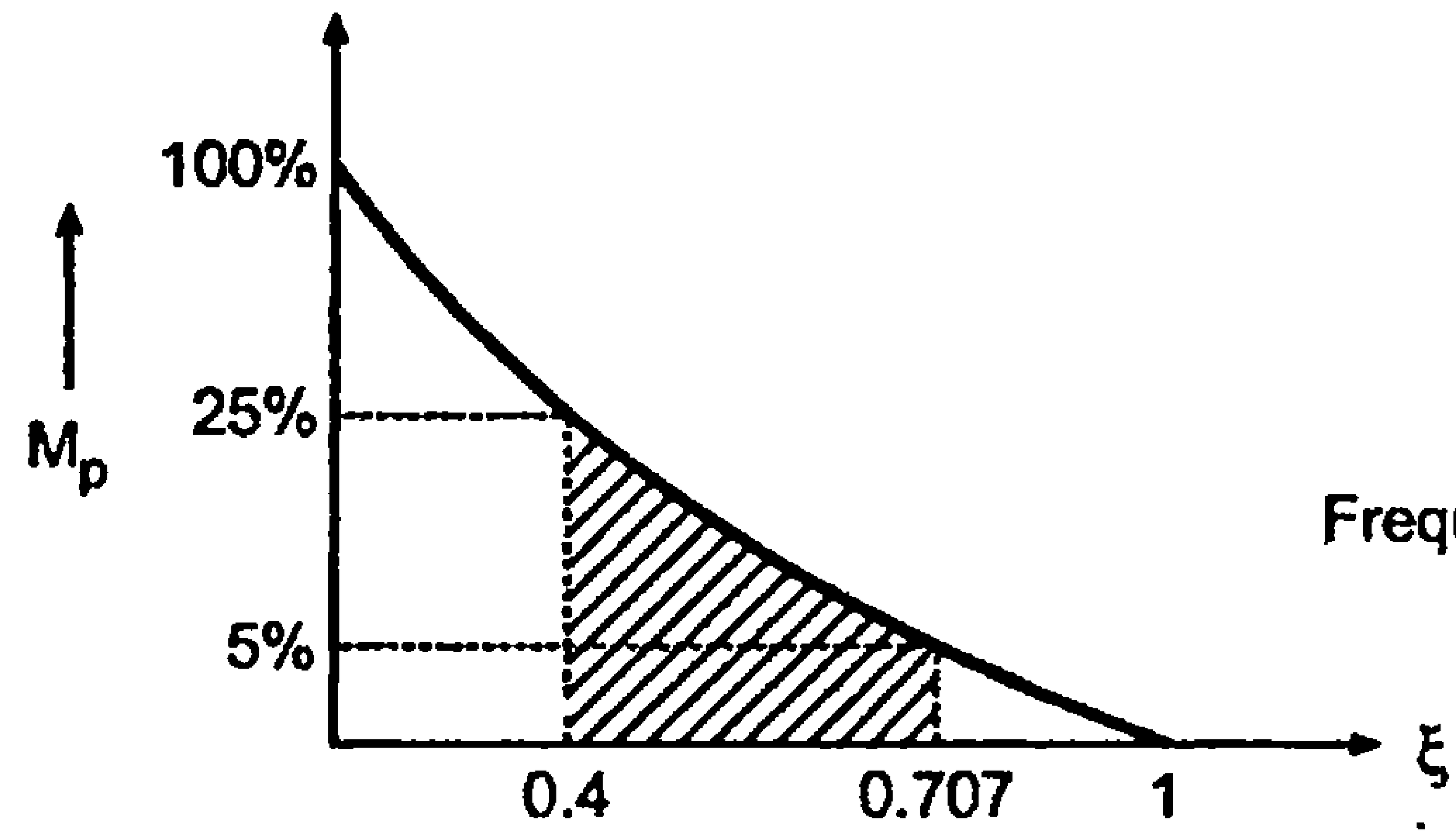


Fig. 10.7

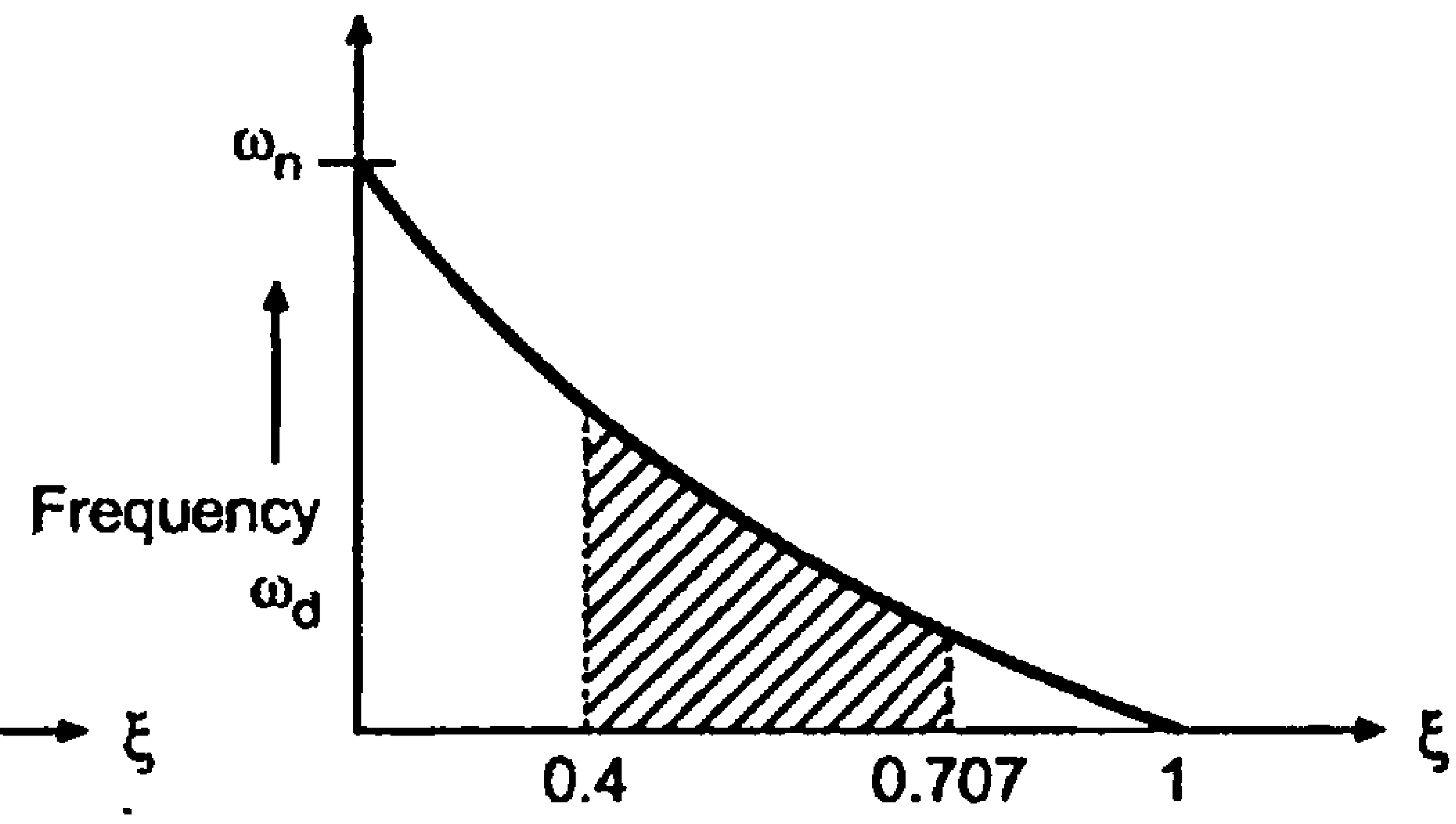


Fig. 10.8

Comparable between $0.4 < \xi < 0.707$, as shown in the Fig. 10.9 and Fig. 10.10.

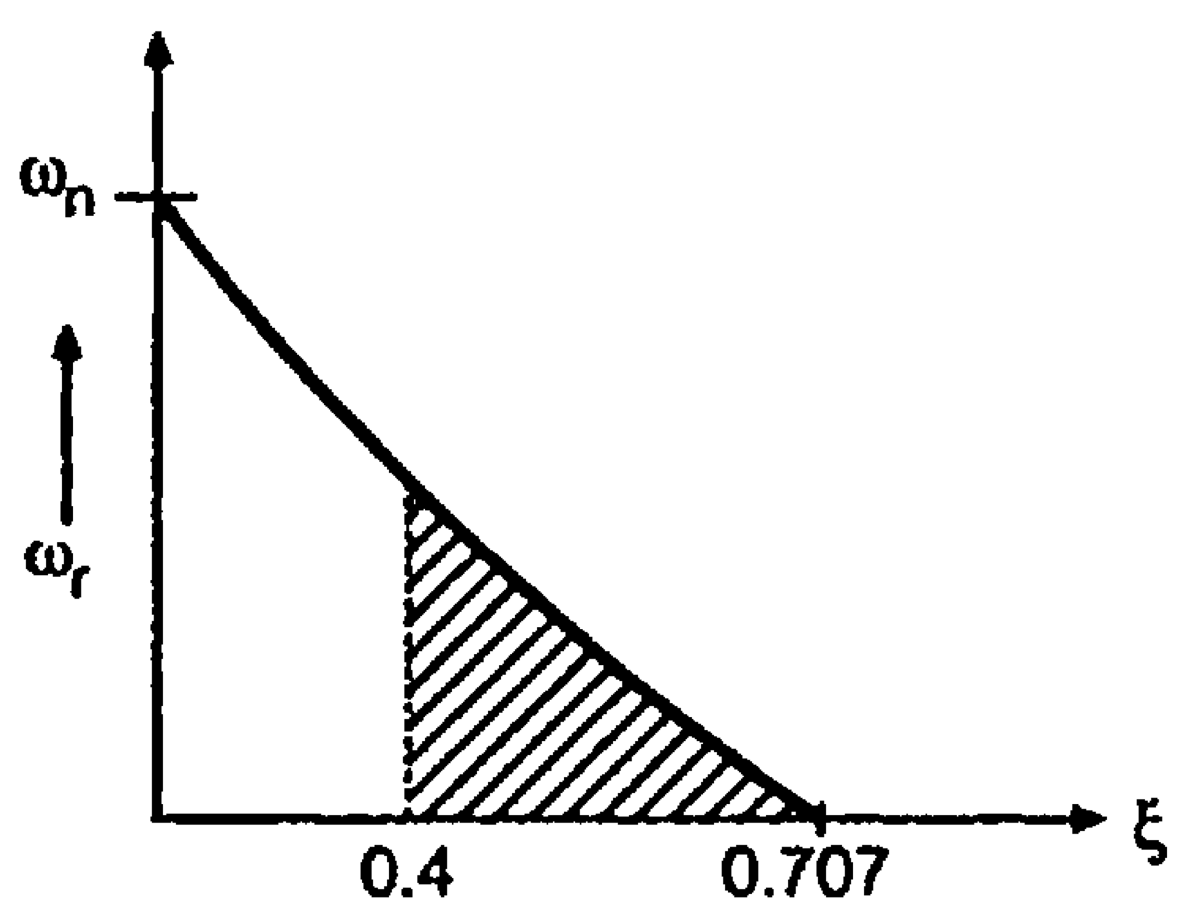


Fig. 10.9

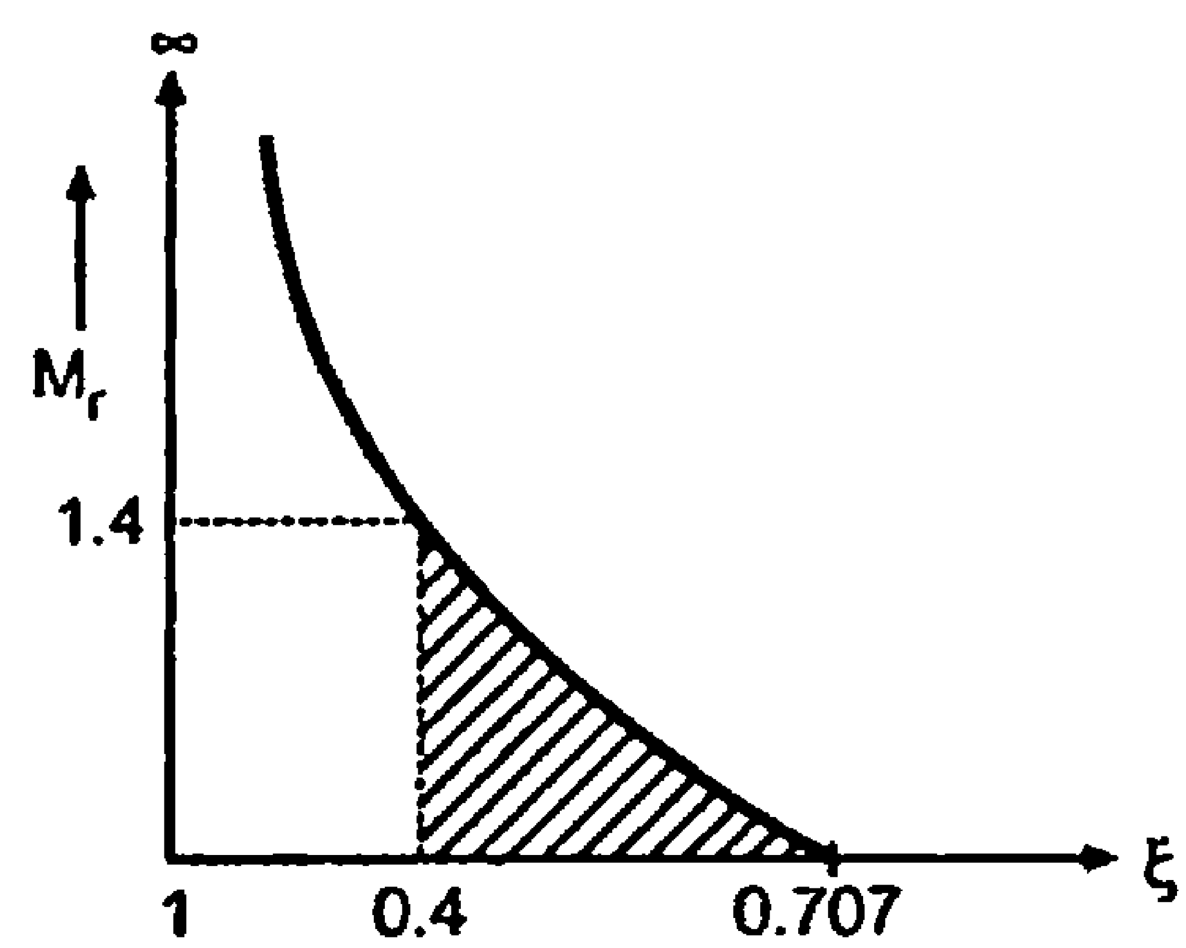


Fig. 10.10

10.10 B.W. (Bandwidth)

It is the range of frequency upto ω_b which is cut-off frequency. At ω_b the magnitude of closed loop transfer function is 3 dB down from its zero frequency level.

For standard second order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n s + \omega_n^2} = \frac{1}{1 + \frac{2\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

$$\therefore \quad \frac{C(j\omega)}{R(j\omega)} = \frac{1}{1 + \frac{2\xi}{\omega_n} j\omega + j^2 \frac{\omega^2}{\omega_n^2}} = \frac{1}{1 + \frac{2\xi\omega}{\omega_n} j - \frac{\omega^2}{\omega_n^2}}$$

$$\text{Let } \frac{\omega}{\omega_n} = x$$

Examples with Solutions

Example 10.1 : *A system of third order shows resonance peak of 2 and resonance frequency 3 rad/sec. Determine the transfer function of the equivalent second order system and hence find the T_r, T_p, T_s, % overshoot, time of oscillations and number of oscillations before settling. Draw a sketch of the step response and frequency response.*

Solution : For the equivalent second order system $M_r = 2$ and $\omega_r = \omega_n \sqrt{1-2\xi^2} = 3$

Now $$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

We are given $$M_r = 2$$

$$\therefore \quad 2 = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \text{i.e.} \quad 4 = \frac{1}{4\xi^2(1-\xi^2)}$$

$$16\xi^2(1-\xi^2) = 1$$

$$16\xi^2 - 16\xi^4 = 1$$

$$16\xi^4 - 16\xi^2 + 1 = 0$$

Hence $$\xi^2 = \frac{16 \pm \sqrt{256-64}}{32} = \frac{16}{32} \pm \frac{\sqrt{192}}{32} = 0.5 \pm 0.433$$

$$\xi^2 = 0.0669 \text{ or } 0.933 \quad \text{and} \quad \xi = 0.2588 \text{ or } 0.966$$

Key Point *: A system with relative damping factor greater than 0.707 does not exhibit any peak in the frequency response hence the acceptable answer is only* $\xi^2 = 0.0669$.

Also $$\omega_r = \omega_n\sqrt{1-2\xi^2} \quad \text{i.e.} \quad 3 = \omega_n\sqrt{1-2(0.2588)^2}$$

$$\therefore \quad \omega_n = 3.223 \text{ rad/sec}$$

The transfer function of equivalent second order system is,

$$M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{(3.223)^2}{s^2 + 2(0.2588)(3.223)s + (3.223)^2}$$

$$= \frac{10.387}{s^2 + 1.668 s + 10.387}$$

Rise time, $$T_r = \frac{\pi - \theta}{\omega_d} \quad \text{where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \text{ radians}$$

In this case, $\theta = \tan^{-1}\left[\dfrac{1-\sqrt{(0.2588)^2}}{0.2558}\right] = \tan^{-1}\left[\dfrac{0.9659}{0.2588}\right] = 1.3$ radians

$\therefore$ Also $\omega_d = \omega_n\sqrt{1-\xi^2} = 3.223\sqrt{1-(0.2588)^2} = 3.11$ rad/sec

$\therefore$ Rise time, $T_r = \dfrac{\pi-1.3}{3.11} = \mathbf{0.592\ sec}$

Time to peak, $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{3.11} = \mathbf{1.0101\ sec}$

Settling time, $T_s = \dfrac{4}{\xi\,\omega_n}$ (For ± 2 % error band) $= \dfrac{4}{0.2588\times 3.223}$

$= \mathbf{4.795\ sec}$

$T_s = \dfrac{3}{\xi\,\omega_n} = 3.596$ sec (for ± 5 % error band)

Period of oscillations,

$$T_{osc.} = \frac{2\pi}{\omega_d} = \frac{2\pi}{3.11} = \mathbf{2.02\ sec.}$$

Number of oscillation during settling,

$$N = \frac{T_s}{T_{osc.}} = \frac{4.795}{2.02} = \mathbf{2.37}$$

Peak overshoot, $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

Since $\dfrac{\xi\pi}{\sqrt{1-\xi^2}} = \dfrac{0.2588\,\pi}{\sqrt{1-(0.2588)^2}} = 0.8417$

$\therefore$ Overshoot $= e^{-0.8417} = \mathbf{0.4309}$

Thus the system exhibits **43.1 %** overshoot.

Sketches of frequency response and estimated time response are shown.

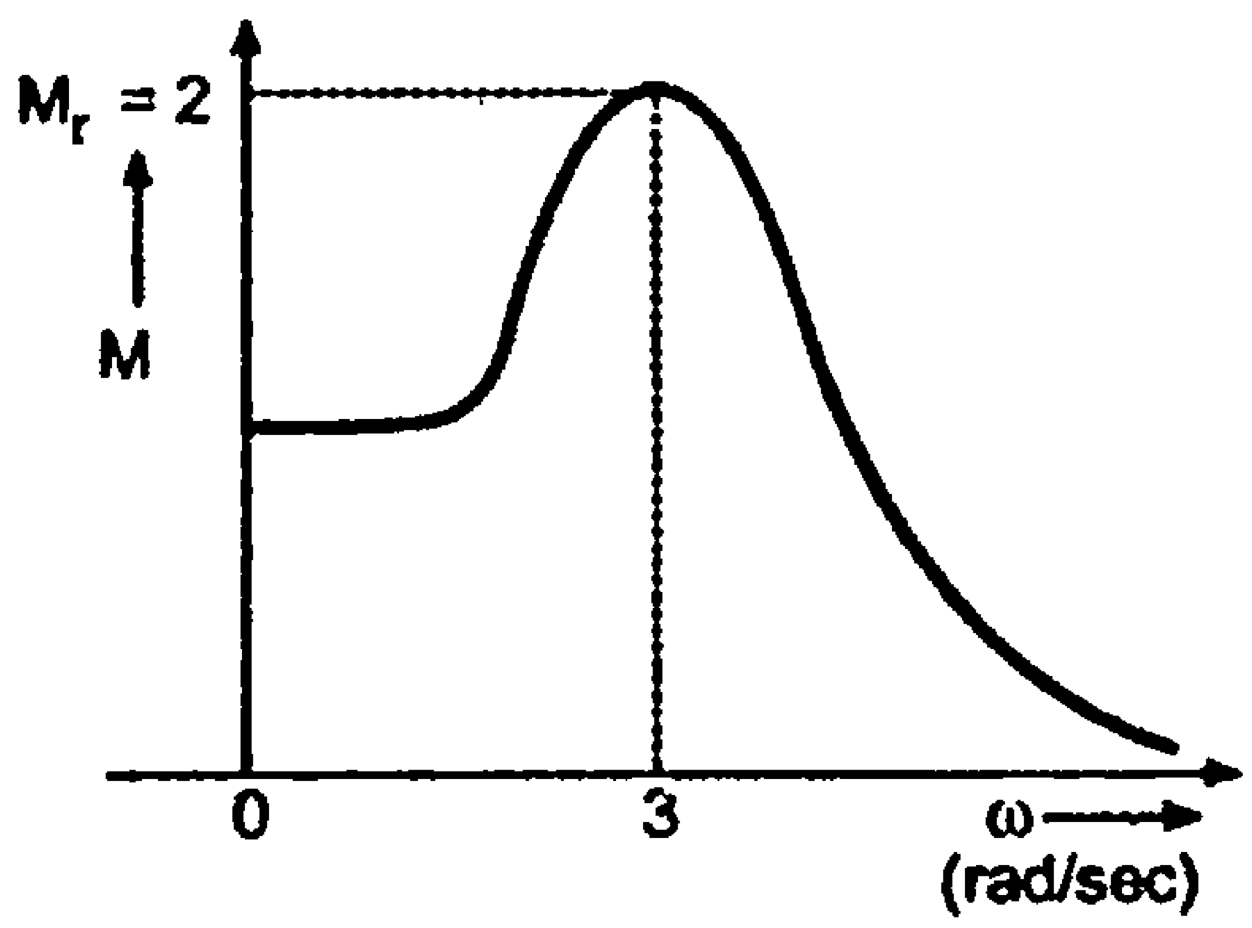


Fig. 10.11 Frequency response

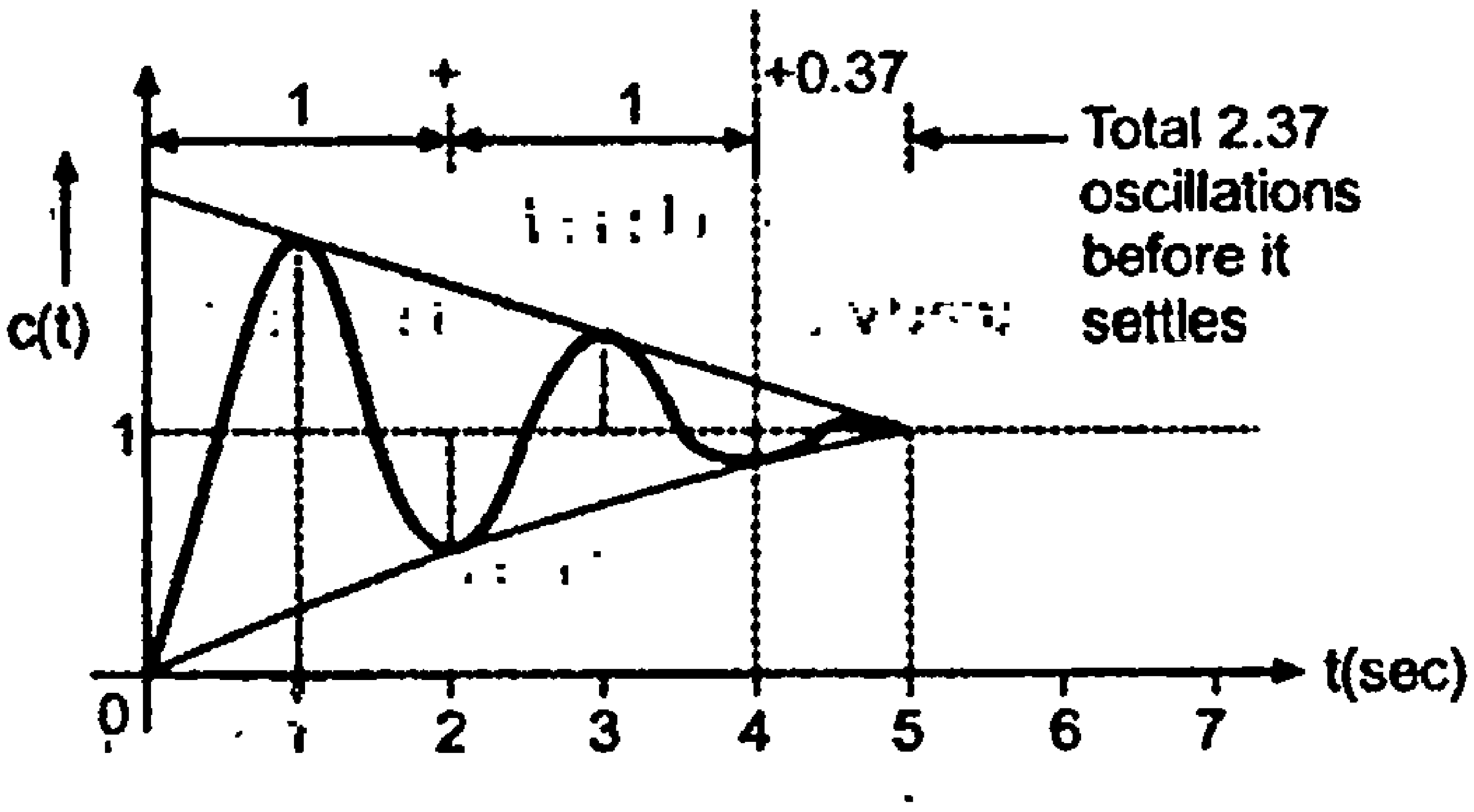


Fig. 10.12 Step response (estimated)

Example 10.2 : *If G(s) of a unity feedback system is $\frac{10}{s(s+10)}$ determine steady state response of the system when the excitation applied is -*

$$r(t) = 10 \sin 8t.$$

(M.U. : May-97, May-98)

Solution :

$$G(s) = \frac{10}{s(s+10)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

i.e.

$$T(s) = \frac{\frac{10}{s(s+10)}}{1+\frac{10}{s(s+10)}}$$

$$\therefore \quad T(s) = \frac{10}{s^2+10s+10}$$

In the frequency domain, replace s by jω

$$\therefore \quad T(j\omega) = \frac{10}{(j\omega)^2+10j\omega+10} = \frac{(10)}{(10-\omega^2)+j10\omega}$$

$$T(j\omega) = M \angle \phi \text{ where}$$

$$M = \frac{|10|}{|(10-\omega^2)+j10\omega|} = \frac{10}{\sqrt{(10-\omega^2)+100\omega^2}}$$

and

$$\phi = -\tan^{-1}\left(\frac{10\omega}{10-\omega^2}\right)$$

Now $r(t) = 10 \sin (8t)$

comparing this with,

$$r(t) = A \sin(\omega t) \text{ we can write}$$

$$A = \text{Amplitude} = 10$$

and $\omega t = 8t$

So if M = Magnitude of closed loop transfer function

and ϕ = phase angle of closed loop transfer function and if it is excited by the input $A \sin \omega t$ then the steady state response is given by,

$$c(t) = (AM) \sin(\omega t + \phi)$$

Hence in this case substituting all the values, we get the steady state response as,

$$\therefore \; c(t) = \frac{10\times 10}{\sqrt{(10-\omega^2)^2+100\omega}} \sin\left(8t - \tan^{-1}\left[\frac{10\omega}{10-\omega^2}\right]\right)$$

$$= \frac{100}{\sqrt{(10-\omega^2)^2 + 100\omega^2}} \sin\left(8t - \tan^{-1}\left[\frac{10\omega}{10-\omega^2}\right]\right)$$

Key Point : *It can be observed that a stable linear time invariant system subjected to a sinusoidal input produces the output of the same frequency as the input.*

But the amplitude and phase of the output will, in general, be different from those of the input.

Example 10.3 : *The closed loop transfer function of a feedback system is given by* $T(s) = \frac{1000}{(s+22.5)(s^2+2.45s+44.4)}$.

(a) *Determine the resonance peak* (M_r) *and resonant frequency* (ω) *of the system by drawing the frequency response curve.*

(b) *What should be the values of damping ratio* (ξ) *and undamped natural frequency* (ω_n) *of an equivalent second order system which will produce the same* M_r *and* ω_r *as determined in part (a) ?*

(c) *Determine the bandwidth of the equivalent second order system.*

Solution : a)

$$T(j\omega) = \frac{1000}{(j\omega+22.5)[(j\omega)^2+2.45j\omega+44.4]}$$

$$\therefore \quad T(j\omega) = \frac{1000}{(j\omega+22.5)[(44.4-\omega^2)+j2.45\omega]} = M\angle\phi$$

$$M = \frac{1000}{\sqrt{\omega^2+(22.5)^2} \times \sqrt{(44.4-\omega^2)^2+(2.45\omega)^2}}$$

$$\phi = -\tan^{-1}\frac{\omega}{22.5} - \tan^{-1}\frac{2.45\omega}{[44.4-\omega^2]}$$

The frequency response data is,

ω	M
0.1	0.99
0.5	1
1	1.019
2	1.085
5	1.884
7	2.39
8	1.51
10	0.668
50	7.4×10^{-3}
∞	0

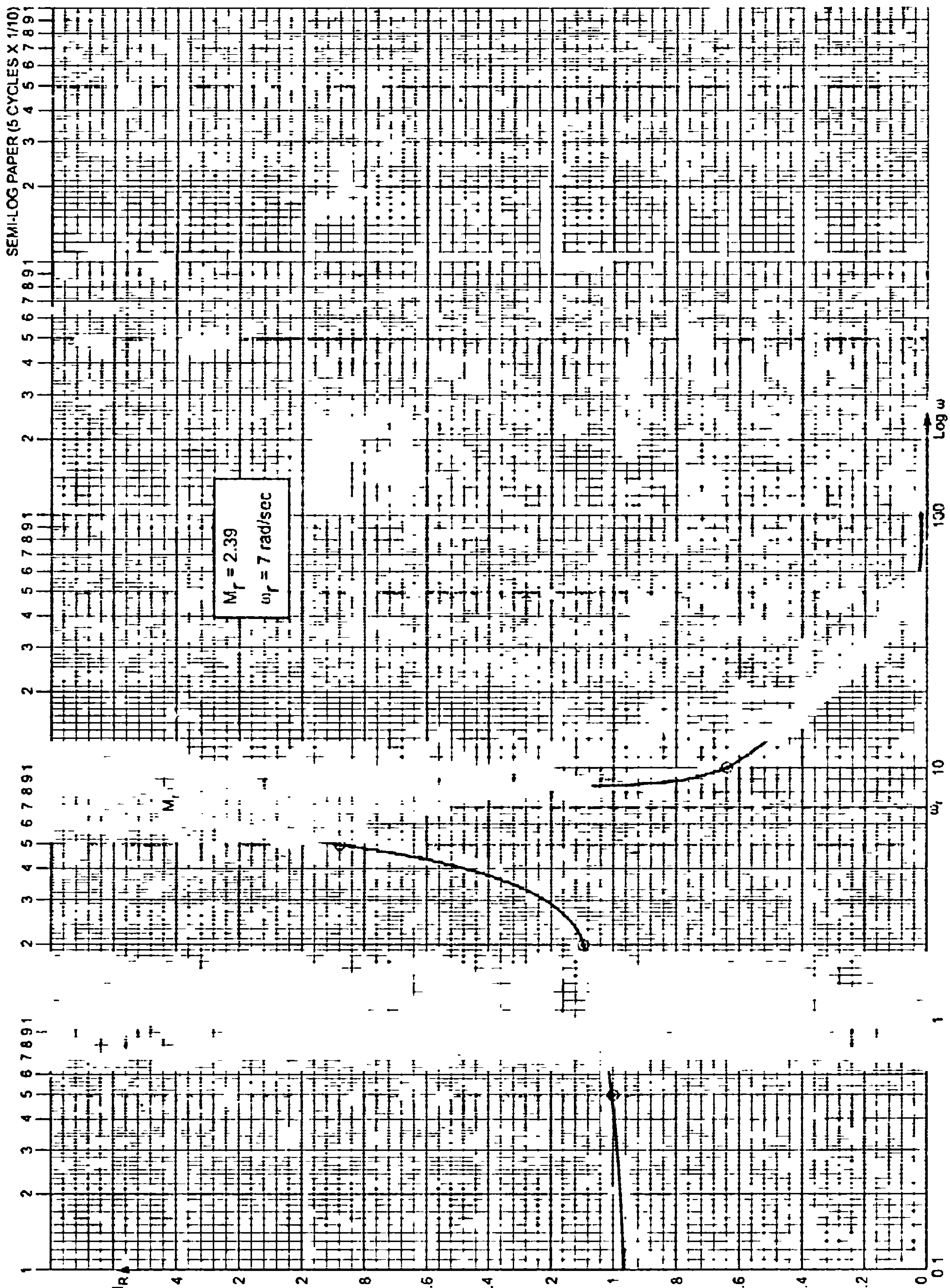

SEMI-LOG PAPER (5 CYCLES X 1/10)
M_r = 2.39
ω_r = 7 rad/sec
Log ω
10
ω_r
M_r
M_R
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0

Fig. 10.13

But M_r can not exit for $\xi > 0.707$ hence $\xi = 0.54$.

$$\text{Now} \quad \omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$\therefore \quad 11.2 = \omega_n\sqrt{1-2\times(0.54)^2}$$

$$\therefore \quad \omega_n = 17.348 \text{ rad/sec}$$

Hence open loop T.F. is,

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)} = \frac{300.953}{s(s+18.7358)}$$

Example 10.5 : *A second order system has overshoot of 50% and period of oscillations 0.2 sec in step response.*

Determine i) Resonant peak ii) Resonant frequency iii) Bandwidth. **(M.U. : Jan.-1993)**

Solution : For step response

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$\therefore \quad 50 = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$0.5 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\therefore \quad \ln(0.5) = -\frac{\pi\xi}{\sqrt{1-\xi^2}} \quad \text{i.e.} \; -0.6931 = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

$$\text{Solving,} \quad \xi^2 = 0.046 \quad \text{i.e.} \; \xi = 0.2154$$

$$\text{Time period} \quad T = 0.2 \text{ sec} \quad \text{i.e.} \; f_d = \frac{1}{T} = 5 \text{ Hz}$$

$$\omega_d = 2\pi f_d = 2\pi\times 5 = 31.41 \text{ rad/sec} = \omega_n\sqrt{1-\xi^2}$$

$$\therefore \quad \omega_n = 32.16 \text{ rad/sec}$$

$$\text{Resonant peak} \quad M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.2154\times\sqrt{1-(0.2154)^2}} = \mathbf{2.377}$$

$$\text{Resonant Frequency} \quad \omega_r = \omega_n\sqrt{1-2\xi^2} = 32.16\sqrt{1-2\times(0.2154)^2} = \mathbf{30.63 \text{ rad/sec}}$$

For bandwidth,

$$-20 \text{ Log } \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}} = -3 \text{ dB}$$

$$\therefore \quad \sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2} = 1.4125$$

Substitute values and squaring.

$$\left(1-\frac{\omega^2}{1034.26}\right)^2 + 1.7944\times 10^{-4}\,\omega^2 = 1.9951$$

$$\therefore \quad 1-\frac{2\times\omega^2}{1034.26} + \frac{\omega^4}{(1034.26)^2} + 1.7949\times 10^{-4}\,\omega^2 = 1.9951$$

$$\therefore \quad \omega^4 + 191.94\,\omega^4 - 2068.52\omega^2 + 1.069\times 10^6 = 2.134\times 10^6$$

$$\therefore \quad \omega^4 - 1876.58\,\omega^2 - 1.065\times 10^6 = 0$$

$$\omega^2 = \frac{1876.58 \pm \sqrt{(1876.58)^2 - 4\times 1\times(-1.065\times 10^6)}}{2}$$

$$\omega^2 = 2333.0617, \; -456.48$$

$$\therefore \quad \omega^2 = 2333.0617$$

$$\therefore \quad \omega_b = 48.30 \text{ rad/sec}$$

$$\therefore \quad \text{B.W.} = 0 \text{ to } 48.30 \text{ rad/sec} = \mathbf{48.3}$$

➡ **Example 10.6 :** *Given that $M_r = 2$ and $\omega_r = 5$ determine the steady state error for a unit ramp for a unity feedback system with a closed loop transfer function of standard second order system.* **(M.U. : May-95, Dec.-95)**

Solution :

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = 2$$

$$4 = \frac{1}{4\xi^2(1-\xi^2)}$$

$$\therefore \quad 16\xi^2(1-\xi^2) = 1$$

$$\therefore \quad 16\xi^4 - 16\xi^2 + 1 = 0$$

$\therefore \quad \xi^2 = 0.0669 \text{ or } 0.933$

$\therefore \quad \xi = 0.2588 \text{ or } 0.966$

Key Point : *For* $\xi = 0.966$, M_r *is not possible, hence neglecting it.*

$\therefore \quad \xi = \mathbf{0.2588}$

$$\omega_r = \omega_n \sqrt{1-2\xi^2} = 5$$

$$\therefore \quad \omega_n = \frac{5}{\sqrt{1-2(0.2588)^2}} = 5.3727$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{28.86}{s^2+2.78s+28.86}$$

$$\therefore \quad C(s) = R(s)\left[\frac{28.86}{s^2+2.78s+28.86}\right], \quad R(s) = \frac{1}{s^2} \text{ for unit ramp input}$$

$$= \frac{1}{s^2}\left[\frac{28.86}{s^2+2.78s+28.86}\right]$$

Now,

$$E(s) = R(s) - C(s)$$

$$= \frac{1}{s^2} - \frac{1}{s^2}\left[\frac{28.86}{s^2+2.78s+28.86}\right]$$

$$= \frac{1}{s^2}\left[1-\frac{28.86}{s^2+2.78s+28.86}\right]$$

$$\therefore \quad e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{s^2}\left[\frac{s(s+2.78)}{s^2+2.78s+28.86}\right]$$

$$= \lim_{s\to 0} \frac{s+2.78}{s^2+2.78s+28.86}$$

$$= \mathbf{0.0963}$$

Example 10.7 : *The specifications of a second order unity feedback control system with the closed loop transfer function*

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

are that the maximum overshoot must not exceed 10% and the rise time must be less than 0.1 sec. Find the corresponding limiting values of Resonance peak (M_r) *and Bandwidth (BW) analytically.* **(M.U. : May-99)**

Solution : The transfer function is,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+a)}}{1+\frac{K}{s(s+a)}} = \frac{K}{s^2+as+K}$$

$\therefore \quad \omega_n^2 = K \quad$ i.e. $\quad \omega_n = \sqrt{K}$

and $\quad 2\xi\omega_n = a$ i.e. $\xi = \dfrac{a}{2\sqrt{K}}$

Now $\quad M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}} = 1.04$

$\therefore \quad 1 = 4.3264\ \xi^2\ (1-\xi^2)$

$\therefore \quad 4.3264\ \xi^4 - 4.3264\ \xi^2 + 1 = 0$

$\therefore \quad \xi^2 = \dfrac{4.3264 \pm \sqrt{(4.3264)^2 - 4\times 4.3264}}{2\times 4.3264} = 0.637,\ 0.362$

$\therefore \quad \xi = 0.7983,\ 0.6016$

For M_r, ξ must be less than 0.707 hence $\xi = 0.6016$.

$\omega_r = \omega_n\sqrt{1-2\xi^2}$

$\therefore \quad 11.55 = \omega_n\sqrt{1-2\times(0.6016)^2}$

$\therefore \quad \omega_n = 21.9788$ rad/sec

$\therefore \quad K = \omega_n^2 = 483.071$

$\therefore \quad a = 26.4445$

ii) $\quad T_s = \dfrac{4}{\xi\omega_n} = 0.3025$ sec

$$B.W. = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}$$

$$= 25.1902$$

Example 10.9 : *Unit step response data of a second order system is given below, obtain the corresponding frequency response indices (M_r, ω_r, ω_b) for the system,*

(M.U. : May-2006)

Solution : For the given pole-zero diagram,

$$T(s) = \frac{K}{(s+2-j\,10)(s+2+j\,10)} = \frac{104}{s^2+4\,s+104}$$

Comparing denominator with $s^2+2\xi\omega_n\,s+\omega_n{}^2$,

$$\omega_n{}^2 = 104 \quad \text{i.e.} \quad \omega_n = 10.198 \text{ rad/sec.}$$

$$2\xi\omega_n = 4 \quad \text{i.e.} \quad \xi = 0.1961$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.1961\times\sqrt{1-(0.1961)^2}} = \mathbf{2.6}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 10.198\times\sqrt{1-2\times(0.1961)^2} = \mathbf{9.4136\ rad/sec.}$$

$$\omega_b = \text{B.W.} = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = \mathbf{15.4066}$$

Example 10.12 : *A second order system has overshoot of 50% and period of oscillations 0.2 sec in step response. Determine : (i) Resonant peak (ii) Resonant frequency (iii)Bandwidth* **(M.U. : Dec.-2007)**

Solution : $M_p = 50\ \%$, T_d = Time period = 0.2 sec.

Now, $$M_p = 100\,e^{-\pi\xi/\sqrt{1-\xi^2}} \quad \text{i.e. } 0.5 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\ln(0.5) = \frac{-\pi\xi}{\sqrt{1-\xi^2}} \quad \text{i.e. } 0.2206 = \frac{\xi}{\sqrt{1-\xi^2}}$$

Solving, $$\xi = \mathbf{0.2154}$$

And $$T_d = 0.2 \text{ sec.} \quad \text{i.e. } f_d = \frac{1}{T_d} = 5\text{ Hz}$$

$$\therefore \quad \omega_d = 2\pi f_d = 10\pi \text{ rad/sec} = \omega_n\sqrt{1-\xi^2}$$

$$\therefore \quad \omega_n = \frac{10\pi}{\sqrt{1-(0.2154)^2}} = \mathbf{32.1711\ rad/sec}$$

i) $$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \mathbf{2.377}$$

ii) $$\omega_r = \omega_n\sqrt{1-2\xi^2} = \mathbf{30.6421}\text{ rad/sec.}$$

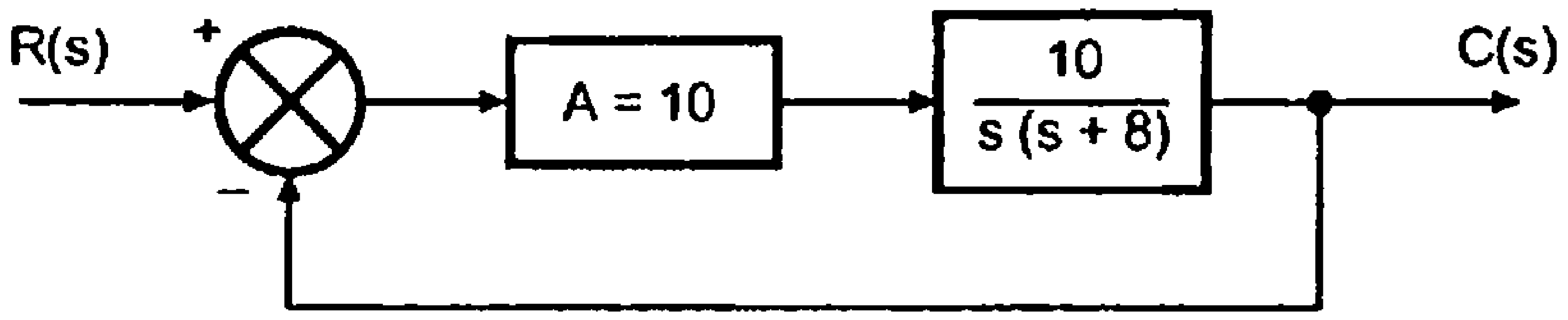
iii) $$\text{B.W.} = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = \mathbf{48.336}$$

Review Questions

1. *State the advantages and limitations of frequency domain approach.*
2. *Write a short note on co-relation between time domain and frequency domain.*
3. *Define bandwidth and derive the expression for bandwidth of a standard second order system.*
4. *Drive the expressions for resonant peak M_r and resonant frequency ω_r for a standard second order system interms of ξ and ω_n.*
5. *A second order system has overshoot of 30% and oscillation period of 0.5 second in the step response. Determine the values of M_r and ω_r and bandwidth in the frequency response.*
6. *A standard second order system has $\omega_n = 8$ and $\xi = 0.3$. Determine the frequency response specifications ω_r and M_r and bandwidth. Also find number of oscillations before settling, in the step response.*
7. *A unit step input is applied to a unity feedback control system whose open–loop transfer function is given by,*

   $$G(s) = \frac{K}{s(sT+1)}$$

   Determine the values of K and T given that maximum overshoot as 26% and resonant frequency $\omega_r = 8$ rad/s. Calculate the resonance peak M_r.
8. *Figure shows the schematic block diagram of a unity feedback control system. For this system calculate M_r and ω_r.*



9. *The overshoot of the step response of a second order feedback system is 30% and settling time is 4 sec. For this system determine the damping ratio, M_r and ω_r.*
10. *For a unity feedback system*

    $$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

    Determine phase margin interms of ξ and ω_n **(M.U. : Dec.-97)**
11. *A second order system has resonant frequency of 10 rad/sec and peak magnitude ratio (M_r) of 4. Determine its transfer and find its the time, peak overshoot and settling time.*
    (Ans. : $T_r = 0.1683$ sec, $T_s = 3.123$ sec, $M_p = 2.7$ %)
12. *A second order system has $M_r = 2$ and $\omega_r = 4$ rad/sec. Find its percent overshoot and settling time.*
    (Ans. : 43.09 %, 3.5959 sec)

11 Bode Plots

11.1 Introduction to Bode Plot

Basic of any frequency response is to plot magnitude M and angle ϕ against input frequency 'ω'. When 'ω' is varied from 0 to ∞ there is wide range of variations in M and ϕ and hence it becomes difficult to accommodate all such variations with linear scale. Hence H.W. Bode suggested the method in which logarithmic values of magnitude are to be plotted against logarithmic values of frequencies. Such plots are called Logarithmic plots which allows us to show wide range of variations in magnitude on a single paper.

So in general Bode plot consists of two plots which are,

1) *Magnitude expressed in logarithmic values against logarithmic values of frequency called Magnitude Plot.*
2) *Phase angle in degrees against logarithmic values of frequency called Phase Angle Plot.*

11.1.1 Magnitude Plot

The magnitude can be expressed in its logarithmic values by finding out the value $20 \log_{10} |G(j\omega)|$, which has a unit as decibel denoted by dB.

For Bode Plot $|G(j\omega)| = 20 \log_{10} |G(j\omega)|$ dB.

Such decibels values are to be plotted against $\log_{10}(\omega)$.

So magnitude plot can be shown as in the Fig. 11.1.

11.1.2 The Phase Angle Plot

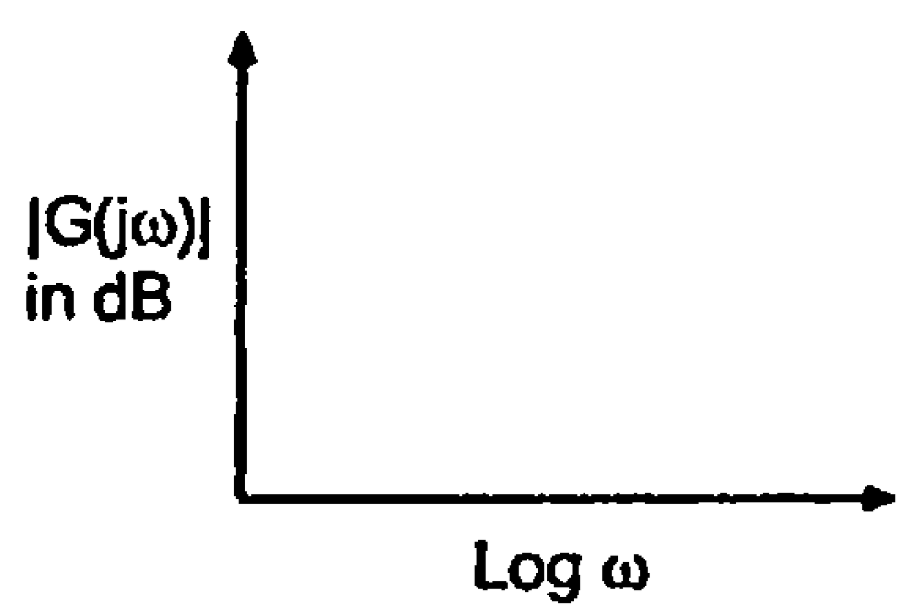


Fig. 11.1 Magnitude plot

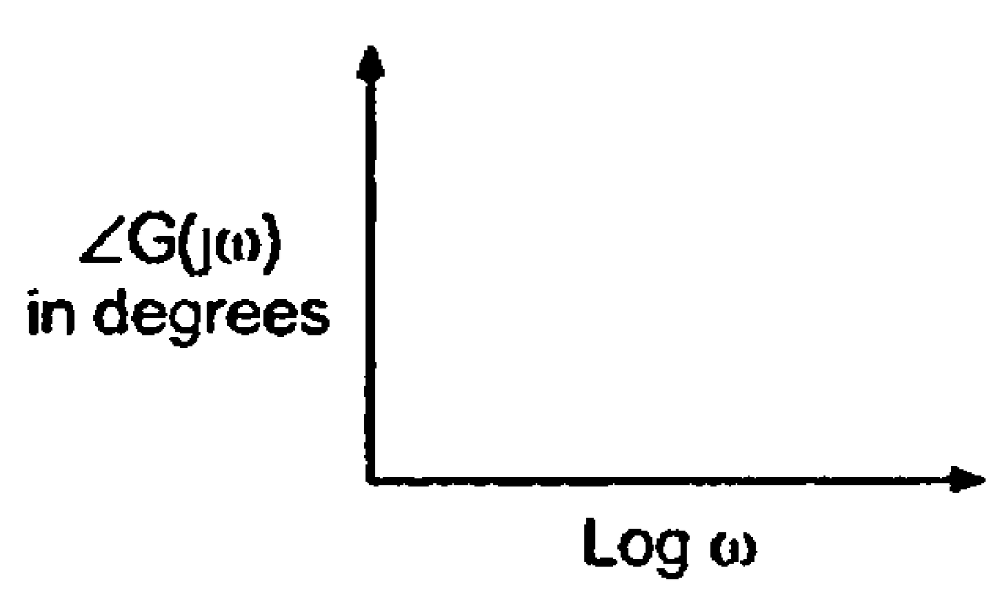


Fig. 11.2 Phase angle plot

In this, angle of $G(j\omega)$ is to be expressed in degrees which is to be plotted against Log ω.

The phase angle plot can be shown as in the Fig. 11.2.

As for both plots X-axis is Log ω both may be drawn on same paper with common X-axis.

Note : To predict the closed loop stability from the frequency response of open loop system, the magnitude and phase angle of open loop transfer function $G(j\omega)H(j\omega)$ is to be plotted against Log ω and not only $G(j\omega)$.

So for Bode plot, magnitude in dB and phase angle in degrees are the magnitudes and phase angles of $G(j\omega)H(j\omega)$, plotted against Log ω.

11.2 Logarithmic Scales (Semilog Papers)

To sketch the magnitude in dB and phase angle in degrees against Log ω, the logarithmic scale is used. This is available on semilog graph paper. In such paper the X-axis is divided into a logarithmic scale which is non linear one. While Y-axis is divided into linear scale and hence it is called ***semilog paper.***

The interesting part about X-axis is the distance between 1 and 2 is greater than distance between 2 and 3 and so on. Similarly, on such scale distance between 1 and 10 is equal to the distance between 10 and 100 or between 100 and 1000 and so on. This distance is called **1 decade.**

This is because Log 1 = 0 and Log 10 = 1. The distance is 1 decade which is divided into 10 parts according to logarithmic scale i.e. Log 2, Log 3, ...

Now Log 10 = 1 and Log 100 = 2. The distance is again (2 − 1) i.e. 1 decade same as between Log 1 and Log 10, further divided into parts as Log 20, Log 30, ...

So X-axis is available, which is divided into two, three, or four such cycles i.e. decades.

So **it is not necessary to find logarithmic value of ω while plotting on X-axis but the logarithmic scale available takes care of logarithmic value of ω.** The advantage of the scale is wide range of frequencies can be accomodated on a single paper.

As Log 0 = − ∞ it is obvious that X-axis cannot be calibrated from 0 but as per requirement the smallest frequency may be selected as starting frequency like 0.01, 0.1 etc. This hardly affects the result of the frequency response.

The Y axis is divided into linear scale similar to standard graph paper.

To clear the idea of semilog paper and decade the graph paper is shown in the Fig.11.3.

Key Point: *The main advantage using the logarithmic representation is that the multiplication and division of magnitudes get replaced by the addition and subtraction respectively.*

The experimental determination of the transfer function is easier if frequency response data is presented in the form of the logarithmic plot. Such plot shows both low frequency and high frequency characteristics in the same diagram.

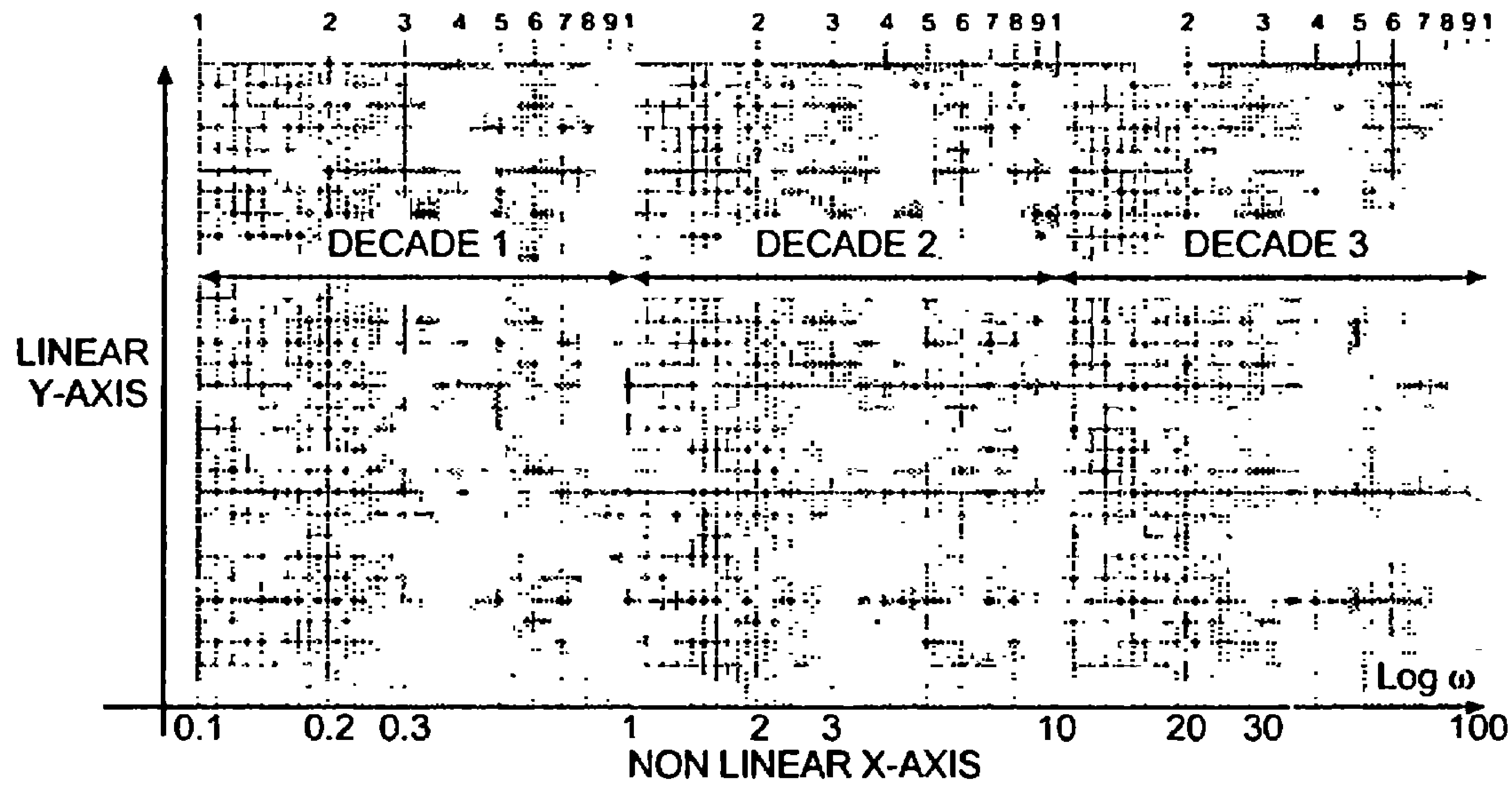


Fig. 11.3 Semilog paper

11.3 Standard Form of Open Loop T.F. G(jω)H(jω)

Consider $G(s)H(s) = \dfrac{K' s^Z (s+Z_1)(s+Z_2) \ldots\ldots}{s^P (s+P_1)(s+P_2) \ldots\ldots\ldots}$

Note **that either** s^Z or s^P **will be present at a time and not both.** But this form is not useful to sketch the Bode Plot.

Hence it is necessary to rewrite the G(s)H(s) in the **time constant form.**

$$G(s)H(s) = \frac{s^Z K (1+T_1 s)(1+T_2 s) \ldots\ldots}{s^P (1+T_a s)(1+T_b s) \ldots\ldots}$$

where $K = \dfrac{Z_1 \times Z_2 \times \ldots\ldots}{P_1 \times P_2 \times \ldots\ldots\ldots} \times K'$

Again either s^Z or s^P is present and not both.

The standard time constant form can be denoted as,

$$\boxed{G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s) \ldots\ldots}{s^P (1+T_a s)(1+T_b s)}}$$

K = Resultant system gain P = Type of the system

$T_1, T_2, T_a, T_b, \ldots\ldots$ = Time constants of different poles and zeros.

11.4.1 Factor 1 : System Gain 'K'

$$G(s)H(s) = K$$

i.e. $$G(j\omega)H(j\omega) = K + j0$$

$$|G(j\omega)H(j\omega)| = \sqrt{K^2 + 0} = K$$

$$\text{Its 'dB' value} = 20 \log_{10} K \text{ dB}$$

As gain 'K' is constant, $20 \log_{10} K$ is always constant though 'ω' is varied from 0 to ∞. So its magnitude plot will be straight line parallel to X-axis.

So magnitude plot for K > 1 is a line parallel to X-axis at a distance of 20 Log K above 0 dB reference line. While for K < 1 it is at a distance of 20 Log K below 0 dB reference line.

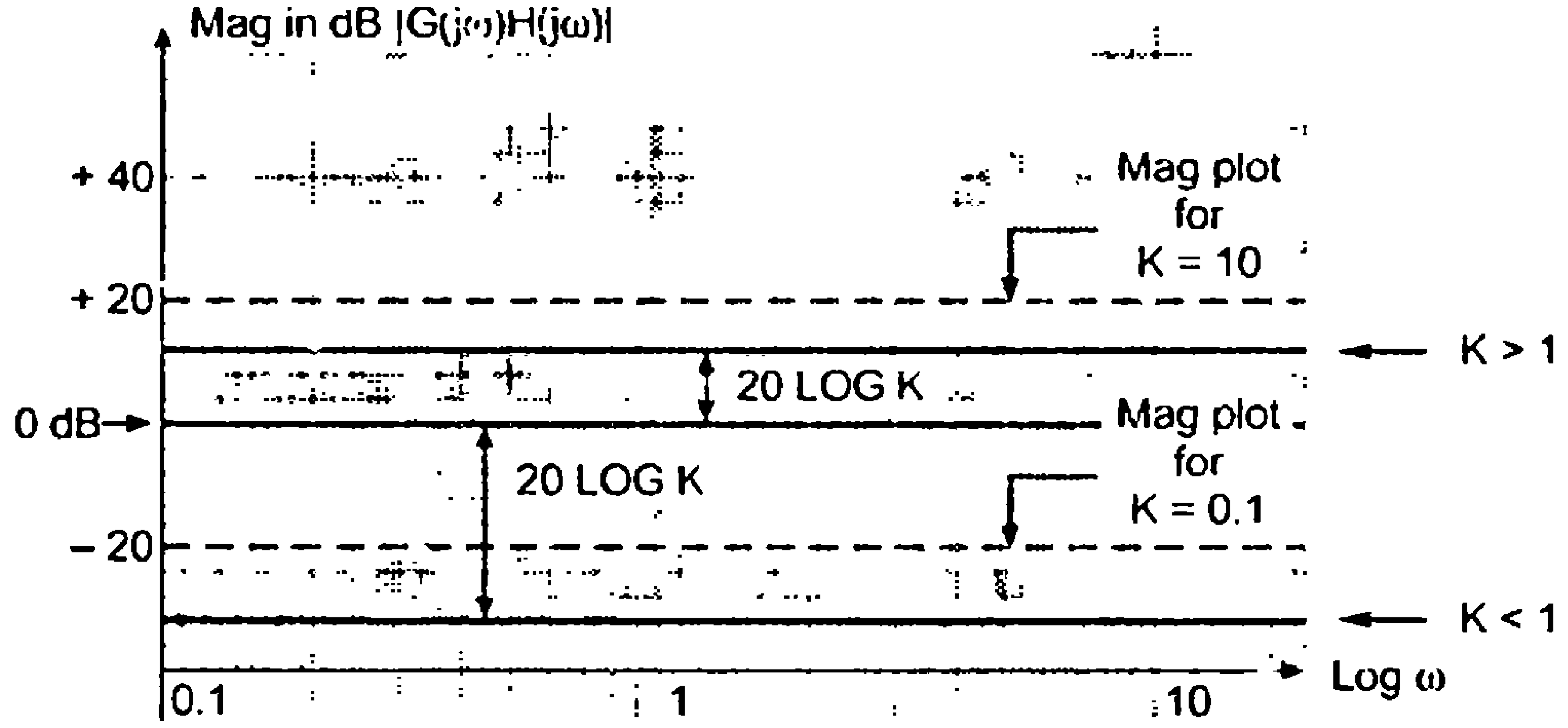


Fig. 11.4 Contribution by K

This means that in the variation of $|G(j\omega)H(j\omega)|$ effect of 'K' is constant equal to 20 Log K dB for all frequencies.

Key Point: *This means 'K' shifts the magnitude plot of $|G(j\omega)H(j\omega)|$ by a distance of 20 Log K dB upwards if K > 1 and downwards if K < 1.*

This fact is useful to design 'K' for the required specification. In such case $|G(j\omega)H(j\omega)|$ plot can be plotted with 'K' as unknown and then it can be just shifted upwards or downwards so as to meet the required specification. This shift is nothing but 20 log K dB, from which required 'K' can be determined.

Phase Angle Plot :

As $$G(j\omega)H(j\omega) = K + j0$$

Corresponding $\phi = \tan^{-1} \frac{\text{imj part}}{\text{real part}} = \tan^{-1} \frac{0}{K} = 0°$

So it does not affect the phase angle plot as its contribution to phase angle plot is 0°.

Key Point: *This means that phase plot specifications remain as it is for any positive value of 'K'.*

But if 'K' is negative, it always contributes – 180° to the phase angle plot independent of frequency.

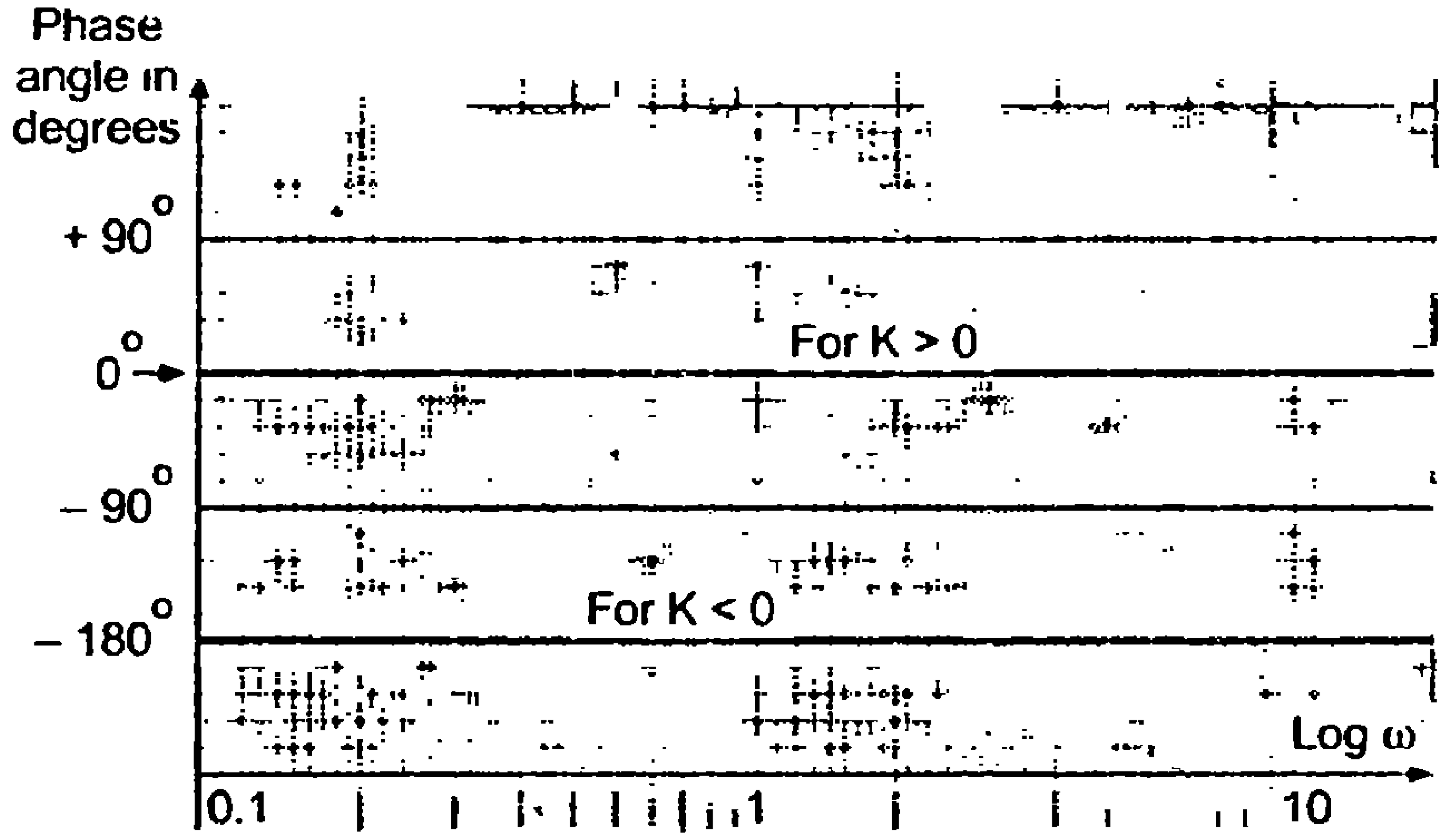


Fig. 11.5

11.4.2 Factor 2 : Poles or Zeros at the Origin $(j\omega)^{\pm P}$

Let us consider for simplicity single pole at the origin

$$G(s)H(s) = \frac{1}{s}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{1}{j\omega} = \frac{1}{0+j\omega}$$

$$\therefore \quad \text{For magnitude plot} \rightarrow |G(j\omega)H(j\omega)| = \frac{1}{\sqrt{0^2+\omega^2}} = \frac{1}{\omega}$$

$$\therefore \text{ Magnitude in dB} = 20 \log \frac{1}{\omega} \text{ dB}$$

$$= 20 \log (\omega)^{-1} \text{ dB}$$

$$= -20 \log \omega \text{ dB}$$

Thus this equation is similar to y = mx i.e. 1 pole at the origin contributes to the magnitude plot according to the equation – 20 log ω i.e. according to the straight line of slope '– 20 '.

Let us see the unit of the slope.

Equation is, $|G(j\omega)H(j\omega)| = -20 \log \omega$

If $\omega = 1 \rightarrow |G(j\omega)H(j\omega)| = 0$ dB

$\omega = 10 \rightarrow |G(j\omega)H(j\omega)| = -20$ dB

$\omega = 100 \rightarrow |G(j\omega)H(j\omega)| = -40$ dB

$\omega = 0.1 \rightarrow |G(j\omega)H(j\omega)| = +20$ dB

Now 10 times changes in frequency range is called 1 decade described earlier i.e. 1 pole at origin reduces the $|G(j\omega)H(j\omega)|$ at the rate of – 20 dB per decade. Hence slope of magnitude plot for 1 pole at the origin is called – 20 dB/decade.

So magnitude plot for 1 pole at origin is a straight line of slope – 20 dB/ decade.

Now at $\omega = 1$, $|G(j\omega)H(j\omega)| = 0$ dB i.e. this line intersects the reference 0 dB line at $\omega = 1$.

At $\omega = 0.1$ it has magnitude +20 dB while at $\omega = 10$ it has magnitude of –20 dB.

As $\omega = 0$ cannot be indicated, the starting frequency may be selected as per the requirement. This contribution is valid for range of 'ω' from 0 to ∞. To sketch such a line of slope –20 dB/decade, first mark the intersection point of $\omega = 1$ with 0 dB line and then go up by 20 dB for each $1/10^{th}$ reduction in frequency from $\omega = 1$ i.e. + 20 dB for $\omega = 0.1$, + 40 dB for $\omega = 0.01$ or go down by 20 dB for each 10 times increase in frequency from $\omega = 1$ i.e. -20 dB for $\omega = 10$, -40 dB for $\omega = 100$ and so on. Then draw a straight line, as shown in the Fig. 11.6.

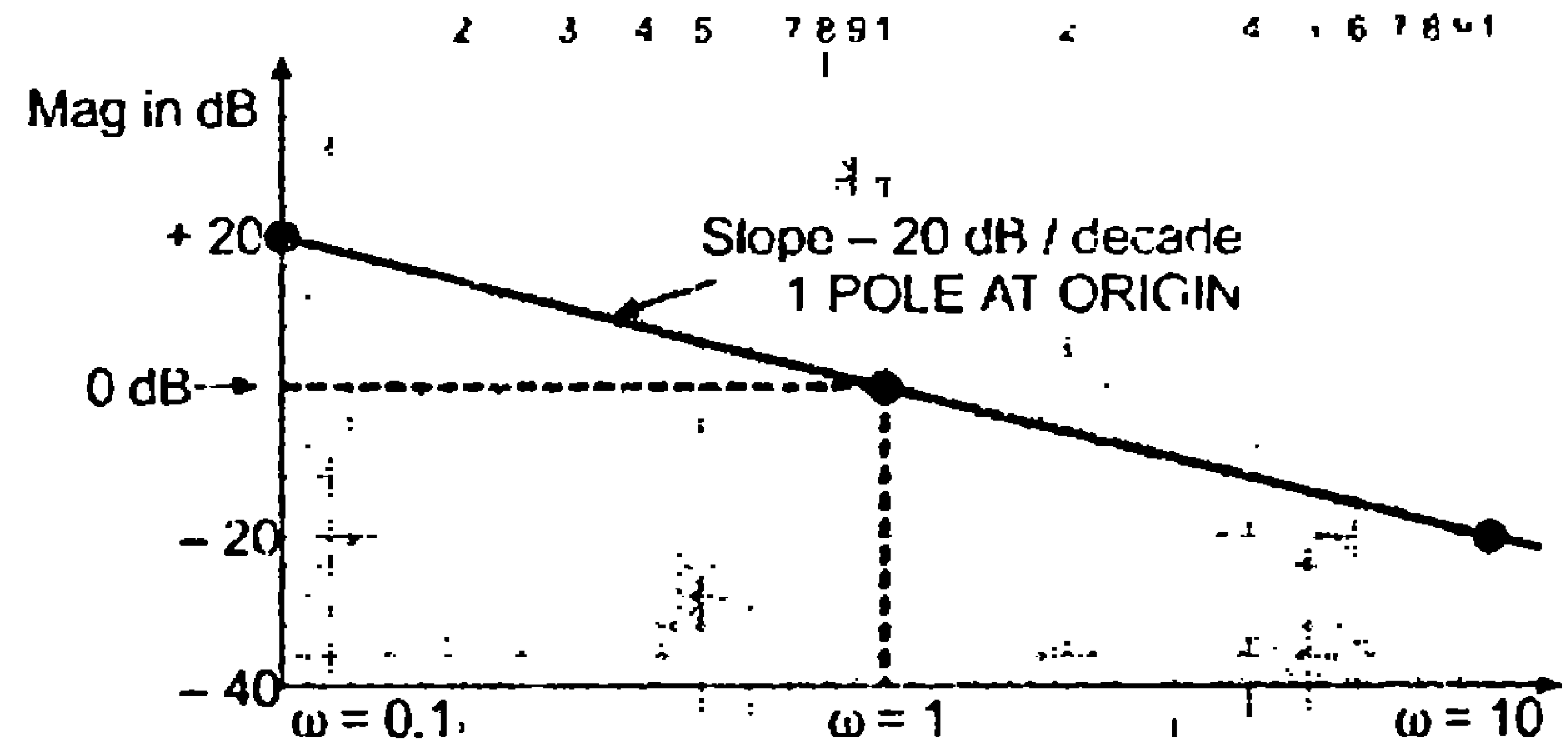


Fig. 11.6 Contribution by 1 pole at origin

Consider two poles at origin

$$G(s)H(s) = \frac{1}{s^2}$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega} \cdot \frac{1}{j\omega}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \frac{1}{\omega} \cdot \frac{1}{\omega} = \frac{1}{\omega^2}$$

$$\therefore |G(j\omega)H(j\omega)| \text{ in dB} = 20 \text{ Log } \frac{1}{\omega^2} = 20 \text{ Log } (\omega)^{-2}$$

$$= -40 \text{ Log } \omega$$

So it is straight line of slope – 40 dB/decade.

$\therefore$ Magnitude in dB $= 20 \text{ Log } \omega \text{ dB.}$

This is the equation of a straight line whose slope is +20 dB/ decade. The only change is the sign of the slope, for pole it is – 20 dB/ decade while for zero it is + 20 dB/ decade but for both, intersection of line with 0 dB occurs at $\omega = 1$ only.

In general for P number of zeros at the origin

$$G(s)H(s) = s^P$$

$$\therefore \quad G(j\omega)H(j\omega) = j\omega \cdot j\omega \cdot j\omega \text{......P times}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \omega^P$$

$$\therefore \text{ Magnitude in dB} = 20 \times P \text{ Log } \omega$$

$$\text{i.e.} \quad \text{slope} = +20 \times P \text{ dB/decade}$$

So it gives family of lines with slopes as +20, + 40 ..... + 20 x P dB/decade passing through intersection point of $\omega = 1$ with 0 dB line as shown in the Fig. 11.8.

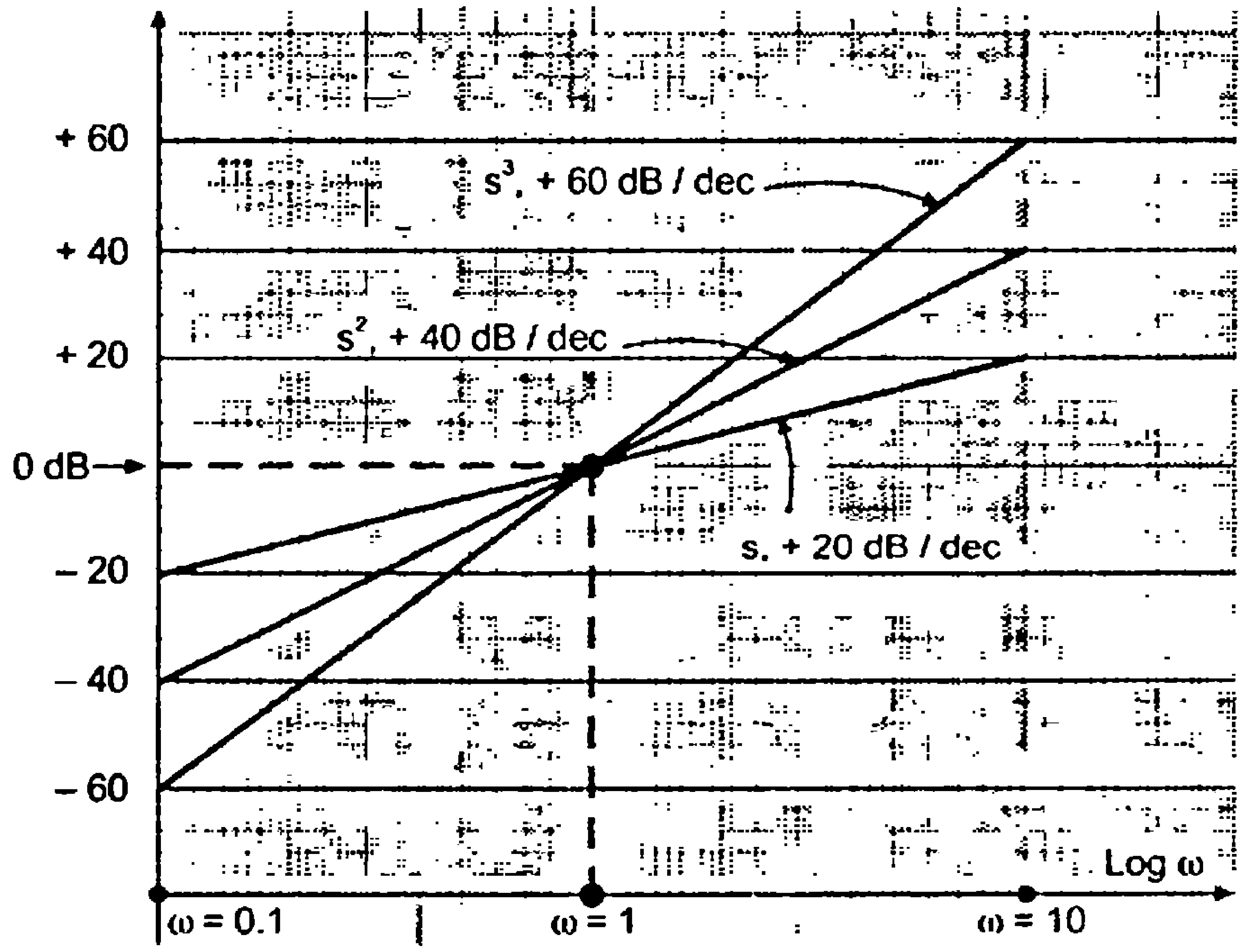


Fig. 11.8 Contribution by zeros at origin

Key Point: *Each zero at the origin increases the magnitude at a rate of + 20 dB/decade.*

Phase Angle Plot : Consider 1 pole at the origin

$$G(s)H(s) = \frac{1}{s} \qquad G(j\omega)H(j\omega) = \frac{1}{j\omega}$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle \frac{1}{j\omega} = \frac{0^\circ}{90^\circ} = -90^\circ$$

This is independent of 'ω'. So phase angle plot of pole at origin is line parallel to X-axis contributing – 90° to phase angle.

For 2 poles at origin,

$$G(s)H(s) = \frac{1}{s^2}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{1}{j\omega} \cdot \frac{1}{j\omega}$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle\frac{1}{j\omega}\angle\frac{1}{j\omega} = \frac{0^\circ}{90^\circ \cdot 90^\circ} = -180^\circ$$

Angle gets added to each other.

Key Point: *In general P number of poles at the origin contribute – 90° × P angle to overall phase angle plot. This contribution is constant irrespective of 'ω'.*

Similarly for a zero at the origin,

$$G(s)H(s) = s \qquad \text{i.e.} \qquad G(j\omega)H(j\omega) = j\omega$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle 0 + j\omega = +\tan^{-1}\frac{\omega}{0} = +90^\circ$$

1 zero at the origin contributes + 90°. The contribution is same as that of pole, the only change is its sign.

Key Point: *In general 'P' number of zeros at the origin, the total angle contribution is +90° × P, irrespective of value of ω.*

This can be shown as in the Fig. 11.9.

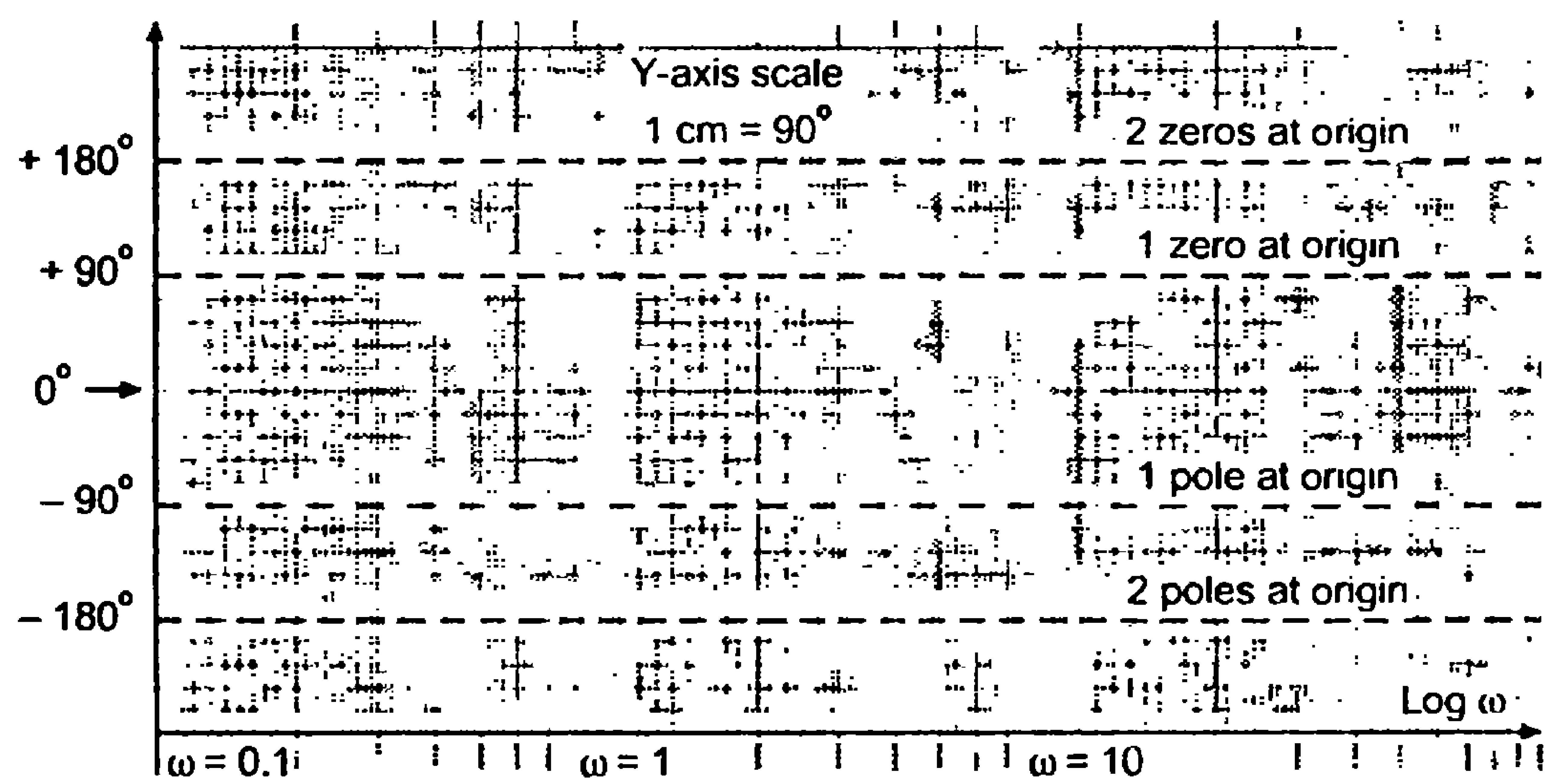


Fig. 11.9 Angle contribution

Before going to next factor, let us see the addition of first two factors on semilog paper. The magnitude plots for poles or zeros at the origin are straight lines having slope $-20 \times P$ dB/decade or $+20 \times P$ dB/decade respectively passing through intersection point of $\omega = 1$ and 0 dB line. Now adding magnitude plot of 'K' to the above means to shift the straight line drawn, upwards or downwards by 20 Log K dB depending on whether K is greater or less than 1. This shift experienced, will be same by all points on the straight line representing poles or zeros at the origin. Hence net addition of 'K' and pole or zeros at origin will be a line parallel to line representing poles or zeros at the origin at a distance of 20 Log K dB upwards or downwards from the 0 dB line.

e.g. Consider $G(s)H(s) = \frac{10}{s}$ so $G(j\omega)H(j\omega) = \frac{10}{j\omega}$

Factors are :

i) Constant K = 10, its contribution to magnitude plot is
20 Log K = 20 Log 10 = + 20 dB.

ii) 1 pole at the origin whose magnitude plot is straight line of slope – 20 dB/decade passing through intersection point of $\omega = 1$ and 0 dB line.

Now at $\omega = 1$, total magnitude will be addition of magnitudes of K and 1/s.

i.e. = 20 dB due to 'K' + 0 dB due to 1 pole at origin

at $\omega = 1$ = 20 dB

i.e. after addition of two lines, intersection point of $\omega = 1$ and 0 dB will shift upwards by 20 dB. So to draw resultant of the two, we can generalize the procedure as,

i) Draw magnitude plot for K.
ii) Draw straight line representing pole at origin i.e. of slope $-20 \times P$ dB/ decade, passing through intersection point of $\omega = 1$ and 0 dB.
iii) Shift intersection point of $\omega = 1$ and 0 dB on the line representing 20 Log K line.
iv) Draw parallel line to the line representing pole at origin from the point obtained in step (iii).

The slope of this line will be same as the slope of line representing poles or zeros at the origin. In this example slope of resultant line will be – 20 dB/ decade. This is because slope of 20 Log K line is 0 dB/decade.

Key Point: *When two lines are added together, the resultant line always has a slope which is algebraic addition of the individual slopes of the two lines which are added.*

So magnitude plot for above G(s)H(s) is as shown in the Fig. 11.10.

Phase Angle Plot :

Prepare the table of individual angle contributions and add them to get resultant phase angles.

i) K = 20 ∴ Its magnitude = 20 Log 20 = + 26 dB

ii) 1 pole at origin. Its magnitude plot is straight line passing through intersection point of $\omega = 1$ and 0 dB with slope – 20 dB/decade.

iii) Simple pole → $\frac{1}{1+0.1\,s}$, comparing with $\frac{1}{1+Ts}$

$$\therefore \quad T = 0.1$$

$$\therefore \quad \omega_C = \frac{1}{T} = \frac{1}{0.1} = 10 \text{ rad/sec}$$

i.e. Asymptotic magnitude plot is 0 dB upto $\omega = \omega_C = 10$ and then straight line of slope – 20 dB/decade. Procedure to plot resultant,

i) Draw 20 Log K line. ii) Draw line for 1 pole at origin.

iii) Shift intersection point of $\omega = 1$ and 0 dB on 20 log K line and from this point draw parallel to a line representing 1 pole at origin. This line will have slope – 20 dB/ decade.

iv) This addition of K and poles at origin will continue, till next factor becomes dominant i.e. at $\omega = \omega_C = 10$.

Hence resultant slope from $\omega = 10$ onwards will be (– 20 dB/decade as starting slope) + (– 20 dB/decade) due to simple pole i.e. resultant – 40 dB/ decade. This will continue upto $\omega \to \infty$ as there is no other factor present in G(s)H(s).

Procedure to draw – 40 dB / decade line :

On the semilog paper itself, draw the lines of different slopes as –20, – 40, – 60, – 80, + 20 dB/decade etc. very light, as shown and then draw parallel to these lines of the required slope in magnitude plot, wherever necessary.

Such lines are shown in the Fig. 11.15.

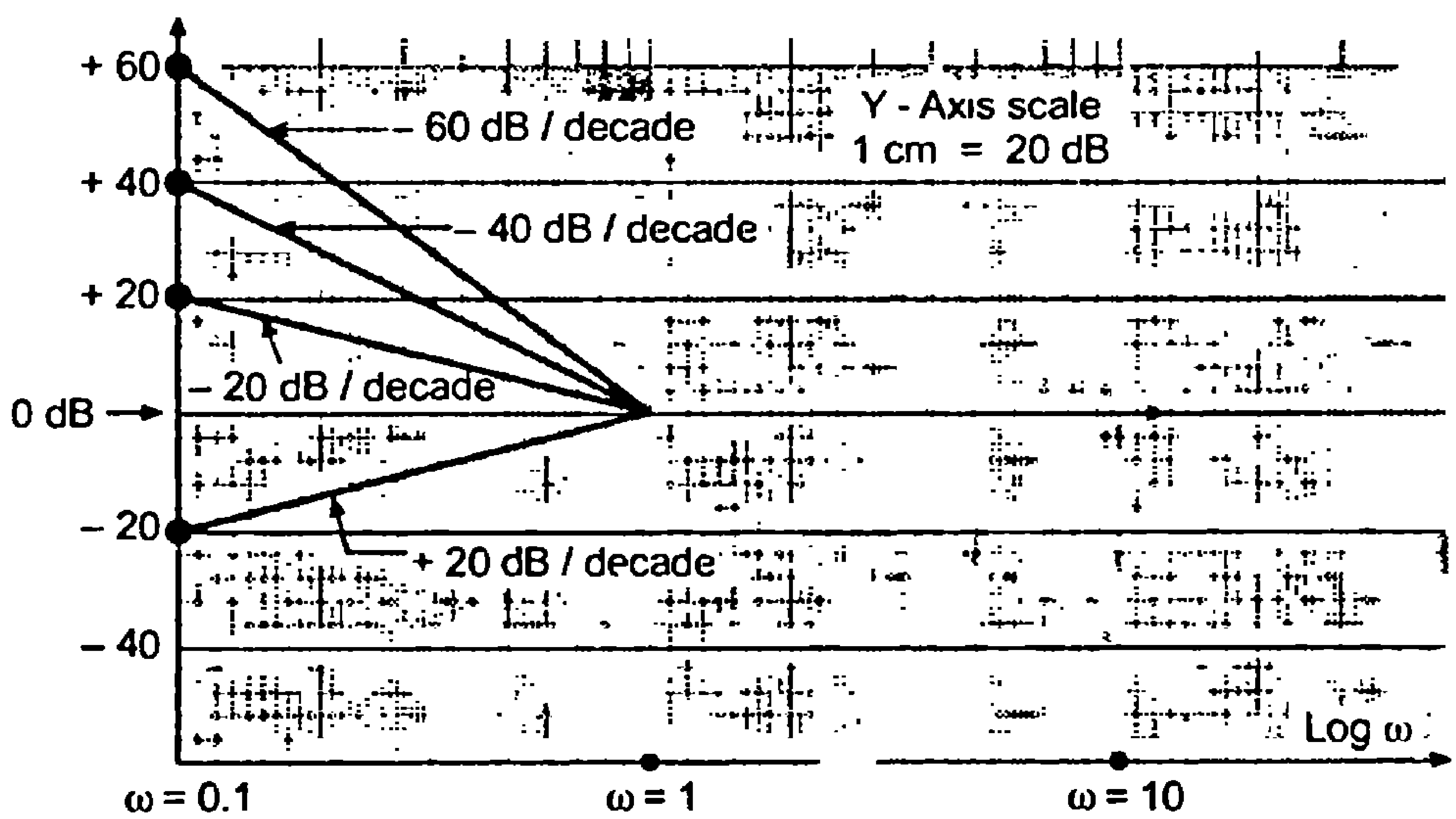


Fig. 11.15

Draw such lines very light and then draw parallel to these lines from the required point of required slope.

For the phase angle plot prepare the table of angles as below :

ω in rad/sec	φ due to 1 pole at origin	φ due to simple pole $= -\tan^{-1} 0.1\omega$	ϕ_R Resultant
0.1	– 90°	– 0.57°	– 90.57°
0.5	– 90°	– 2.86°	– 92.86°
1	– 90°	– 5.7°	– 95.7°
2	– 90°	– 11.3°	– 101.3°
10	– 90°	– 45°	– 135°
50	– 90°	– 78.79°	– 168°

φ due to simple pole $= -\tan^{-1} \omega T = -\tan^{-1} 0.1\omega$

φ due to 1 pole at the origin is always – 90°. If required, more 'ω' values may be selected to draw the smooth curve. Let us combine all the things on semilog paper to complete the Bode plot.

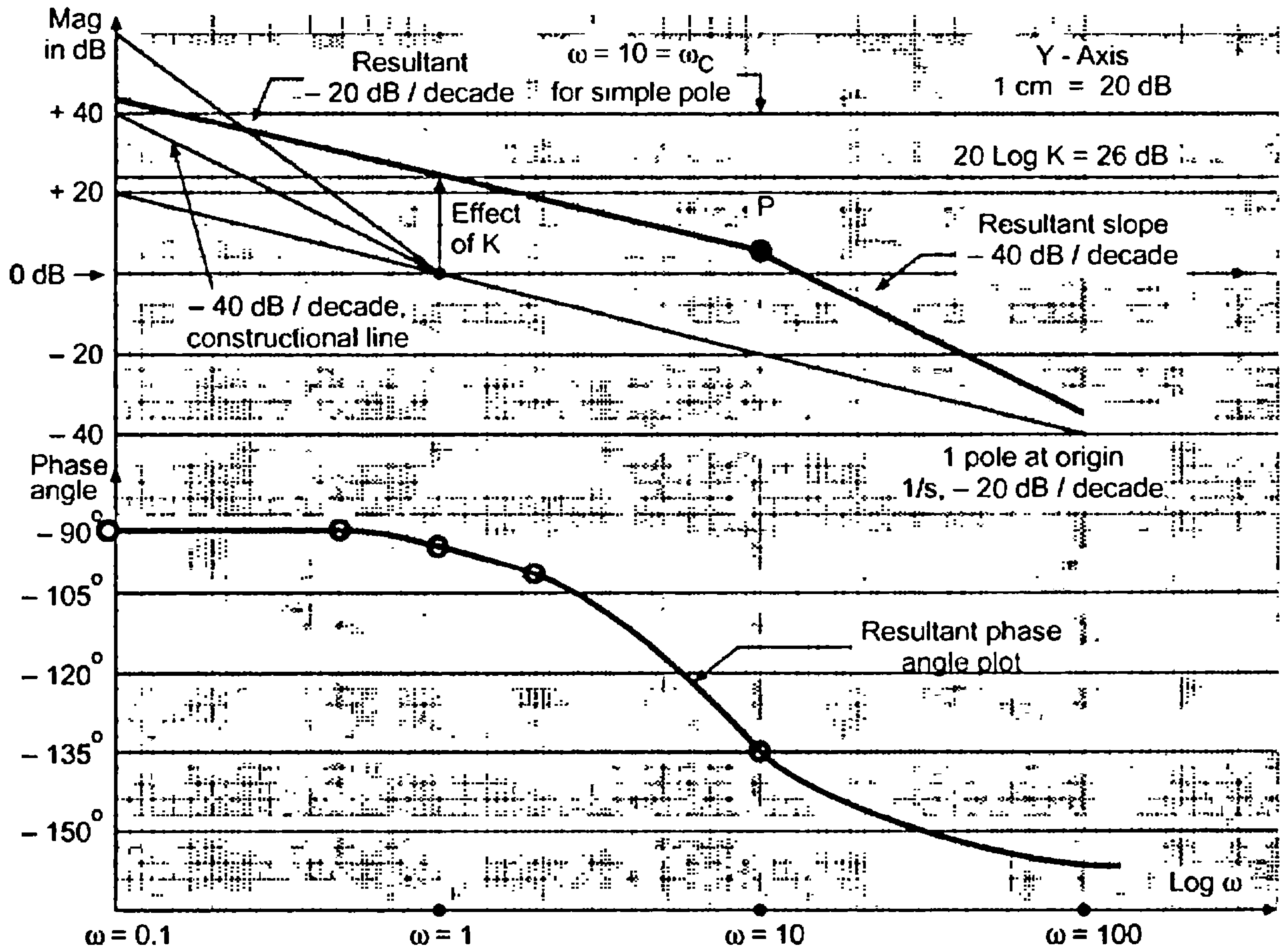


Fig. 11.16

- Now $\angle|G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$ can be read out from phase angle plot. Extend $\omega = \omega_{gc}$ line downwards till it intersects phase angle plot. This point of intersection is point C as shown in the Fig. 11.21.

Key Point: *The distance between this point C and –180° line i.e. point D is nothing but the phase margin. If point C is above –180° line, P.M. is positive and if it is below –180° line, P.M. is negative.*

System is said to be stable when P.M. and G.M. are positive while system is said to be unstable when both P.M. and G.M. are negative. Now when system is on the verge of instability i.e. marginally stable in nature then G.M. and P.M. both are zero. This is possible when $\omega_{gc} = \omega_{pc}$. This condition $\omega_{gc} = \omega_{pc}$ is useful to design the marginally stable systems.

For G.M. and P.M. positive i.e. stable system, $\omega_{gc} < \omega_{pc}$. While for G.M. and P.M. negative i.e. unstable system, $\omega_{gc} > \omega_{pc}$.

In some absolutely stable systems G.M. may be obtained as $+\infty$ while for inherently unstable systems G.M. may be obtained as $-\infty$.

All such G.M. and P.M. conditions are shown in figures below.

Stability Conditions :

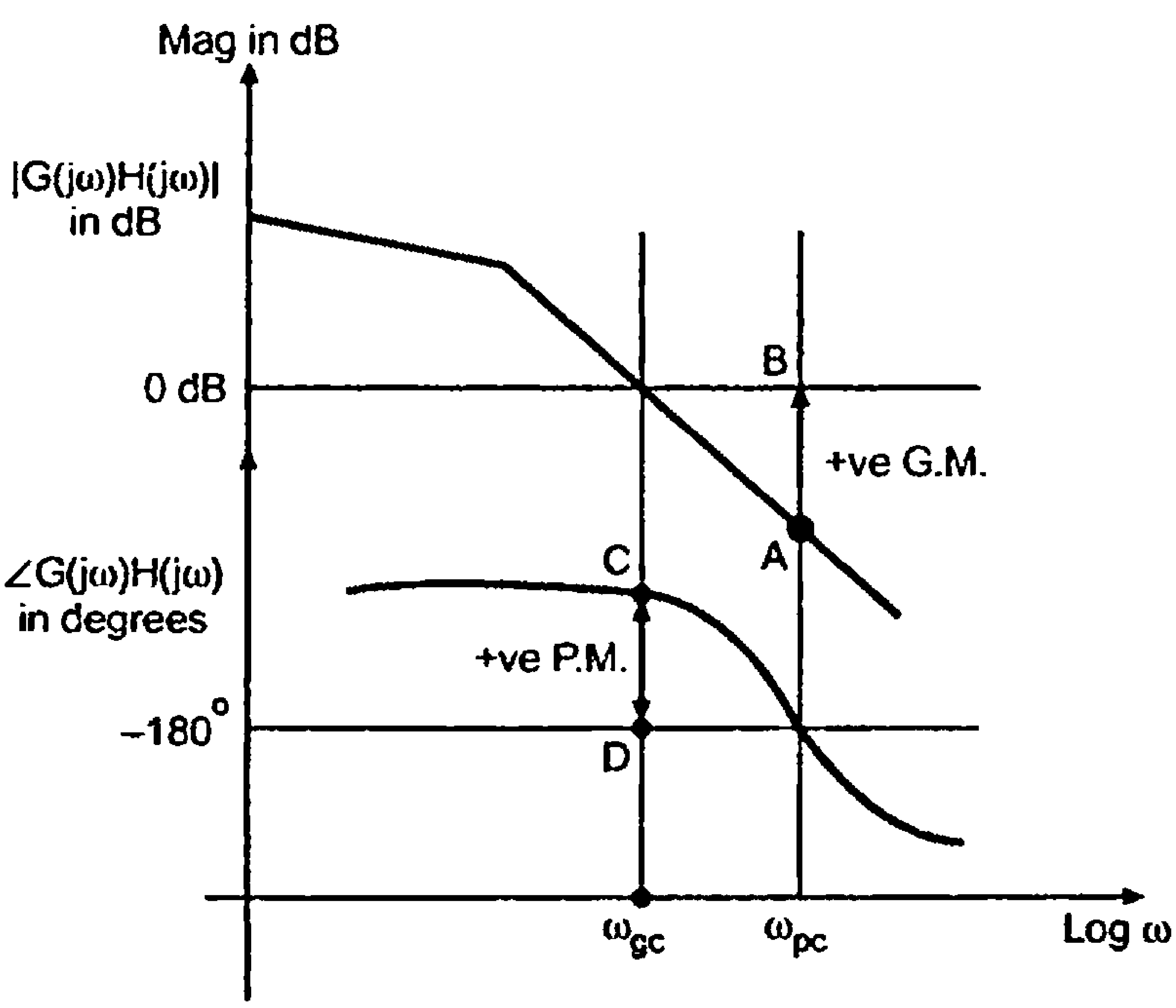


Fig. 11.21 $\omega_{gc} < \omega_{pc}$ **G.M. and P.M. positive, stable system**

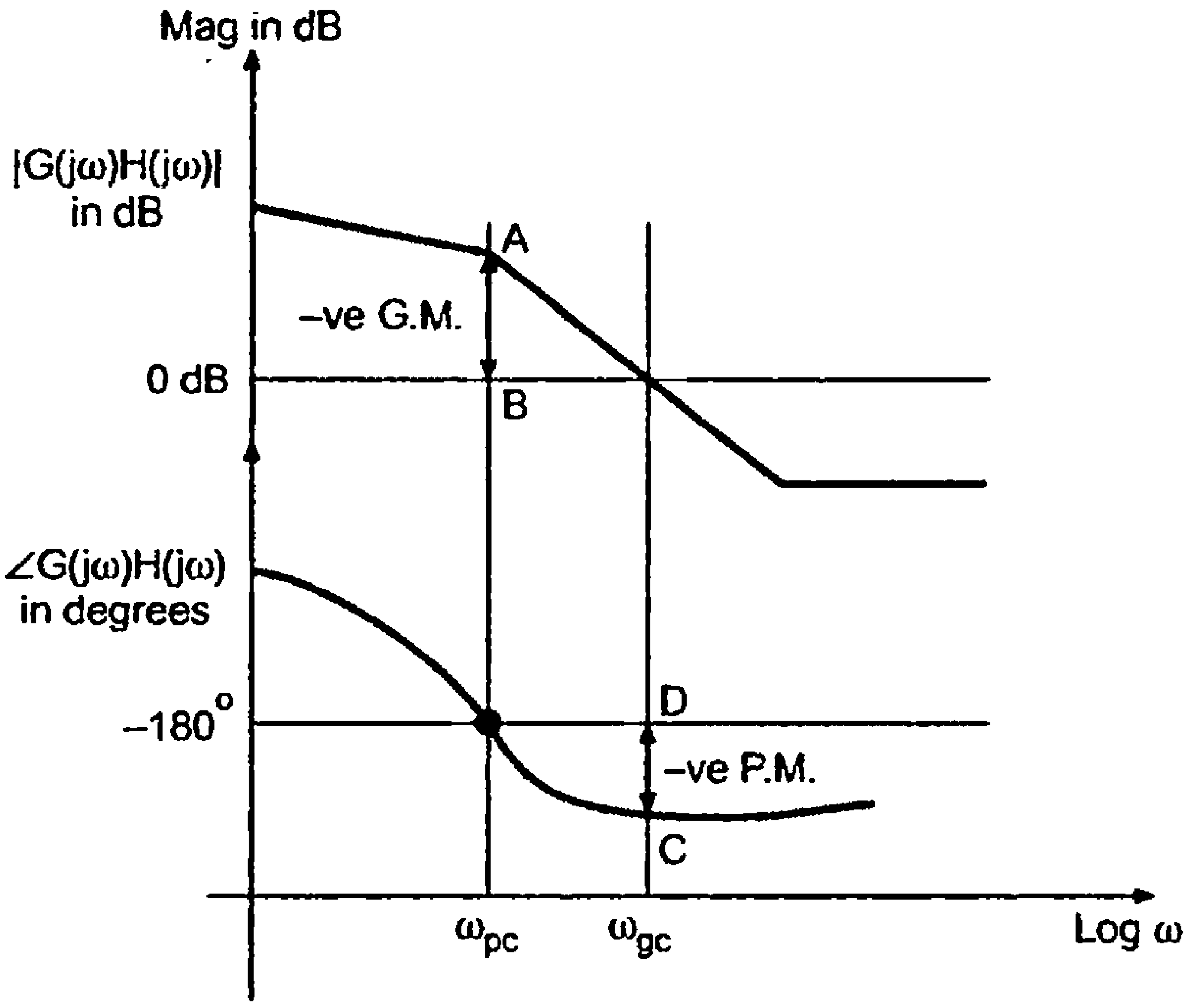


Fig. 11.22 $\omega_{gc} > \omega_{pc}$ G.M. and P.M. negative, unstable system

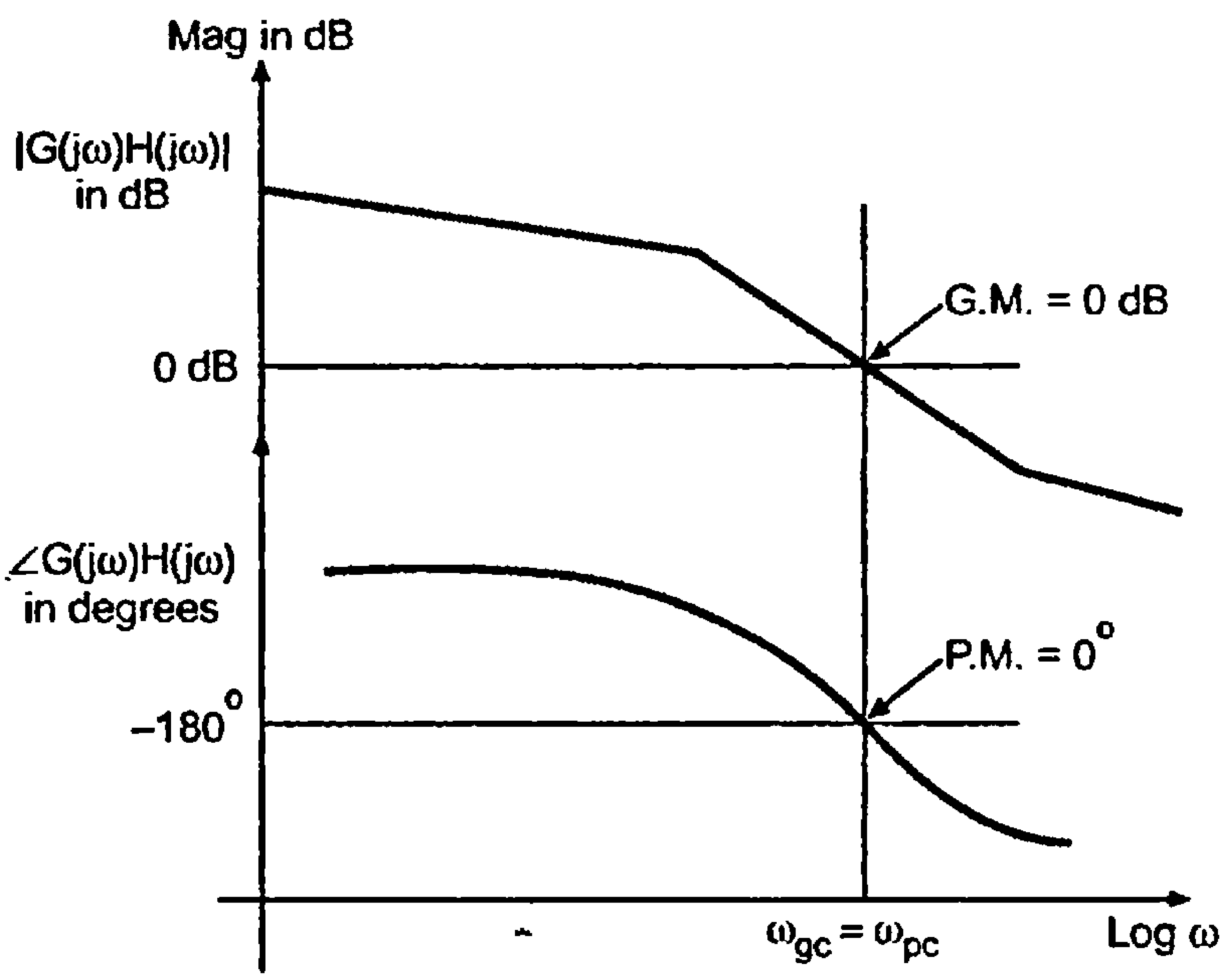


Fig. 11.23 $\omega_{gc} = \omega_{pc}$ G.M. and P.M. zero, marginally stable system

11.8 What should be Values of G.M. and P.M. of a Good System ?

It is obvious that a system should have gain which is lower than critical value so that the system is far removed from unstable conditions.

This is necessary because for most of the systems the transfer function of components and systems changes with variation in temperature and pressures of surrounding environment. Moreover the gains are also dependent on supply frequency, supply voltage, loading conditions, variations in control energy sources such as pneumatic air pressure (in pneumatic systems).

$$\therefore \quad G(j\omega)H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega} = \frac{\omega_n^2}{-\omega^2 + 2\xi\omega_n j\omega}$$

ω_{gc} is the gain crossover frequency for which magnitude of $G(j\omega)H(j\omega)$ is 0 dB i.e. $20 \log |G(j\omega)H(j\omega)| = 0$ dB

$$\therefore \quad |G(j\omega)H(j\omega)| = 1 \quad \text{for } \omega = \omega_{gc}$$

For ω_{gc}, $|G(j\omega)H(j\omega)| = 0$ dB i.e. $|G(j\omega)H(j\omega)| = 1$

$$\text{Now} \quad |G(j\omega)H(j\omega)| = \frac{\omega_n^2}{\sqrt{(-\omega^2)^2 + (2\xi\omega\omega_n)^2}} = \frac{\omega_n^2}{\sqrt{\omega^4 + 4\xi^2\omega^2\omega_n^2}} = 1$$

Squaring both sides.

$$\frac{\omega_n^4}{\omega^4 + 4\xi^2\omega^2\omega_n^2} = 1$$

$$\therefore \quad \omega_n^4 = \omega^4 + 4\xi^2\omega^2\omega_n^2$$

$$\therefore \quad \omega^4 + 4\xi^2\omega^2\omega_n^2 - \omega_n^4 = 0$$

Now ω_n is constant, we want to determine 'ω' which is ω_{gc} as $|G(j\omega)H(j\omega)|$ is equated to 1. It is quadratic in ω^2 i.e. ω_{gc}^2.

$$\therefore \quad \omega_{gc}^2 = \frac{-4\xi^2\omega_n^2 \pm \sqrt{(4\xi^2\omega_n^2)^2 - 4\times 1\times(-\omega_n^4)}}{2\times 1}$$

$$= \frac{-4\xi^2\omega_n^2 \pm \sqrt{16\xi^4\omega_n^4 + 4\omega_n^4}}{2} = -2\xi^2\omega_n^2 \pm \frac{\omega_n^2\sqrt{16\xi^4 + 4}}{2}$$

$$\omega_{gc}^2 = -2\xi^2\omega_n^2 \pm \omega_n^2\sqrt{4\xi^4 + 1}$$

Now ω_{gc} cannot be negative.

$$\therefore \quad \omega_{gc} = \sqrt{-2\xi^2\omega_n^2 + \omega_n^2\sqrt{4\xi^4+1}} = \omega_n\sqrt{-2\xi^2 + \sqrt{4\xi^4+1}}$$

Now let us determine phase margin.

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2\xi\omega\omega_n j}$$

$$\text{P.M.} = 180° + \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle \omega_n^2 + j0}{\angle -\omega^2 + 2\xi\omega\omega_n j}$$

Now, $\angle\omega_n^2 + j0 = 0°$ as ω_n is positive

$$\frac{1}{\angle -\omega^2 + 2\xi\omega\omega_n j} = -\tan^{-1}\left\{\frac{2\xi\omega\omega_n}{-\omega^2}\right\} = -\tan^{-1}\left\{\frac{2\xi\,\omega_n}{-\omega}\right\}$$

Mathematically this angle can be written as positive with 180° subtracted from it. To absorb negative sign it is required to subtract 180° from it.

$$\therefore \quad \angle G(j\omega)H(j\omega)\,|_{\omega=\omega_{gc}} = +\tan^{-1}\left\{\frac{2\xi\,\omega_n}{\omega_{gc}}\right\} - 180^\circ$$

Substitute ω_{gc} as derived.

$$\angle G(j\omega)H(j\omega)\,|_{\omega=\omega_{gc}} = +\tan^{-1}\left\{\frac{2\xi\,\omega_n}{\omega_n\sqrt{-2\xi^2+\sqrt{4\xi^4+1}}}\right\} - 180^\circ$$

$$\therefore \quad P.M. = 180^\circ + \angle G(j\omega)H(j\omega)\,|_{\omega=\omega_{gc}}$$

$$= 180^\circ + \tan^{-1}\left\{\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{4\xi^4+1}}}\right\} - 180^\circ$$

$$\therefore \quad \boxed{P.M. = +\tan^{-1}\left\{\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{4\xi^4+1}}}\right\}}$$

Example 11.2 : *A unity feedback control system has* $G(s) = \dfrac{80}{s(s+2)(s+20)}$. *Draw the Bode Plot. Determine G.M. P.M.* ω_{gc} *and* ω_{pc}. *Comment on the stability.*

Solution : Step 1 : Arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{80}{s(s+2)(s+20)} \quad , \; H(s) = 1$$

$$= \frac{80}{s(2)\left(1+\frac{s}{2}\right)(20)\left(1+\frac{s}{20}\right)} = \frac{2}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{20}\right)}$$

Step 2 : Identify factors :

i) K = 2 ii) 1 pole at origin

iii) Simple pole $\dfrac{1}{\left(1+\frac{s}{2}\right)}$ with $T_1 = \dfrac{1}{2}$ $\therefore\; \omega_{C1} = \dfrac{1}{T_1} = 2$

iv) Simple pole $\dfrac{1}{\left(1+\frac{s}{20}\right)}$ with $T_2 = \dfrac{1}{20}$ $\therefore\; \omega_{C2} = \dfrac{1}{T_2} = 20$

Step 3 : Magnitude Plot Analysis

i) For K = 2, 20 Log K = 20 Log 2 = 6 dB

ii) For 1 pole at origin. Straight line of slope − 20 dB/decade passing through intersection point of $\omega = 1$ an 0 dB.

To draw straight lines of – 40 dB/decade and – 60 dB/decade from ω_{C1} = 2 and ω_{C2} = 20, draw faint lines of slopes – 20, – 40, – 60 dB/decade from intersection point of ω = 1 and 0 dB line and just draw parallel to them from respective points.

Step 5 : Bode plot and solution.

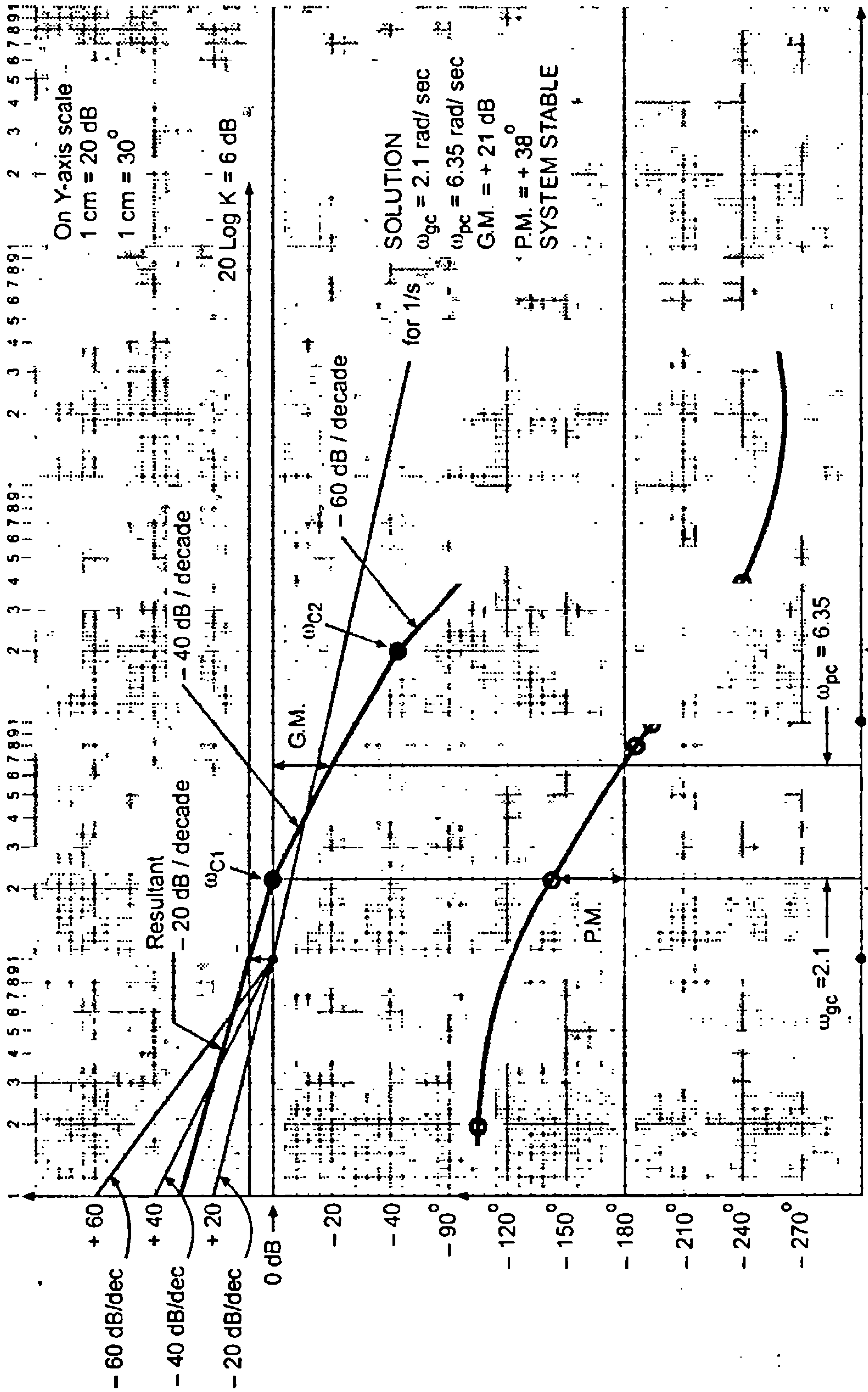
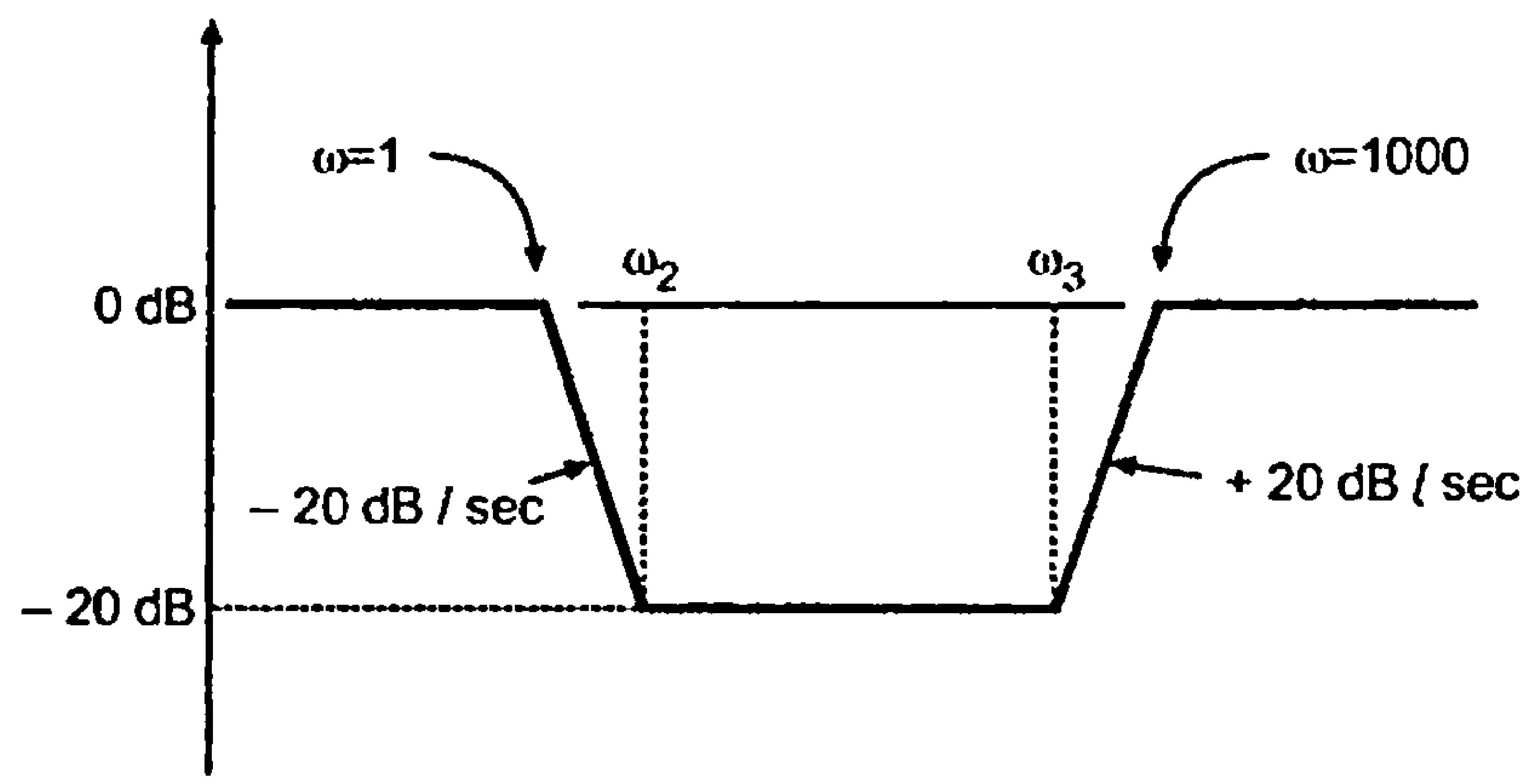


Fig. 11.24

7) It indicates how system should be compensated to get the desired response.
8) The value of system gain K can be designed for required specifications of G.M. and P.M. from Bode Plot.
9) Without the knowledge of the transfer function the Bode Plot of stable open loop system can be obtained experimentally.

Example 11.3 : *Determine the transfer function, of a system whose asymptotic gain plot is given below.*



Solution : Starting slope is 0 dB. So there is no pole or zero at the origin. The first change in slope occurs at $\omega = 1$

i.e. $\omega_{C1} = 1$

Change in slope = – 20 – 0 = – 20 dB/dec , there is simple pole with $\omega_{C1} = 1$.

i.e. $T_1 = \frac{1}{\omega_{C1}} = 1$

$\therefore$ Factor $\frac{1}{1+T_1 s} = \frac{1}{1+s}$

The shift at $\omega = 1$ is 0 dB so 20 Log K = 0 dB $\therefore K = 1$

The further change in slope occurs at ω_2 which is unknown but we can determine it. The change in slope occurs is 0 – (– 20) = + 20 dB/dec. i.e. there is simple zero. Now magnitude corresponding to ω_2 is – 20 dB and magnitude at $\omega = 1$ is 0 dB. So there is change of – 20 dB and slope of the line is – 20 dB/decade. i.e. ω_2 is decade away from $\omega = 1$.

i.e. $\omega_{C2} = \omega_2 = 10$

$T_2 = \frac{1}{\omega_{C2}} = 0.1$

$\therefore$ Factor is $(1 + T_2\ s) = (1 + 0.1\ s)$ as simple zero.

2 poles at the origin. So initial slope of magnitude plot is – 40 dB/dec and equation of this line is

$$|G(j\omega)H(j\omega)| \text{ in dB} = -20 \log \omega^2 + 20 \log K_a$$

$$\text{at } \omega = 1, -20 \log \omega^2 = 0 \text{ dB/decade}$$

$$|G(j\omega)H(j\omega)| = 20 \log K_a \text{ dB}$$

This is shown in the Fig. 11.27. When extended, this line will intersect 0 dB line at say ω_a

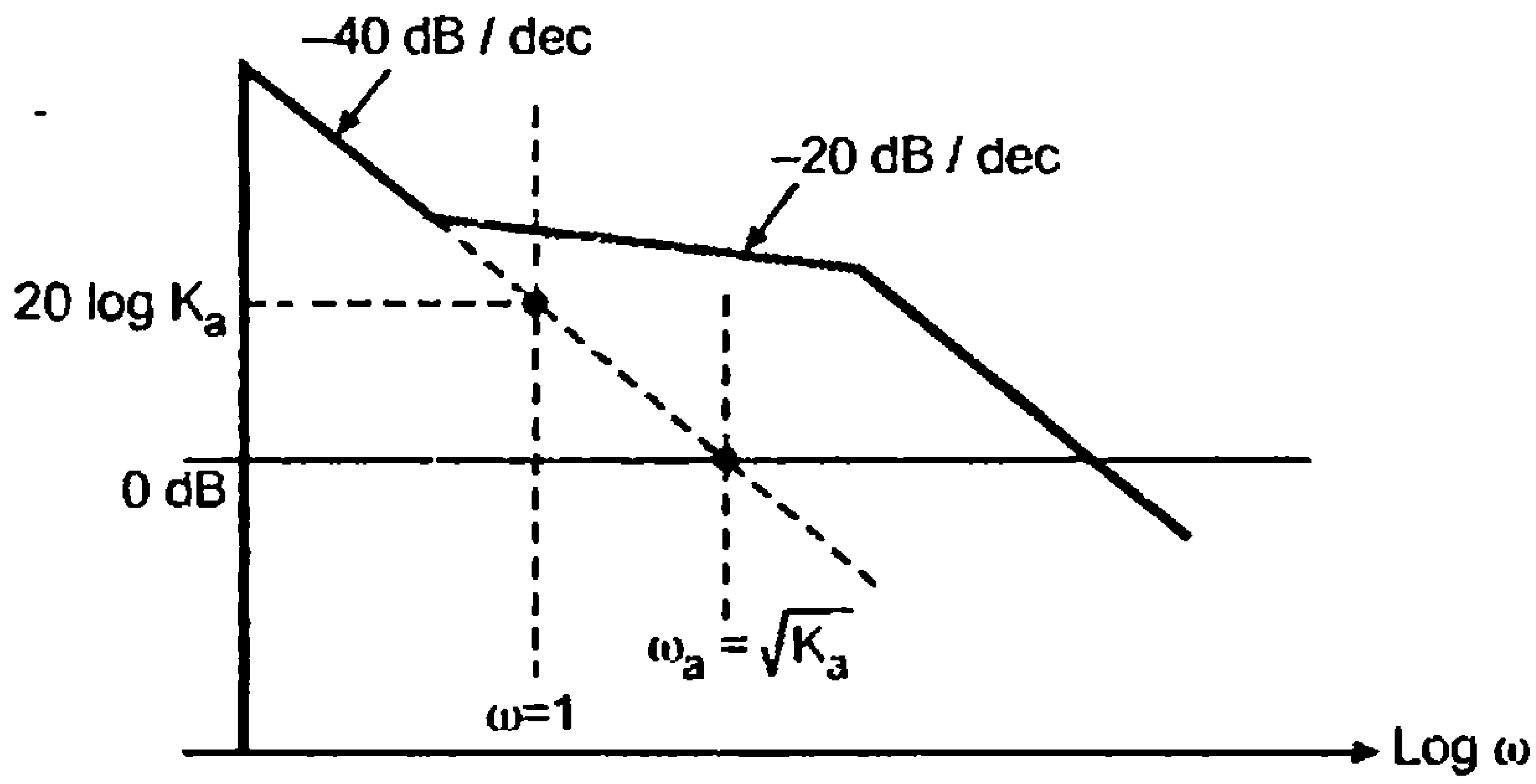


Fig. 11.27

$$0 \text{ dB} = -20 \log (\omega_a)^2 + 20 \log K_a$$

$$\text{i.e.} \quad +20 \log \frac{K_a}{(\omega_a)^2} = 20 \log 1$$

$$\therefore \quad K_a = (\omega_a)^2$$

Key Point: *So frequency at which initial line of slope – 40 dB/decade intersects with 0 dB line, gives the value of square root of acceleration error coefficient.*

$$\boxed{\omega_a = \sqrt{K_a}}$$

11.14 Bode Plot of Systems with Transportation Lag

Transportation lag is also called as **dead time or time delay.** In practical systems due to several reasons it is necessary to stop certain action in a system for some time. Such time delay is called as transportation lag. For example in modern systems using micro, controllers it is difficult to match the speed of peripherals with micro controller. In such a case it is necessary to provide purposely a time delay to micro controllers to adjust with the speeds of other supporting peripherals. Such time delay is called as transportation lag. This has an excessive phase lag with no attenuation at high frequencies.

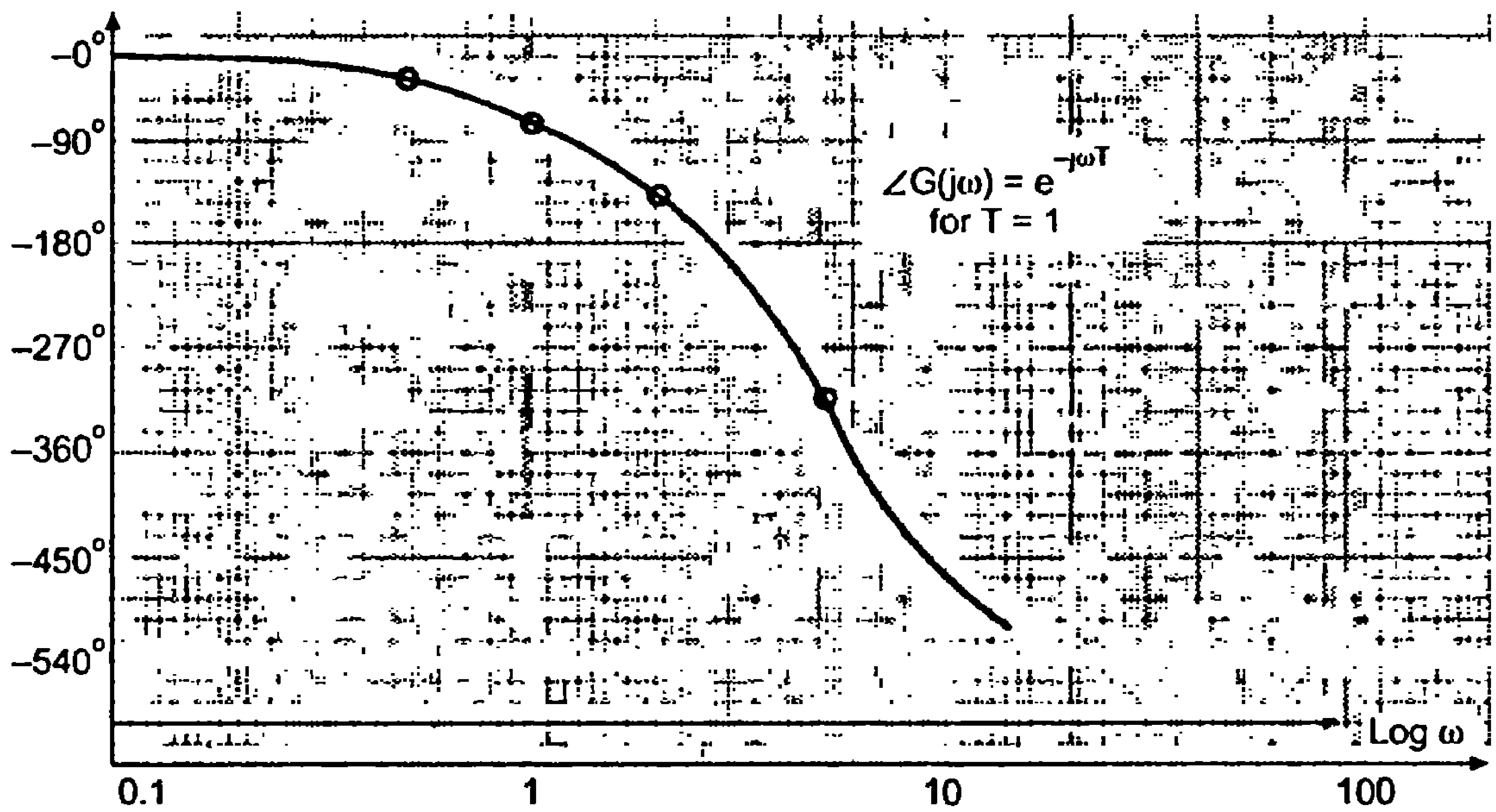


Fig. 11.29

Examples with Solutions

Example 11.4 : *For a particular unity feedback system,* $G(s) = \dfrac{242\,(s+5)}{s\,(s+1)\,(s^2+5s+121)}$. *Sketch the Bode Plot. Find* ω_{gc} *and* ω_{pc} *, G.M., P.M. Comment on stability.*

Solution : Step 1 : Arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{242\times5\times(1+\frac{s}{5})}{s\times(1+s)\,(121)\,(1+\frac{5}{121}s+\frac{s^2}{121})} = \frac{10\,(1+\frac{s}{5})}{s\,(1+s)\,(1+0.041\,s+\frac{s^2}{121})}$$

As G(s)H(s) includes a quadratic pole, comparing it with $\dfrac{1}{s^2+\xi\,\omega_n\,s+\omega_n^2}$, decide ξ and ω_n. These values give us proper correction to be applied at ω_n while sketching , magnitude plot of it.

∴ Comparing

$$\frac{1}{s^2+5s+121} \approx \frac{1}{s^2+2\,\xi\,\omega_n\,s+\omega_n^2}$$

$$\omega_n^2 = 121 \qquad \therefore\ \omega_n = 11$$

and $$2\,\xi\,\omega_n = 5 \qquad \therefore\ \xi = \frac{5}{2\times11} = 0.22$$

Referring to correction Table 11.2 discussed earlier in quadratic factor analysis, approximate correction required at $\omega_n = 11$ is + 7.95 dB upwards, for $\xi = 0.2$

Or correction can be calculated precisely as

$$\text{Correction} = -20\ \text{Log}\sqrt{4\xi^2} = -20\ \text{Log}\ 2\xi = -20\ \text{Log}\ (2\times0.22)$$

$$= +7.13\ \text{dB}$$

$$\angle \frac{1}{1+j\omega} = -\tan^{-1}\omega,$$

$$\angle \frac{1}{1+0.041\,j\omega-\frac{\omega^2}{121}} = -\tan^{-1}\left\{\frac{0.041\,\omega}{1-\frac{\omega^2}{121}}\right\}$$

∴ Phase Angle Table

ω	$\frac{1}{j\omega}$	$+\tan^{-1}\frac{\omega}{5}$	$-\tan^{-1}\omega$	$-\tan^{-1}\left\{\frac{0.041\,\omega}{1-\frac{\omega^2}{121}}\right\}$	ϕ_R
0.1	− 90°	+ 1.14°	− 5.7°	− 0.23°	− 94.7°
1	− 90°	+ 113°	− 45°	− 2.36°	− 126.0°
5	− 90°	+ 45°	− 78.6°	− 14.4°	− 138°
8	− 90°	+ 58°	− 82.8°	− 34.8°	− 149.8°
10	− 90°	+ 63.4°	− 84.2°	− 67.0°	− 177.8°
20	− 90°	+ 75.9°	− 87.13°	+ 19.5° − 180° = −160.4°	− 261.63°
∞	− 90°	+ 90°	− 90°	− 180°	− 270°

Step 5 : Sketch the Bode Plot and obtain the solution. (See the Fig. 11.30)

➠ **Example 11.5 :** *For a unity feedback system* $G(s) = \frac{K}{s\,(s+2)\,(s+10)}$. *Determine marginal value of 'K' for which system will be marginally stable, using Bode plot.*

Solution : Step 1 : Arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{K}{s\,(2)\,(1+\frac{s}{2})\,(10)\,(1+\frac{s}{10})}$$

$$= \frac{\frac{K}{20}}{s\,(1+\frac{s}{2})\,(1+\frac{s}{10})} = \frac{K'}{s\,(1+\frac{s}{2})\,(1+\frac{s}{10})}$$

$$\text{where } K' = \frac{K}{20}$$

To avoid the confusion, assume new constant $K' = \frac{K}{20}$ and calculate $K'_{marginal}$ by Bode plot and then corresponding $K_{marginal}$ can be determined by the above relation.

iv) At $\omega_{C_1}=1$, simple zero occurs, contributing + 20 dB/decade individually hence resultant will have slope − 40 + 20 = − 20 dB/dec. Hence '1' onwards slope of resultant will be − 20 dB/dec contributing upto next corner frequency ω_{C2} = 6.

v) At ω_{C2} = 6 , another simple zero occurs contributing + 20 dB/dec individually making the slope of the resultant will become − 20 + 20 = 0 dB/dec from 6 onwards i.e. line parallel to x-axis till next corner frequency ω_{C3} = 20.

vi) At ω_{C3} = 20 , quadratic pole occurs contributing − 40 dB/dec individually hence the slope of resultant will becomes 0 − 40 = − 40 dB/decade from 20 onwards and will continue upto '∞' as there is no other factor. But at ω_{C3} = 20 it will show overshoot of + 2 dB.

Step 4 : Phase Angle Plot.

$$G(j\omega)H(j\omega) = \frac{0.045\ (1+j\omega)(1+j\frac{\omega}{6})}{(j\omega)^2\left(1+0.045\,j\omega+\frac{(j\omega)^2}{400}\right)} = \frac{0.045\ (1+j\omega)(1+j\frac{\omega}{6})}{(j\omega)^2\ (1+0.045\,j\omega-\frac{\omega^2}{400})}$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 0.045\ \ \angle 1+j\omega \angle 1+j\frac{\omega}{6}}{\angle (j\omega)^2\ \angle 1+0.045\,j\omega-\frac{\omega^2}{400}}$$

$$\angle 0.045+j0 = 0^\circ\ ,\ \ \angle 1+j\omega = +\tan^{-1}\omega,\ \ \ \angle 1+j\frac{\omega}{6} = +\tan^{-1}\frac{\omega}{6}$$

$$\angle\frac{1}{(j\omega)^2} = -2\times 90^\circ = -180^\circ \quad \text{as 2 poles at origin}$$

$$\angle\frac{1}{1+0.045\,j\omega-\frac{\omega^2}{400}} = -\tan^{-1}\left\{\frac{0.045\ \omega}{1-\frac{\omega^2}{400}}\right\}$$

∴ Phase Angle Table

ω	$\frac{1}{(j\omega)^2}$	$+\tan^{-1}\omega$	$+\tan^{-1}\frac{\omega}{6}$	$-\tan^{-1}\left\{\frac{0.045\omega}{1-\frac{\omega^2}{400}}\right\}$	ϕ_R
0.1	− 180°	+ 5.7°	+ 0.95°	− 0.25°	− 173.6°
1	− 180°	+ 45°	+ 9.46°	− 2.58°	− 128.1°
6	− 180°	+ 80.53°	+ 45°	− 16.52°	− 70.9°
10	− 180°	+ 84.28°	+ 59°	− 30.96°	− 67.6°
20	− 180°	+ 87.13°	+ 73.3°	− 90°	− 109.57°

50	-180°	$+88.85^\circ$	$+83.15^\circ$	$+23.19-180^\circ=-156.8^\circ$	-164.8°
100	-180°	$+89.42^\circ$	$+86.56^\circ$	$+10.61-180^\circ=-169.38^\circ$	-173.4°
∞	-180°	$+90^\circ$	$+90^\circ$	-180°	-180°

As two zeros are always contributing more than a quadratic pole phase angle plot cannot cross -180° but at the end will run parallel to it.

Step 5 : Sketch the Bode plot and obtain the solution.

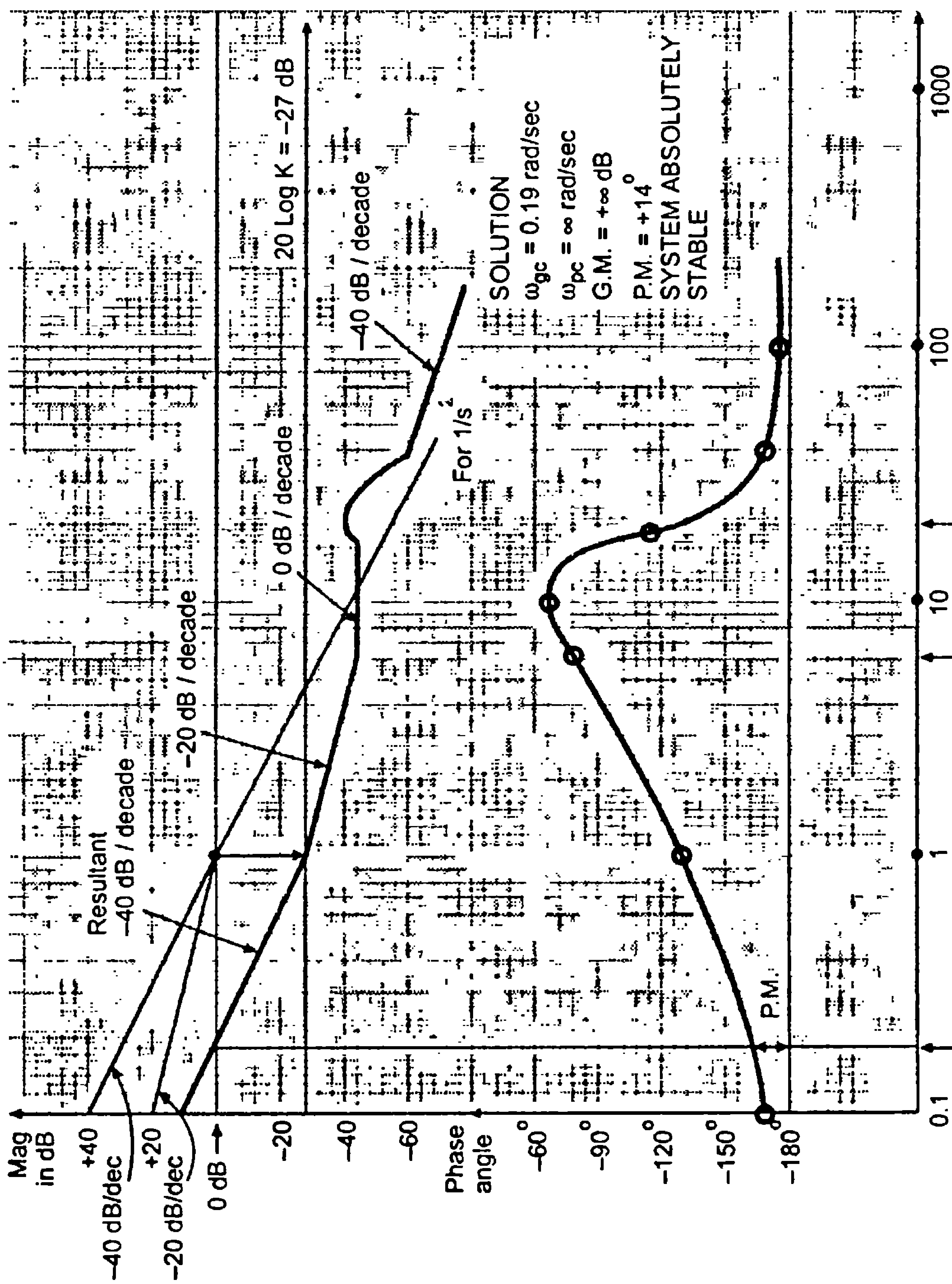


Fig. 11.32

Example 11.9 : *Sketch the Bode plot for the transfer function*

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}.$$

Determine the value of K for the gain cross-over frequency to be 5 rad/sec.

(M.U.: May-96)

Solution : Step 1 : G(s) is in the time constant form.

Step 2 : Analysis of factors

1. K is unknown and its effect is to shift the entire magnitude plot by 20 log K dB.
2. Two zeros at origin, so straight line of slope + 40 dB/dec, passing through intersection point of $\omega = 1$ and 0 dB.
3. Simple pole, 1/1 + 0.2 s, $T_1 = 0.2$

   $\therefore \quad \omega_{C1} = \frac{1}{T_1} = \frac{1}{0.2} = 5$ rad/sec

   So straight line of slope – 20 dB/dec for $\omega \geq 5$.
4. Simple pole, 1/1 + 0.02s, $T_2 = 0.02$

   $\therefore \quad \omega_{C2} = \frac{1}{T_2} = \frac{1}{0.02} = 50$ rad/sec

Resultant slope table :

Starting slope	+ 40 dB/dec
$0 < \omega < 5$	+ 40 dB/dec
$5 < \omega < 50$	+ 20 dB/dec
$50 < \omega < \infty$	0 dB/dec

Step 3 : Phase angle table

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)\,(1+0.02j\omega)}$$

ω	$(j\omega)^2$	$-\tan^{-1} 0.2\omega$	$-\tan^{-1} 0.02\omega$	ϕ_R
0.1	+ 180°	– 1.14°	– 0.11°	+ 178.74°
1	+ 180°	– 11.3°	– 1.14°	+ 167.55°
10	+ 180°	+ 63.4°	– 11.3°	+ 105.29°
100	+ 180°	– 87.1°	– 63.43°	+ 29.46°
∞	+ 180°	– 90°	– 90°	0°

Step 4 : The Bode plot is shown in the Fig. 11.35.

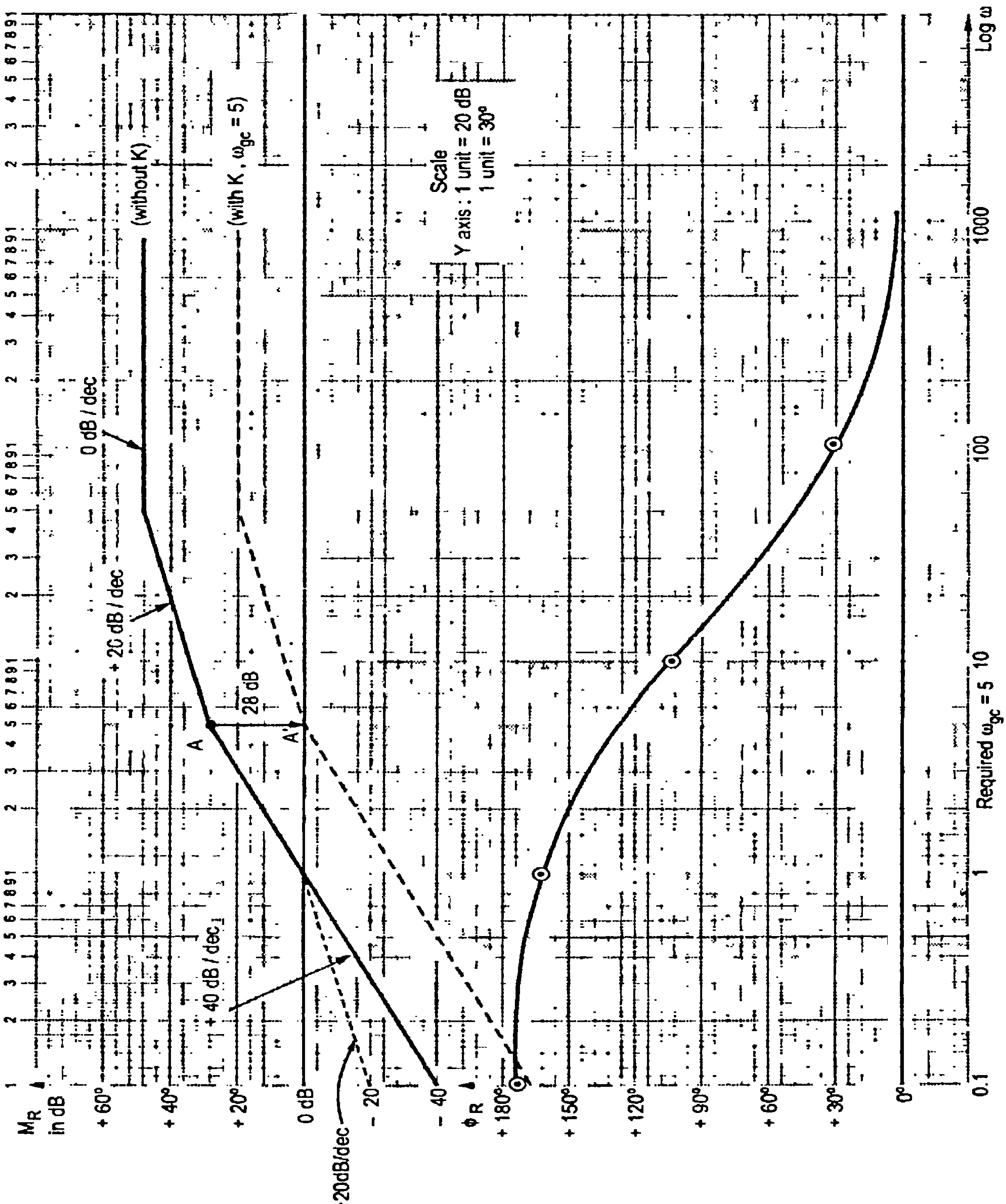


Fig. 11.35

From the Bode plot to get $\omega_{gc} \gg 5$, the plot must intersect 0 dB line at ω=5 rad/sec.

But at ω=5, the point on plot without K is 28 dB away from the 0 dB line. This is point A as shown. It should be on 0 dB at A′ as shown for ω_{gc} =5. So shift A to A′ must

Magnitude in dB = 15 dB given

$\therefore$ Substituting $15 = 20 \text{ Log } \omega_1$

$\therefore \quad \omega_1 = 5.623 \text{ rad/sec} = \omega_{C2}.$

$\therefore \quad T_2 = \dfrac{1}{\omega_{C2}} = \dfrac{1}{5.623} = 0.177$

$\therefore$ Factor is $\dfrac{1}{1+T_2 s} = \dfrac{1}{1+0.177s}$

Next change in slope is at ω_2.

$\therefore$ $\omega_{C3} = \omega_2$. Change is $-20 - 0 = -20$ dB/dec

$\therefore$ Factor is simple pole.

To find ω_2 write equation for that line having slope – 20 dB/dec.

$$\text{mag in dB} = m \text{ Log } \omega + C$$

$$\therefore \quad \text{mag in dB} = -20 \text{ Log } \omega + C.$$

At $\omega = 1000$, mag in dB = 0

Substituting $0 = -20 \text{ Log } 1000 + C$

$\therefore \quad C = +60$ dB

$\therefore$ Equation is, mag in dB = $-20 \text{ Log } \omega + 60$

At ω_2, mag.in dB = $15 = -20 \text{ Log } \omega_2 + 60$

$\therefore \quad -20 \text{ Log } \omega_2 = -45$

$\therefore \quad \omega_2 = 177.82 = \omega_{C3}$

$\therefore \quad T_3 = \dfrac{1}{\omega_{C3}} = 0.0562$

$\therefore$ Factor is $\dfrac{1}{1+0.00562\, s}$

The last change in slope is at

$$\omega = 1000 = \omega_{C4}$$

Change is $0 - (-20) = +20$ dB/dec so factor is simple zero

$$\omega_{C4} = 1000$$

$\therefore \quad T_4 = \dfrac{1}{\omega_{C4}} = 0.001$

$\therefore$ Factor $(1+T_4\, s) = (1 + 0.001s)$

$\therefore$ The total transfer function is product of all of them

$$\boxed{G(s)H(s) = \frac{(1+s)(1+0.001\, s)}{(1+0.177\, s)(1+0.00562\, s)}}$$

Example 11.11 : *What are i) gain margin and ii) phase margin ? Determine these two analytically for a system with* $G(s)H(s) = \dfrac{1}{s(s+1)\left(s+\frac{1}{2}\right)}$ *given that the gain crossover frequency is 0.82 rad/sec. Is the system stable ?*

Solution : ***Gain Margin :*** It is margin in gain allowable by which gain can be increased till system reaches on the verge of instability.

$$G.M. = 1 - 20 \log_{10} |G(j\omega)\, H(j\omega)|_{\omega=\omega_{gc}}$$

Phase Margin : The amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability is called phase margin.

$$P.M. = 180° + \angle G(j\omega)\, H(j\omega)|_{\omega=\omega_{gc}}$$

$$G(s)H(s) = \frac{1}{s(s+1)(s+0.5)} \quad \text{and } \omega_{gc} = 0.82 \text{ rad/sec}$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)(0.5+j\omega)}$$

$$P.M. = 180° + \angle G(j\omega)\, H(j\omega)\; |_{\omega=\omega_{gc}}$$

$$= 180° + -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{0.5}\Big|_{\omega=0.82}$$

$$= 180° - 90° - 39.35° - 58.62° = \mathbf{-\,7.97°}$$

To find ω_{pc}, rationalize $G(j\omega)H(j\omega)$ and equate imaginary part to zero.

$$G(j\omega)\, H(j\omega) = \frac{-j\omega(1-j\omega)(0.5-j\omega)}{\omega^2 \times (1+\omega^2) \times (0.25+\omega^2)} = \frac{-j\omega[0.5 - 1.5\, j\omega - \omega^2]}{\omega^2\,(1+\omega^2)(0.25+\omega^2)}$$

$$= \frac{-1.5\,\omega^2}{\omega^2\,(1+\omega^2)(0.25+\omega^2)} - \frac{j\,\omega[0.5-\omega^2]}{\omega^2\,(1+\omega^2)(0.25+\omega^2)}$$

$$\therefore \quad 0.5 - \omega^2 = 0$$

$$\therefore \quad \omega_{pc} = 0.7071$$

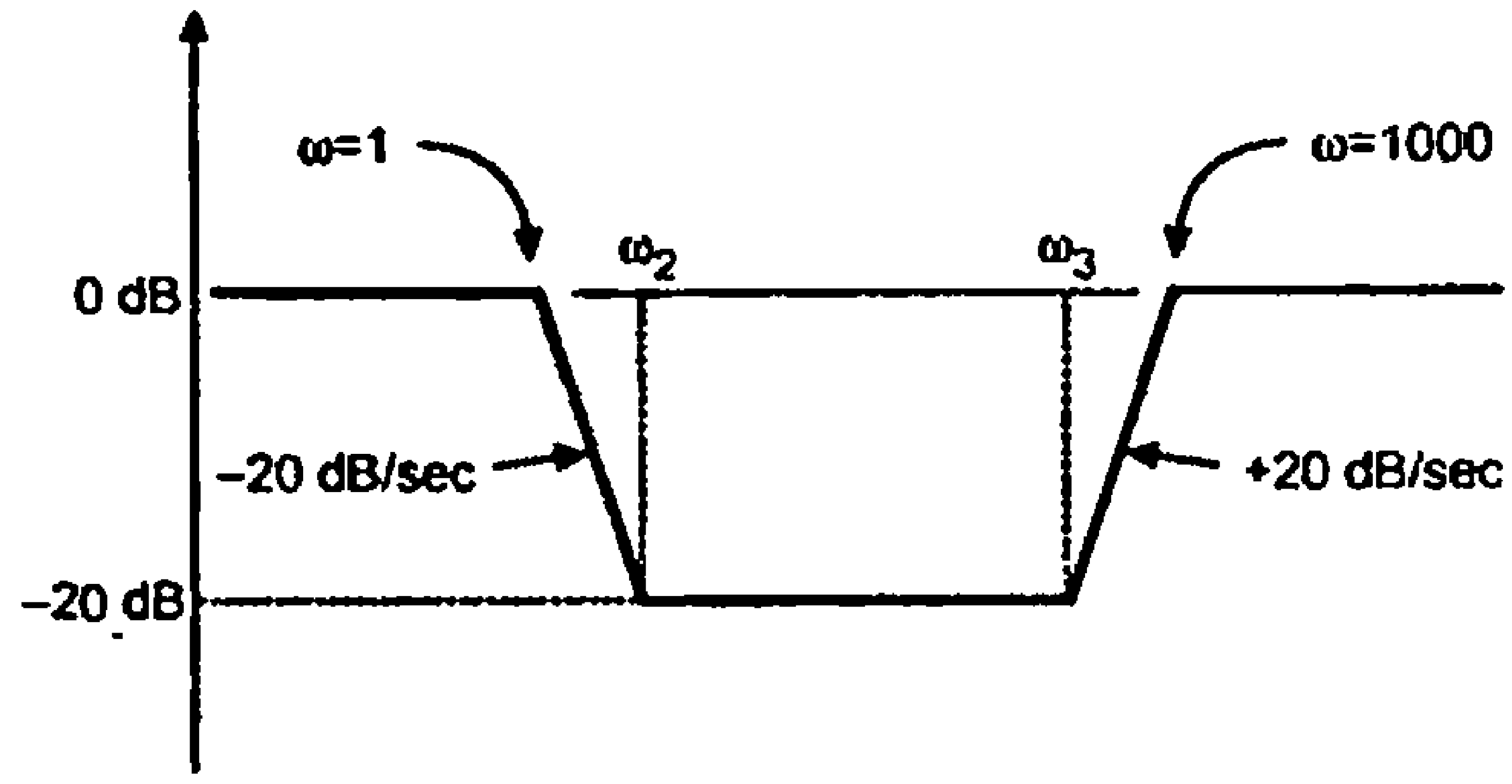
$$\therefore \quad G.M. = -20 \log_{10} |G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}$$

$$= -20 \log\left[\frac{1}{\omega \cdot \sqrt{1+\omega^2} \cdot \sqrt{0.25+\omega^2}}\right]_{\omega=0.7071} = -\,20 \log 1.333$$

$$= \mathbf{-\,2.498\ dB}$$

As both G.M. and P.M. are negative the system is unstable in nature.

Example 11.12 : *Determine the transfer function of a system whose asymptotic gain plot is given below.*



Solution : Starting slope is 0 dB. So there is no pole or zero at the origin. The first change in slope occurs at $\omega = 1$

i.e. $$\omega_{C1} = 1$$

Change in slope = – 20 – 0 = – 20 dB/dec , there is simple pole with $\omega_{c1} = 1$.

i.e. $$T_1 = \frac{1}{\omega_{c1}} = 1$$

$$\therefore \quad \text{Factor } \frac{1}{1+T_1 s} = \frac{1}{1+s}$$

The shift at $\omega = 1$ is 0 dB so 20 Log K = 0 dB $\therefore K = 1$

The further change in slope occurs at ω_2 which is unknown but we can determine it First change in slope occurs is $\frac{0 \text{ dB}}{\text{dec}} = -(-20) = +20$ dB/dec. i.e. there is simple zero.

Now magnitude corresponding to ω_2 is – 20 dB and magnitude at $\omega = 1$ is 0 dB. So there is change of – 20 dB and slope of the line is – 20 dB/decade. i.e. ω_2 is decade away from $\omega = 1$

i.e. $$\omega_{C2} = \omega_2 = 10$$

$$T_2 = \frac{1}{\omega_{c2}} = 0.1$$

$\therefore$ Factor is $(1 + T_2 s) = (1 + 0.1\, s)$.

Next change in slope is at ω_3 so ω_3 is ω_{c3}. Change is + 20 – 0 = + 20 dB/dec. So there is simple zero with corner frequency ω_{c3}.

Now magnitude at ω_{C3} = – 20 dB and magnitude at $\omega = 1000$ is 0 dB i.e. change of + 20 dB is there from ω_{C3} to 1000 and slope is + 20 dB/decade. i.e. ω_{C3} and 1000 are decade away.

i.e. $\omega_{C3} = 100$

i.e. $T_3 = \frac{1}{\omega_{C3}} = 0.01$

$\therefore$ Factor is $(1 + T_3\, s) = \mathbf{(1 + 0.01\, s)}$.

Next change in slope occurs at $\omega = 1000$ which is $0 - 20 = -20$ dB/dec i.e. there is simple pole with $\omega_{C4} = 1000$

i.e. $T_4 = \frac{1}{\omega_{C4}} = 0.001$

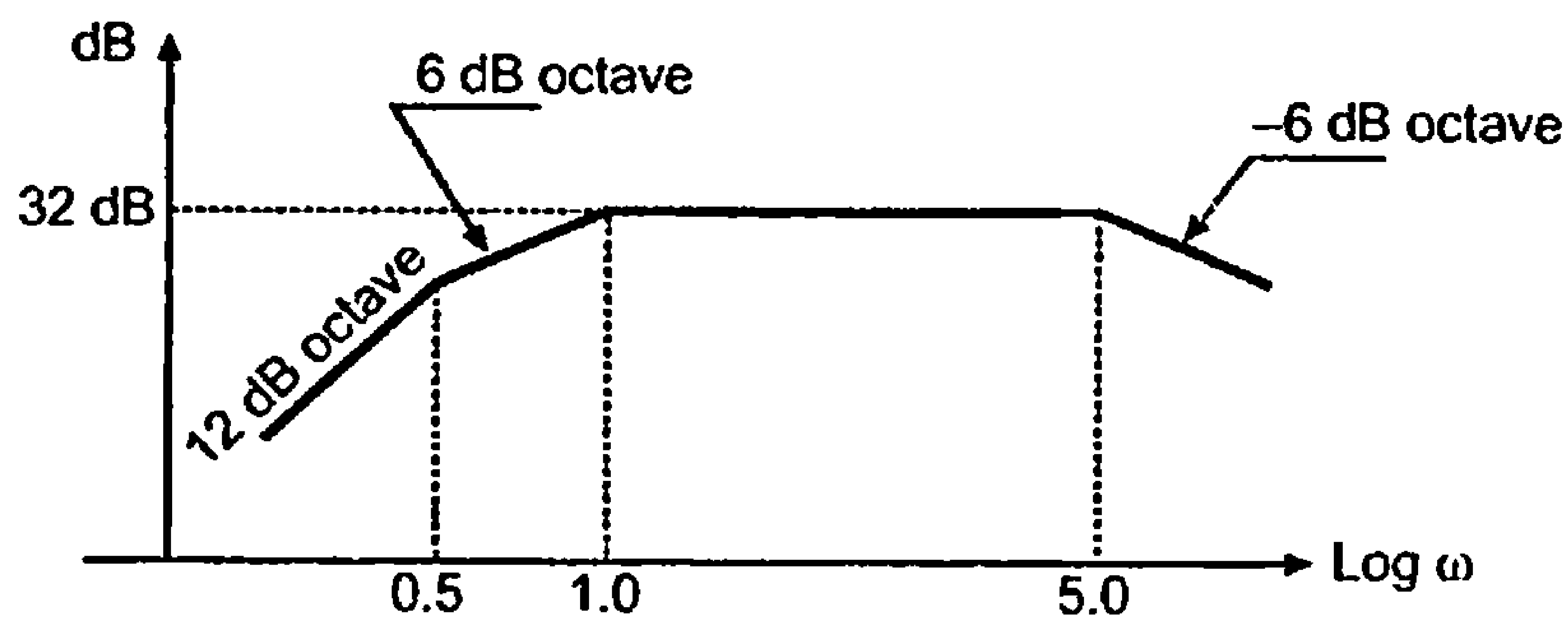
$\therefore$ Factor is $\frac{1}{1+T_4 s} = \frac{\mathbf{1}}{\mathbf{1+0.001\, s}}$

$\therefore$ Transfer function is product of all factors

$$G(s)H(s) = \frac{(1+0.1s)(1+0.01s)}{(1+s)(1+0.001s)}$$

Example 11.13 : *For the plot shown below determine the transfer function.*

(M.U.: Dec-97)



Solution : The slope dB/octave is to be converted to dB/decade. Initial slope is +12 dB/octave, hence there are two zeros at the origin. This is because

6 dB/octave	= 20 dB/decade
12 dB/octave	= 40 dB/decade

Now the equation of line after $\omega = 0.5$ is say

$$M = 20 \log \omega + C$$

at $\omega = 1,\ M = +32$ dB shown

$\therefore$ $32 = 20 \text{ Log } 1 + C$

$\therefore \quad C = 32$

At $\quad \omega = 0.5, M = 20 \text{ Log } 0.5 + 32 = +26 \text{ dB}$

Now $\omega = 0.5$, M = 26 dB is also on the initial line whose equation is

$$M = +40 \log \omega + C_1$$

At $\quad \omega = 0.5, +26 = 40 \log 0.5 + C_1$

$\therefore \quad C_1 = 38.0412 \text{ dB}$

Now this line must have M = 0 dB at ω = 1 for K =1. But at ω =1, M = 40 log 1+ 38.0412

i.e. $\quad M = 38.0412 \text{ dB}$

This is due to contribution of system gain constant K.

$\therefore \quad 20 \text{ Log } K = 38.0412$

$\therefore \quad K = 79.8$

At $\omega_C = 0.5$, slope changed by –20, there is simple pole.

$$\text{Factor} = \frac{1}{(1+T_1 s)}, \text{ where } T_1 = \frac{1}{\omega_C} = \frac{1}{0.5} = 2$$

$$= \frac{1}{(1+2s)}$$

At $\omega_C = 1$, slope changed by –20, there is simple pole.

$$\text{Factor} = \frac{1}{1+T_2 s}, \text{ where } T_2 = \frac{1}{\omega_C} = \frac{1}{1} = 1$$

$$= \frac{1}{1+s}$$

At $\omega_C = 5$, slope further changed by –20, there is simple pole.

$$\text{Factor} = \frac{1}{1+T_3 s}, \text{ where } T_3 = \frac{1}{\omega_C} = \frac{1}{5} = 0.2$$

$$= \frac{1}{(1+0.2s)}$$

Hence the transfer function is

$$G(s)H(s) = \frac{79.8 s^2}{(1+2s)(1+s)(1+0.2s)}$$

Example 11.14 : *Use Bode plot to determine the range of K within which a unity-feedback system with open loop transfer function G(s) is stable.*

$$G(s) = \frac{K}{s(1+0.2s)(1+0.02s)}$$

(M.U.: Dec.-98)

Solution : The open loop transfer function is

$$G(s) = \frac{K}{s(1+0.2s)(1+0.02s)}$$

The various factors are :

i) Constant K is unknown

ii) One pole at the origin.

The magnitude plot is the straight line of slope – 20 dB/decade passing through intersection point of $\omega = 1$ and 0 dB line.

iii) The simple pole, $\frac{1}{(1+0.2s)}$

$$\omega_{C1} = \frac{1}{0.2} = 5$$

Magnitude plot is straight line of 0 dB upto 5 and straight line of slope – 20 dB/decade after 5.

iv) The simple pole, $\frac{1}{(1+0.02s)}$

$$\omega_{C2} = \frac{1}{0.02} = 50$$

Magnitude plot is straight line of 0 dB upto 50 and straight line of slope –20dB/decade after 50.

The resultant slope table is,

Range of ω	Slope
start < ω < 5	– 20 dB/decade
5 < ω < 50	– 40 dB/decade
50 < ω < ∞	– 60 dB/decade

Example 11.15 : *If* $G(s)H(s) = \frac{\left(1+\frac{s}{A}\right)^2}{s^3}$ *determine the value of A, that gives the system a phase margin of 50°.* **(M.U. : May-98)**

Solution :

$$G(s)H(s) = \frac{\left(1+\frac{s}{A}\right)^2}{s^3}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{\left(1+j\frac{\omega}{A}\right)^2}{(j\omega)^3}$$

$$\text{Now} \quad |G(j\omega)H(j\omega)| = M = \frac{\left(\sqrt{1+\left(\frac{\omega}{A}\right)^2}\right)^2}{\omega^3} = \frac{\left(1+\frac{\omega^2}{A^2}\right)}{\omega^3}$$

For phase margin, $\omega = \omega_{gc}$, and at ω_{gc} $M = 1$

$$\therefore \quad 1 = \frac{\left[1+\left(\frac{\omega_{gc}^2}{A^2}\right)\right]}{\omega_{gc}^3}$$

$$\therefore \quad \omega_{gc}^3 = 1+\frac{\omega_{gc}^2}{A^2}$$

$$\therefore \quad \omega_{gc}^3 A^2 = A^2 + \omega_{gc}^2 \quad \text{... (1)}$$

$$\text{Now} \quad \text{P.M.} = 180° + \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$$

$$\therefore \quad 50° = 180° + 2\tan^{-1}\frac{\omega_{gc}}{A} - 270°$$

$$\therefore \quad 2\tan^{-1}\frac{\omega_{gc}}{A} = 140°$$

$$\therefore \quad \tan^{-1}\frac{\omega_{gc}}{A} = 70°$$

$$\frac{\omega_{gc}}{A} = 2.7474 \quad \text{... (2)}$$

So factor is $(1 + 0.01\ s)^2$

At $\omega_{C2} = 10^5$ slope changes by – 20 dB/dec to become straight line. So factor is simple pole.

$$\therefore \quad T_2 = \frac{1}{\omega_{C2}} = \frac{1}{10^5} = 10^{-5}$$

$$\therefore \quad \text{Factor} = \frac{1}{(1+10^{-5}s)}$$

Hence transfer function is,

$$\boxed{\text{T.F.} = \frac{100(1+0.01s)^2}{s(1+10^{-5}s)}}$$

Example 11.17 : *Obtain semi-log magnitude and phase plot. Hence obtain -*

i) Gain crossover frequency

ii) Phase crossover frequency

iii) Gain margin

iv) Phase margin

v) Stability of the system.

$$G(s)H(s) = \frac{4000(s+0.05)(s+4)}{s(s+1)(s+10)}$$

(M.U.: May - 2005)

Solution : Step 1 : Obtain G(s)H(s) in the time constant form.

$$G(s)H(s) = \frac{4000\times 0.05\times 4\times(1+\frac{s}{0.05})(1+\frac{s}{4})}{s(1+s)\times 10\times(1+\frac{s}{10})} = \frac{80(1+\frac{s}{0.05})(1+\frac{s}{4})}{s(1+s)(1+\frac{s}{10})}$$

Step 2 : Analysis of factors

i) K = 80, 20 Log K = 38.06 dB, Straight line parallel to Log ω axis.

ii) $\frac{1}{s}$, One pole at the origin

The straight line of slope –20 dB/dec, passing through intersection of $\omega = 1$ and 0 dB.

iii) $\left(1+\frac{s}{0.05}\right)$, simple zero, $T_1 = \frac{1}{0.05}$, $\omega_{C1} = \frac{1}{T_1} = 0.05$

The straight line of slope +20 dB/dec for $\omega \geq 0.05$.

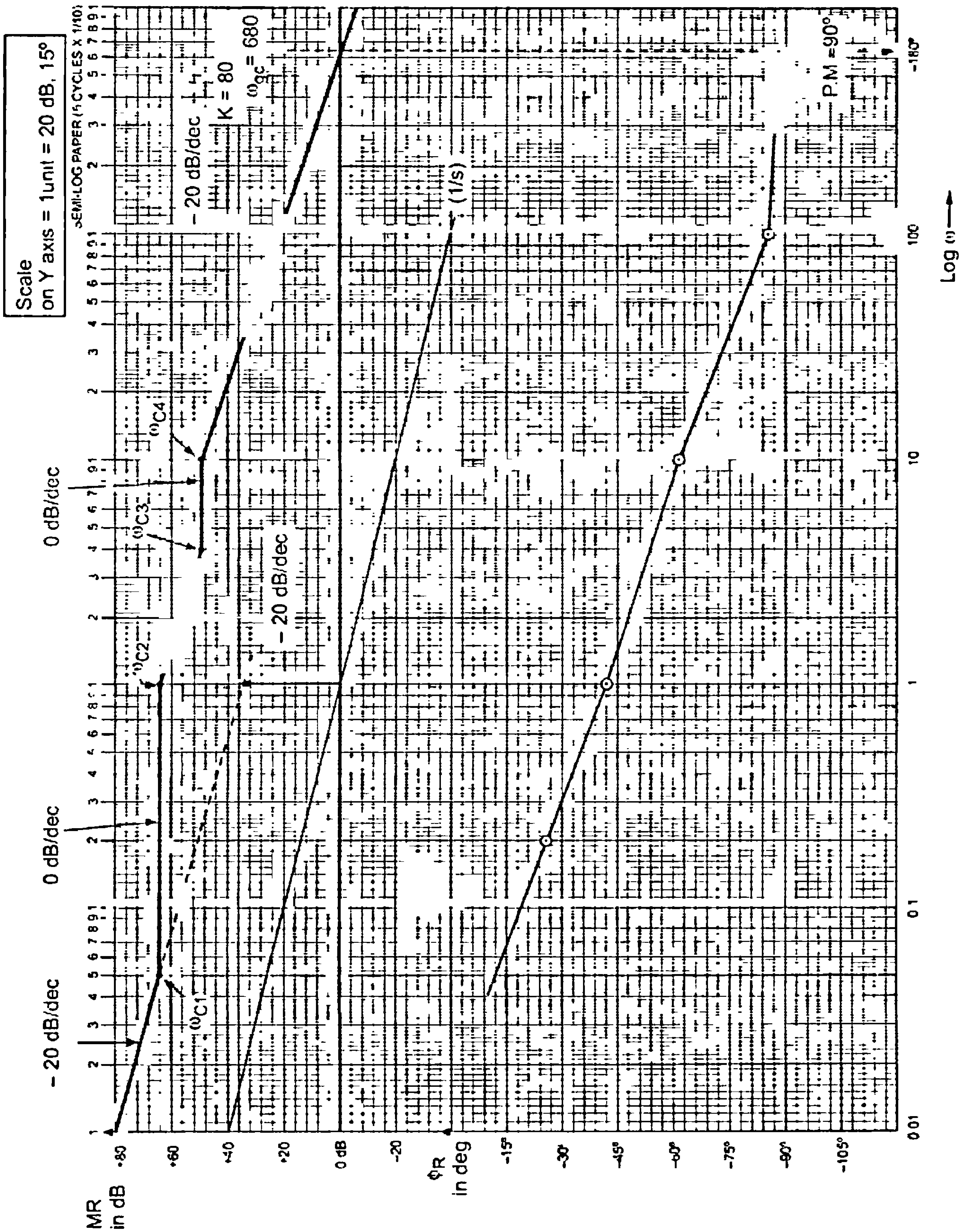

Scale
on Y axis = 1unit = 20 dB, 15°
- 20 dB/dec
0 dB/dec
0 dB/dec
- 20 dB/dec
- 20 dB/dec
K = 80
ω_{gc} = 680
(1/s)
P.M.=90°
ω_{C1}
ω_{C2}
ω_{C3}
ω_{C4}
MR
in dB
ϕ_R
in deg
Log ω
0.01
0.1
1
10
100
0 dB
-15°
-30°
-45°
-60°
-75°
-90°
-105°
-180°

Fig. 11.38

v) $\dfrac{1}{1+\frac{s}{20}}$, simple pole , $T_2 = \frac{1}{20}$, $\omega_{C2} = \frac{1}{T_2} = 20.$

The straight line of slope – 20 dB/dec for $\omega \geq 20$.

vi) $(1+\frac{s}{30})$, simple zero, $T_3 = \frac{1}{30}$, $\omega_{C3} = \frac{1}{T_3} = 30.$

The straight line of slope + 20 dB/dec for $\omega \geq 30$.

vii) $\dfrac{1}{(1+\frac{s}{90})}$, simple pole , $T_4 = \frac{1}{90}$, $\omega_{C4} = \frac{1}{T_4} = 90$

The straight line of slope – 20 dB/dec for $\omega \geq 90$.

Resultant slope table :

Range of ω	Resultant slope
$0 < \omega < 1$ (ω_{C1})	– 20 dB/dec
$1 < \omega < 20$ (ω_{C2})	– 20 –20 = – 40 dB/dec
$20 < \omega < 30$ (ω_{C3})	– 40 – 20 = – 60 dB/dec
$30 < \omega < 90$ (ω_{C4})	– 60 + 20 = – 40 dB/dec
$90 < \omega < \infty$	– 40 – 20 = – 60 dB/dec

Step 3 : Phase angle table

$$G(j\omega) = \frac{K e^{-0.1 j\omega} (1 + j\frac{\omega}{30})}{j\omega (1 + j\omega) (1 + j\frac{\omega}{20}) (1 + j\frac{\omega}{90})}$$

ω	$\frac{1}{J\omega}$	–0.1 ω rad	$-\tan^{-1}\omega$	$-\tan^{-1}\frac{\omega}{20}$	$+\tan^{-1}\frac{\omega}{30}$	$-\tan^{-1}\frac{\omega}{90}$	ϕ_R
0.5	–90°	– 0.05 rad = – 2.86°	– 26 .56°	– 1.43°	+ 0. 88°	– 0.31°	– 120.28°
10	–90°	– 1 rad = – 57.3°	– 84.28°	– 26.56°	+ 18.43°	– 6.34°	– 246.05°
5	–90°	– 0.5 rad = – 28.6°	– 78.69°	– 14.03°	+9.46°	– 3.18°	– 211.31°
2	–90°	– 0.2rad = – 11.4°	– 63.43°	– 5.71°	+ 3.81°	– 1.27°	– 168°
4	–90°	– 0.4 rad = – 22.9°	– 75.96°	– 11.3°	+ 7.59°	– 2.54°	– 194.75°

At $\omega = 1$, the magnitude is 36.123 dB as obtained by equation (1).

$\therefore$ $36.123 = -60 \text{ Log } \omega + C_2$ i.e. $C_2 = 36.123$.

$\therefore$ Magnitude in dB = $-60 \text{ Log } \omega + 36.123$. ... (2)

At $\omega = 5$, the slope changes by + 20 dB/dec hence there is simple zero at $\omega_{C2} = 5$. Thus $T_2 = \frac{1}{\omega_{C2}} = 0.2$. The factor is $(1 + 0.2s)$.

The magnitude at $\omega = 5$ can be ontained by equation (2).

Magnitude in dB (at $\omega = 5$) = $-60 \text{ Log } 5 + 36.123 = -5.815$ dB

The equation of the third line is $-40 \text{ Log } \omega + C_3$.

$\therefore$ $-5.815 = -40 \text{ Log } 5 + C_3$ i.e. $C_3 = +22.143$ dB

$\therefore$ Magnitude in dB = $-40 \text{ Log } \omega + 22.143$. ... (3)

At $\omega = 10$, the slope changes by -40 dB/dec. As correction is zero ($\xi = 0$), it is a quadratic factor without middle term. The corner frequency is $\omega_{C3} = 10$. Hence $\omega_n = \omega_{C3} = 10$. Thus the factor is $\left[1 \Big/ \left(1 + \frac{s^2}{\omega_n^2}\right)\right]$ i.e. $\left[1 \Big/ \left(1 + \frac{s^2}{100}\right)\right]$.

Hence the overall transfer function is,

$$G(s)H(s) = \frac{64\,(1 + 0.2\,s)}{s^2\,(1 + s)\left(1 + \frac{s^2}{100}\right)}$$

ii) With correction ($\xi = 0.316$)

Let the quadratic factor is $s^2 + 2\xi\omega_n s + \omega_n^2$.

As corner frequency is 10, $\omega_n = 10$ rad/ sec.

$$\therefore \quad 2\xi\omega_n = 2 \times 0.316 \times 10 = 6.32$$

$$\therefore \quad \text{Factor} = \left[\frac{1}{s^2 + 6.32\,s + 100}\right]$$

$$\text{In time constant form} = \frac{1}{[1 + 0.0632\,s + \frac{s^2}{100}]}$$

Other analysis remains same. Hence the overall transfer function is,

$$G(s)H(s) = \frac{64\,(1 + 0.2\,s)}{s^2\,(1 + s)\,(1 + 0.0632\,s + 0.01\,s^2)}$$

Example 11.22 : *Derive the transfer function from the given Bode plot.*

(M.U. : Dec.-2003)

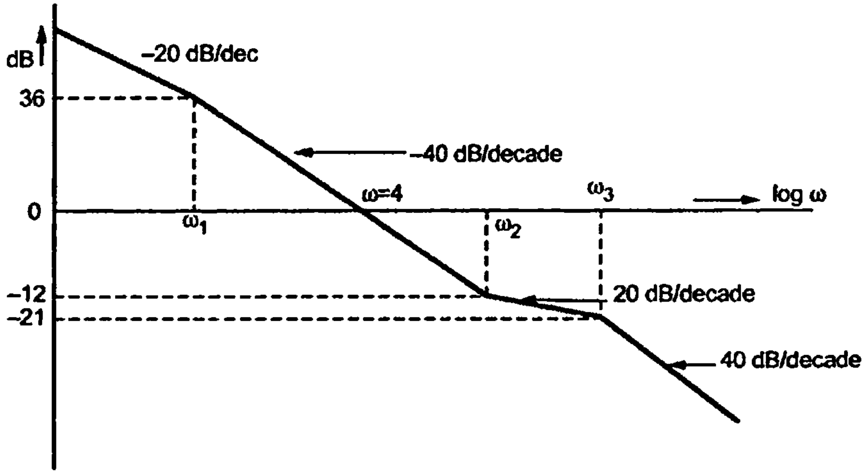


Fig. 11.43

Solution :

The inital slope is – 20 dB/dec hence there is one pole at the origin (1/s). The equation of the initial line is $y = -20 \text{ Log } \omega + C_1$ where y = mag in dB.

$$\therefore \quad y = -20 \text{ Log } \omega + C_1 \qquad \text{... (1)}$$

Consider second line whose equation is $y = -40 \text{ Log } \omega + C_2$. And on this line y = 0 dB at ω = 4.

$$\therefore \quad 0 = -40 \text{ Log } 4 + C_2 \qquad \text{i.e. } C_2 = 24.0823$$

$$\therefore \quad y = -40 \text{ Log } \omega + 24.0823 \qquad \text{... (2)}$$

At $\omega = \omega_1$, y = 36 dB which is common to both the lines.

$$\therefore \quad 36 = -40 \text{ Log } \omega_1 + 24.0823 \text{ i.e. } \omega_1 = \mathbf{0.5035.}$$

Using ω_1 in (1), $36 = -20 \text{ Log } 0.5035 + C_1$ i.e. $C_1 = 30.04$

$$\therefore \quad y = -20 \text{ Log } \omega + 30.04 \qquad \text{... (1a)}$$

At ω = 1, the equation (1a) gives y = 30.04 dB which is the contribution by K i.e. 20 Log K.

$$\therefore \quad 20 \text{ Log K} = 30.04 \qquad \text{i.e. } K = \mathbf{31.768.}$$

The slope changes by – 20 dB/dec at $\omega = \omega_1 = 0.5035$. Thus there is simple pole with $\omega_{C1} = 0.5035$. Thus $T_1 = \frac{1}{\omega_{C1}} = 1.986$. The factor is $\left[\frac{1}{(1+1.986s)}\right]$

Now on the second line, at $\omega = \omega_2$, the magnitude y = – 12 dB.

Using in (2),

$-12 = -40 \text{ Log } \omega_2 + 24.0823$ i.e. $\omega_2 = 7.981 \approx 8$

The slope changes by +20 dB/dec at $\omega_2 = 8$. Thus there is simple zero with $\omega_{C2} = 8$. Thus $T_2 = \frac{1}{\omega_{C2}} = 0.125$ The factor is **(1+0.125s)** .

The equation of third line is $y = -20 \text{ Log } \omega + C_3$. At $\omega_2 = 8$, y = – 12 dB hence,

$$-12 = -20 \text{ Log } 8 + C_3 \quad \text{i.e. } C_3 = 6.0618$$

$$\therefore \quad y = -20 \text{ Log } \omega + 6.0618 \qquad \text{... (3)}$$

On this line, at $\omega = \omega_3$ the magnitude is – 21 dB.

$$\therefore \quad -21 = -20 \text{ Log } \omega_3 + 6.0618 \quad \text{i.e. } \omega_3 = 22.547$$

The slope changes by – 20 dB/dec at ω_3. Thus there is simple pole with $\omega_{C3} = 22.547$.

Thus $T_3 = \frac{1}{\omega_{C3}} = 0.0443$. The factor is $\left[\frac{1}{1+0.0443s}\right]$.

Thus the overall transfer function is,

$$G(s)H(s) = \frac{31.768\,(1+0.125s)}{s(1+1.986s)(1+0.0443s)} = \frac{45.135\,(s+8)}{s(s+0.5035)(s+22.547)}$$

Example 11.23 : *Obtain the Bode plot for,*

$$G(s)H(s) = \frac{10\,(1-s)}{s(s+2)\,(s^2+2s+25)}$$

Hence find G.M. and P.M. **(M.U.: May - 2006)**

Solution : Step 1 : Time constant form of G(s)H(s).

$$G(s)H(s) = \frac{10\,(1-s)}{s \times 2 \times (1+\frac{s}{2}) \times 25\,(1+\frac{2}{25}s+\frac{s^2}{25})} = \frac{0.2(1-s)}{s(1+0.5s)\,(1+0.08s+0.04s^2)}$$

Step 2 : Analysis of factors

i) K = 0.2, 20 Log K = – 13.98 dB, straight line parallel to the Log ω axis.

ii) $\frac{1}{s}$, one pole at origin, straight line of slope –20 dB/dec passing through the intersection of $\omega = 1$ and 0 dB.

iii) (1– s), simple zero, $T_1 = 1$, $\omega_{C1} = \frac{1}{T_1} = 1$

The straight line of slope + 20 dB/des for $\omega \geq 1$.

Corresponding to $\omega_{pc1}=20$, AB = + 64 dB

$\therefore \quad 20 \text{ Log } K = 64$

$\therefore \quad K = 1585$

Corresponding to $\omega_{pc2} = 400$, CD = 100 dB

$\therefore \quad 20 \text{ Log } K = 100$

$\therefore \quad K = 100000$

So range of values of K is,

$$1585 < K < 100000$$

Review Questions

1. *What are Bode plots?*
2. *State the advantages of Bode plots.*
3. *Explain the nature of Bode plots for*

   i) Poles at origin ii) Simple pole iii) Simple zero
4. *Explain the concept of gain margin and phase margin. Explain how these values help in studying relative stability.*
5. *Write a note on frequency domain specifications.*
6. *Draw the Bode diagram for*

   $$G(s) = \frac{100(0.02s+1)}{(s+1)(0.1s+1)(0.01s+1)^2}$$

   Mark the following on the Bode diagram, recording the numerical values.

   1) Gain cross-over frequency

   2) Phase margin

   3) Phase cross-over frequency

   4) Gain margin **(Ans. : $\omega_{gc} = 32$; $\omega_{pc} = 60$; G.M. = 10; P.M. – 15° stable)**

7. *Given*

   a) Draw the Bode diagram.

   b) Is the system stable? **(Ans. : $\omega_{gc} = 45$; $\omega_{pc} = \infty$; G.M. = ∞; P.M. = 18° stable)**

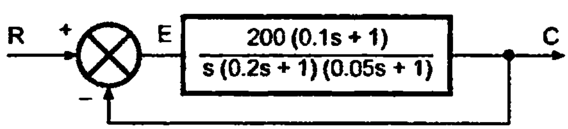


Fig. 11.46

12 Polar and Nyquist Plots

12.1 Background

In the last two chapters we have discussed what is frequency response, its advantages, limitations and Bode plot method of stability analysis. In continuation of the frequency response discussion, in this chapter we are going to discuss the polar plot which is base of Nyquist plot and the stability analysis using Nyquist plot method. We are going to experience the powerfulness of this method.

12.2 Polar Plot

As seen earlier, to sketch the frequency response means to plot the variations in magnitude and phase angle versus the input frequency. The two plots constituting the frequency response are called **gain plot** or **magnitude plot** and the **phase plot**. The scientist Bode has suggested to use a logarithmic scale to sketch frequency response.

In polar plot, the magnitude of $G(j\omega)H(j\omega)$ is plotted against the phase angle of $G(j\omega)H(j\omega)$ for various values of ω. In frequency response we have,

$$M = |G(j\omega)H(j\omega)| = \text{Magnitude}$$

$$\phi = \angle G(j\omega)H(j\omega) = \text{Phase}$$

We can obtain the values of M and ϕ by varying the input frequency ω from 0 to ∞. The result can be tabulated as below.

ω	$M = \lvert G(j\omega)H(j\omega) \rvert$	$\phi = \angle G(j\omega)H(j\omega)$
0	M_0	ϕ_0
ω_1	M_1	ϕ_1
:	:	:
:	:	:
:	:	:
∞	M_∞	ϕ_∞

This is the data required for the polar plot.

Key Point: *It is not necessary to convert magnitude to its dB value or to find logarithm of the frequencies.*

Now each value of M and ϕ corresponding to particular frequency ω decides a point as per the polar co-ordinate system. i.e. for $\omega = \omega_1$, $M = M_1$ and $\phi = \phi_1$. So it decides a point having polar co-ordinates as $M_1 \angle \phi_1$. This is the point which is tip of the phasor of magnitude M_1 plotted at an angle ϕ_1.

Key Point: *The positive angles are measured in anticlockwise direction while negative are measured in clockwise direction.*

So without actually having an indication of frequency, for various values of frequencies from 0 to ∞ the magnitudes are plotted against phase angles.

Key Point: *Polar plot is the locus of tips of the phasors of various magnitudes plotted at the corresponding phase angles for different values of frequencies from 0 to ∞.*

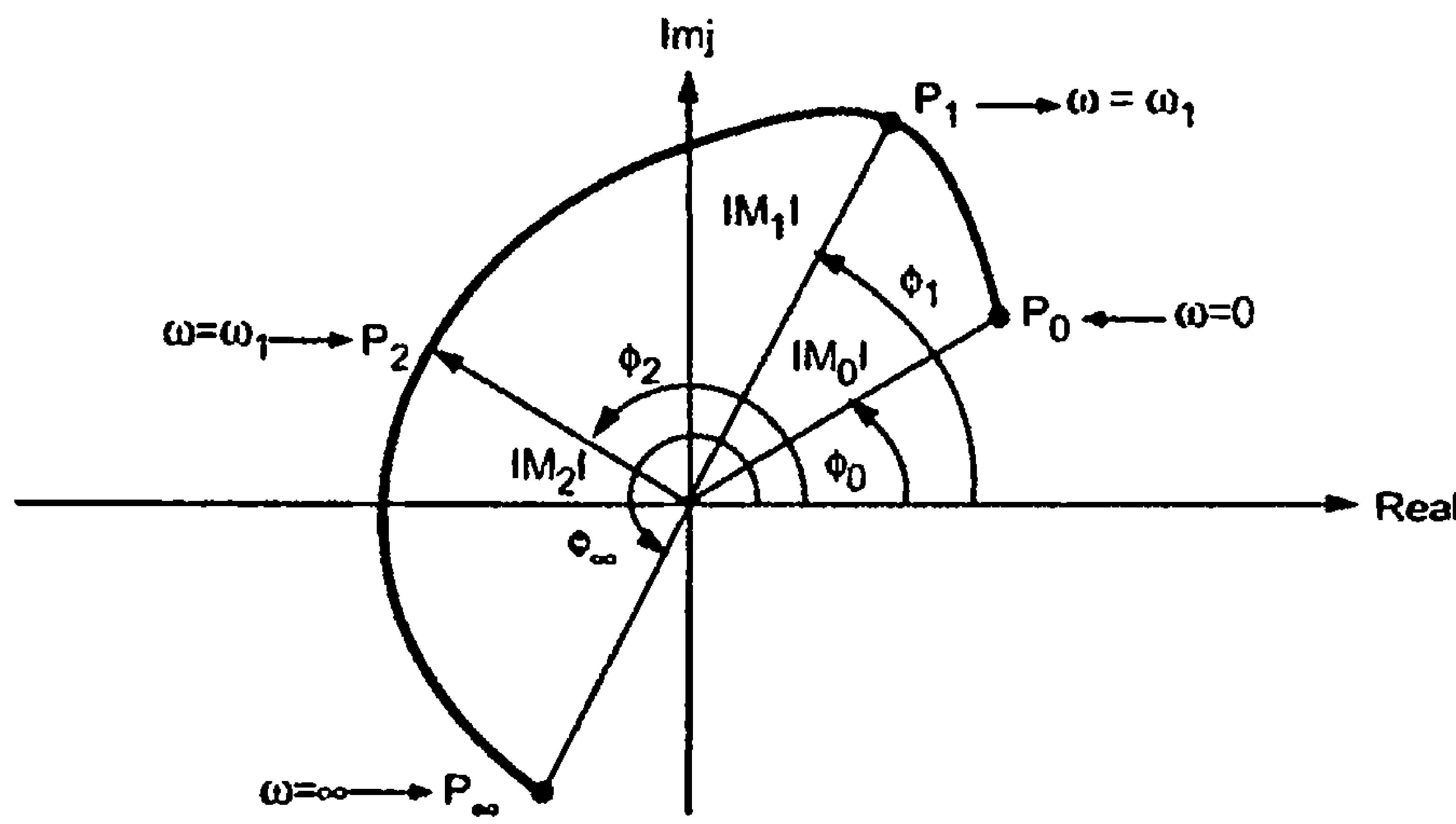


Fig. 12.1 Polar plot

This is shown in the Fig. 12.1.

So polar plot starts at point representing magnitude and phase angle for $\omega = 0$. While it terminates at a point representing magnitude and phase angle for $\omega = \infty$.

Example 12.1 : *Consider a system with open loop transfer function as* $G(s)H(s) = \frac{10}{s}$. *Obtain its polar plot .*

Solution : Now to obtain its polar plot, obtain frequency domain transfer function by replacing s by $j\omega$.

$$G(j\omega)\,H(j\omega) = \frac{10}{j\omega} = \frac{10 + j0}{0 + j\omega}$$

$$\therefore \quad |G(j\omega)\,H(j\omega)| = M = \frac{10}{\omega}$$

$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}\left(\frac{0}{10}\right)}{\tan^{-1}\left(\frac{\omega}{0}\right)} = \frac{0^\circ}{90^\circ} = -90^\circ$$

For various values of ω, M is changing but angle remains constant as –90°.

ω	M	φ
0	∞	– 90°
10	1	– 90°
100 : :	0.1	– 90°
∞	0	– 90°

So plot starts at ∞ at angle – 90° and terminates at origin along the axis of angle – 90° i.e. negative imaginary axis.

Hence we get the corresponding polar plot as shown in the Fig. 12.2 which is negative imaginary axis.

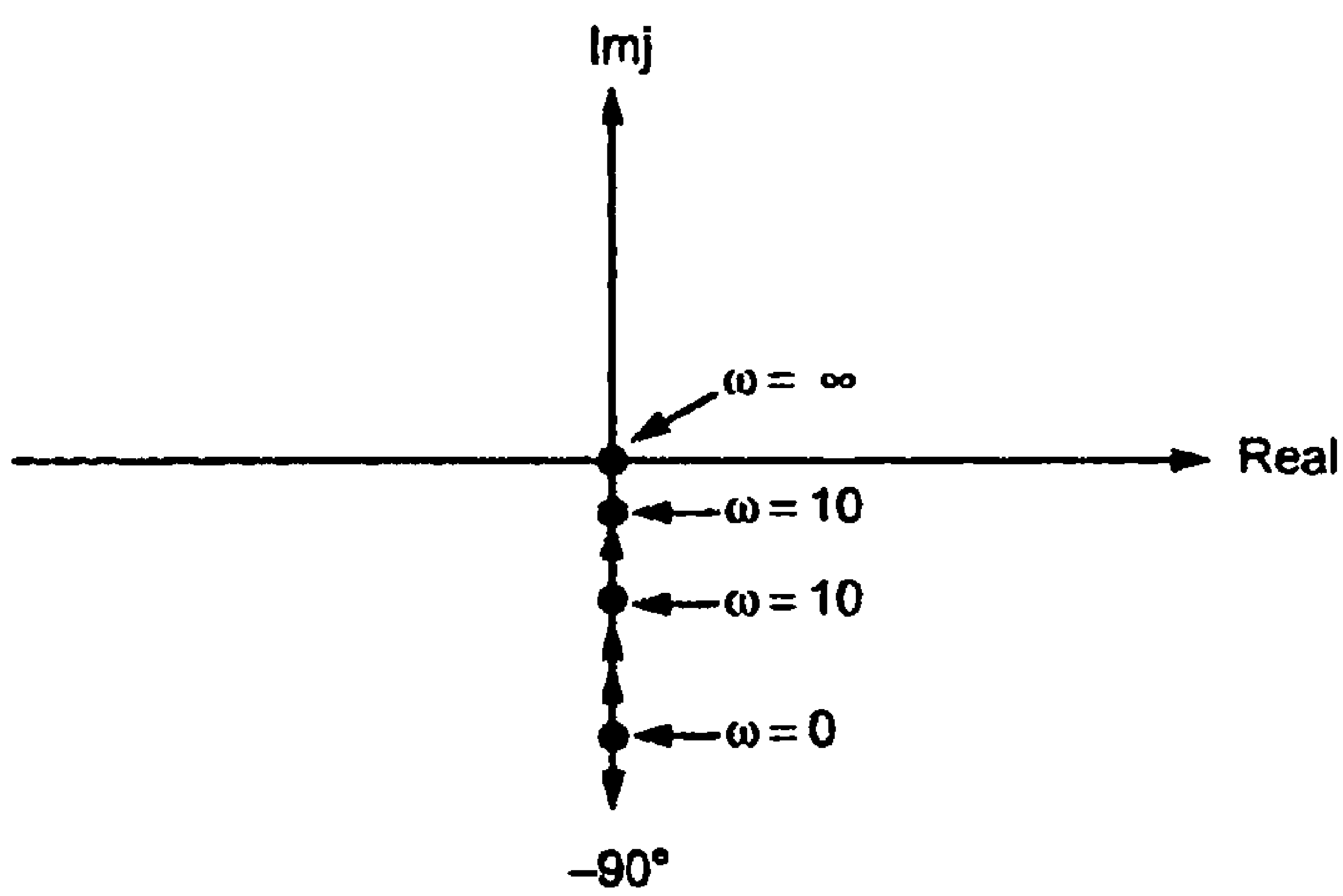


Fig. 12.2

Example 12.2 : *Consider a Type 0 system with open loop transfer function*

$G(s)H(s) = \frac{1}{1+Ts}$ *where T is constant. Obtain its polar plot.*

Solution : The frequency domain transfer function is,

$$G(j\omega)H(j\omega) = \frac{1}{1+Tj\omega} = \frac{1+j0}{1+j\omega T}$$

$$|G(j\omega)H(j\omega)| = M = \frac{1}{\sqrt{1+\omega^2 T^2}}$$

$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}\left(\frac{\omega T}{1}\right)} = \frac{0^\circ}{(\tan^{-1}\omega T)} = -\tan^{-1}(\omega T)$$

For various values of ω, the result can be tabulated as,

ω	M	φ
0	1	0°
$\frac{1}{T}$	$\frac{1}{\sqrt{2}}$	− 45°
$\frac{10}{T}$	$\frac{1}{\sqrt{101}}$	− 84.2°
:	:	:
:	:	:
∞	0	− 90°

This shows that plot starts at point 1 ∠0° corresponding to ω = 0 and ends at 0 ∠ −90°. i.e. at origin tangential to the axis of angle −90° corresponding to ω = ∞.

The corresponding polar plot is shown in the Fig. 12.3. The plot is semicircular.

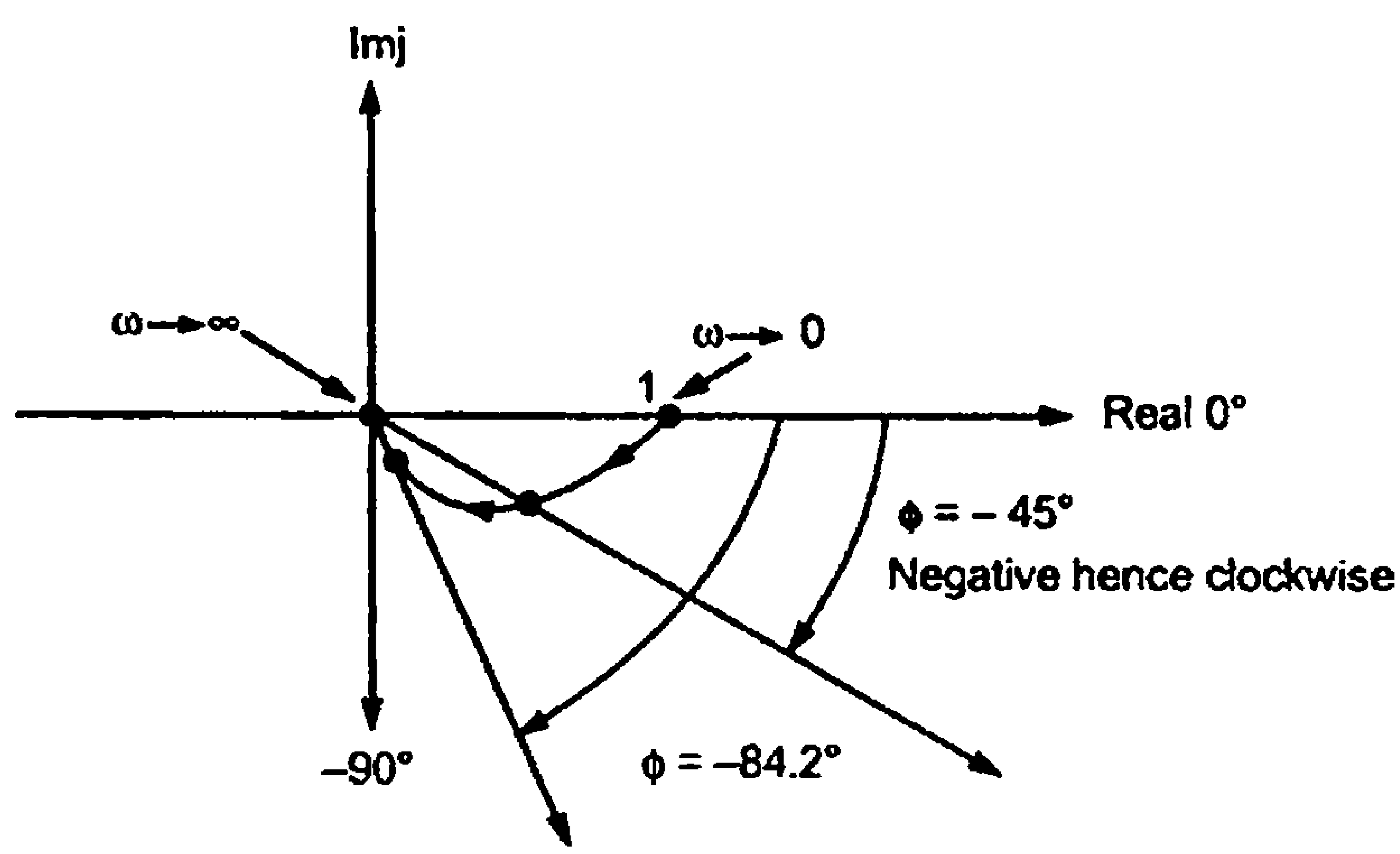


Fig. 12.3

Solution : The frequency domain transfer function is

$$G(j\omega)H(j\omega) = \frac{1}{j\omega \cdot j\omega \cdot (1 + j\omega T)} = \frac{(1 + j0)}{(0 + j\omega)(0 + j\omega)(1 + j\omega T)}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = M = \frac{1}{\omega^2 \cdot \sqrt{1 + T^2\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}\left(\frac{\omega}{0}\right)\tan^{-1}\left(\frac{\omega}{0}\right)\tan^{-1}\left(\frac{\omega T}{1}\right)} = \frac{0^\circ}{90^\circ \cdot 90^\circ \tan^{-1}\omega T}$$

$$\therefore \quad \phi = -180^\circ - \tan^{-1}\omega T$$

Starting point	$\omega \to 0$	$\infty \angle -180^\circ$	Rotation of plot = $-270^\circ - (-180^\circ) = -90^\circ$
Terminating point	$\omega \to \infty$	$0 \angle -270^\circ$	

Again rotation of plot is 90° clockwise but starting point has further moved to the axis of angle – 180°. So polar plot can be sketched as shown in the Fig. 12.7.

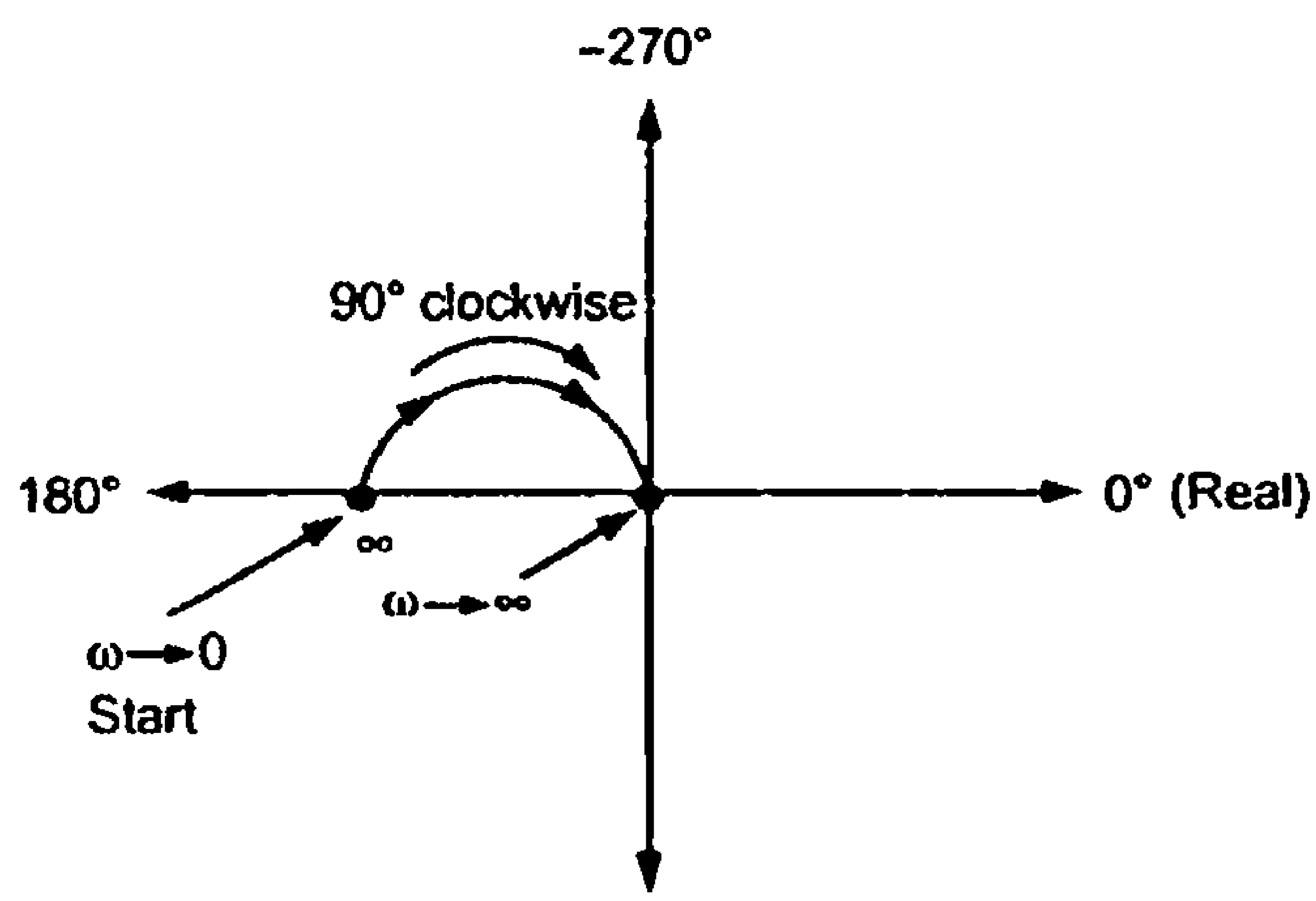


Fig. 12.7

> **Key Point**: *It can be concluded that a pole at origin shifts the starting point of polar plot by 90° in clockwise direction keeping the rotation of plot same.*

Example 12.5 : *Let us add a simple pole and see its effect on polar plot.*

$$G(s)H(s) = \frac{1}{(1 + T_1 s)(1 + T_2 s)}$$

Solution : The frequency domain, transfer function is

$$G(j\omega)H(j\omega) = \frac{1}{(1+T_1\, j\omega)(1+T_2\, j\omega)}$$

$$|G(j\omega)H(j\omega)| = M = \frac{1}{\sqrt{1+T_1^2\omega^2} \times \sqrt{1+T_2^2\omega^2}}$$

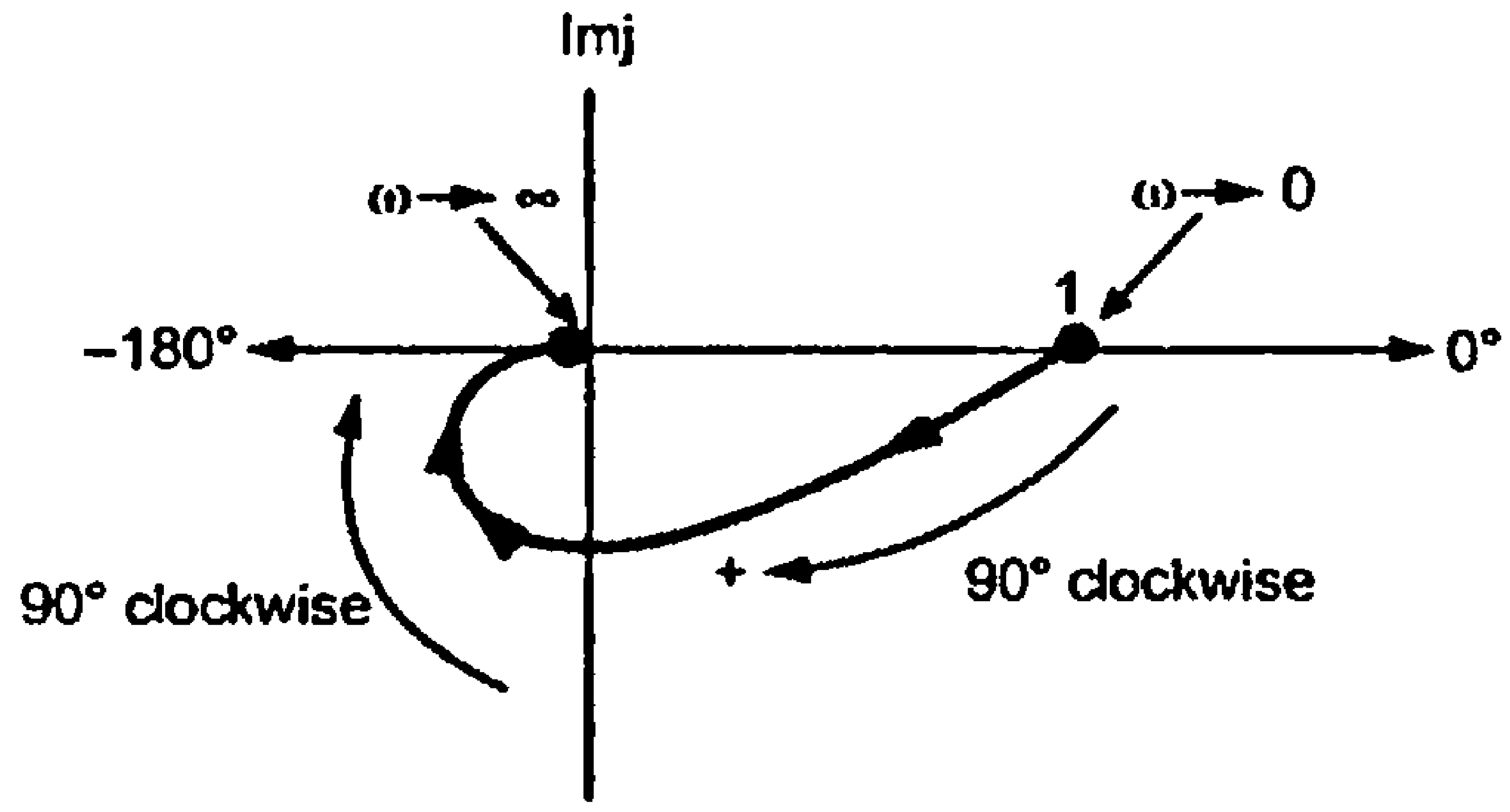
$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}\left(\frac{\omega T_1}{1}\right)\tan^{-1}\left(\frac{\omega T_2}{1}\right)}$$

$$\therefore \quad \phi = -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Starting point	$\omega \to 0$	$1\angle 0°$	Rotation of plot = $-180° - 0° = -180°$ clockwise
Terminating point	$\omega \to \infty$	$0\angle -180°$	

Rotation of plot = $-180°$ i.e. $180°$ in clockwise direction.

So polar plot is as shown in the Fig. 12.8.



Total 180° clockwise rotation of point

Fig. 12.8

> **Key Point:** *It can be concluded that a simple pole keeps the starting point same but adds a 90° clockwise rotation to the polar plot.*

12.3 ω_{gc} and ω_{pc} in Polar Plot

In Bode plot we have defined that the frequency at which $|G(j\omega)H(j\omega)| = 0$ dB is called gain crossover frequency. But in polar plot we are not using dB values. So $|G(j\omega)H(j\omega)| = 1$ corresponding to 0 dB at $\omega = \omega_{gc}$ from polar plot point of view. Now how to locate a point on polar plot having M = 1? Consider polar plot shown in the Fig. 12.9. To get a point with M = 1, draw a circle with radius 1 and centre as origin. The point where this circle intersects polar plot is the point where $|G(j\omega)H(j\omega)| = 1$ and corresponding frequency is $\omega = \omega_{gc}$.

Now point P is the point on polar plot corresponding to $\omega = \omega_{gc}$ obtained by drawing unit radius circle. So if ϕ is the angle of point P corresponding to $\omega = \omega_{gc}$ the P.M. can be calculated as shown in the Fig. 12.10.

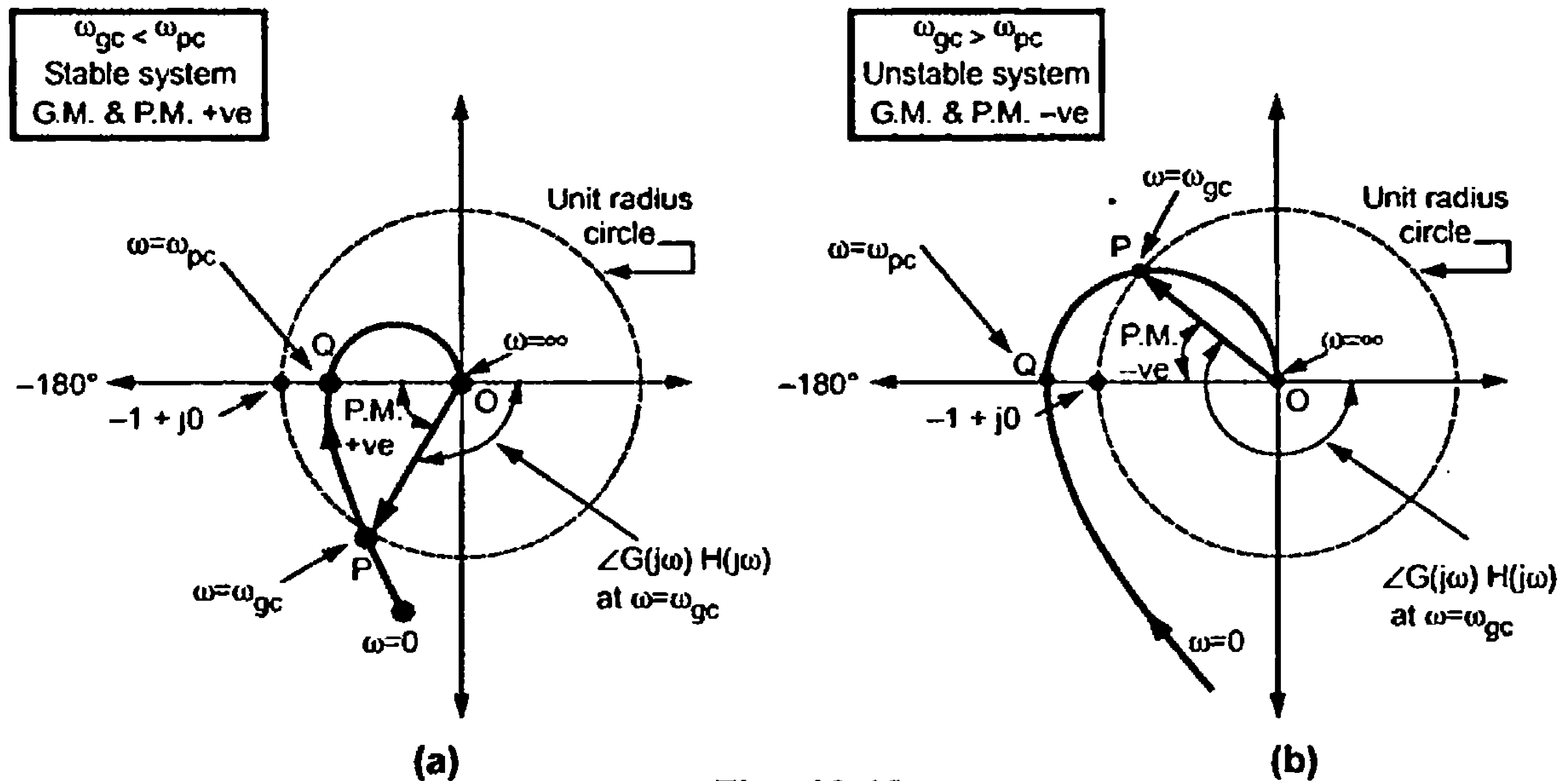


Fig. 12.10

Note : In practice, polar plot is required to be drawn precisely and to the scale to obtain accurate value of P.M. In examination P.M. cannot be obtained accurately from polar or Nyquist plot. But G.M. can be obtained. Because point Q which is intersection of polar plot with negative real axis can be obtained mathematically.

12.4.1 Determining ω_{pc} Mathematically

At $\omega = \omega_{pc}$, there occurs intersection of polar plot with negative real axis say at point Q in the Fig. 12.9. Now the rectangular co-ordinates of point Q are $-\sigma + j0$ where $-\sigma$ is its real part but its imaginary part is zero as it is located on negative real axis.

> **Key Point**: *So $\omega = \omega_{pc}$, is such a frequency at which imaginary part of $G(j\omega)H(j\omega)$ becomes zero, when $G(j\omega)H(j\omega)$ is expressed in rectangular co-ordinates.*

Mathematically this can be obtained as below :

i) Rationalize the open loop transfer function $G(j\omega)H(j\omega)$.

ii) Separate real and imaginary parts of $G(j\omega)H(j\omega)$. Both are the function of ω.

iii) Equate imaginary part to zero to get equation as $f(\omega) = 0$. Solve this to get value of ω which is making this imaginary part zero i.e. $\omega = \omega_{pc}$. This frequency **should be positive finite and greater than zero.** Otherwise it can be concluded that there is no intersection of polar plot with negative real axis.

iv) Substitute this value of ω_{pc} in the real part to get the actual co-ordinates of an intersection of point of polar plot with negative real axis.

So we can conclude,

> **Key Point**: *Zeros of 1 + G(s)H(s) = Closed loop poles of a system*

e.g. Consider the same system with $G(s)H(s) = \frac{10}{s(s+2)}$

Now $$F(s) = 1 + G(s)H(s) = 1 + \frac{10}{s(s+2)} = \frac{s^2 + 2s + 10}{s(s+2)} = \frac{P(s)}{Q(s)}$$

So Q(s) = 0 gives poles of 1 + G(s)H(s) i.e. s = 0, – 2 which are the open loop poles.

While P(s) = 0 gives zeros of 1 + G(s)H(s) i.e. roots of $s^2 + 2s + 10$ which are the closed loop poles.

Key Point: *So from Nyquist point of view, the system is absolutely stable if **all the zeros** of 1 + G(s)H(s). i.e. closed loop poles of the system are located in the left half of s-plane.*

The stability criterion remains the same, only the way to express it is changed by analyzing the function 1 + G(s)H(s) for its own poles and zeros. This new approach is important in the understanding of a Nyquist criterion.

12.7 Encirclement

Earlier we have defined enclosement. It is applicable for the paths wh not closed like polar plots. But for a closed path, it is necessary to under of an encirclement.

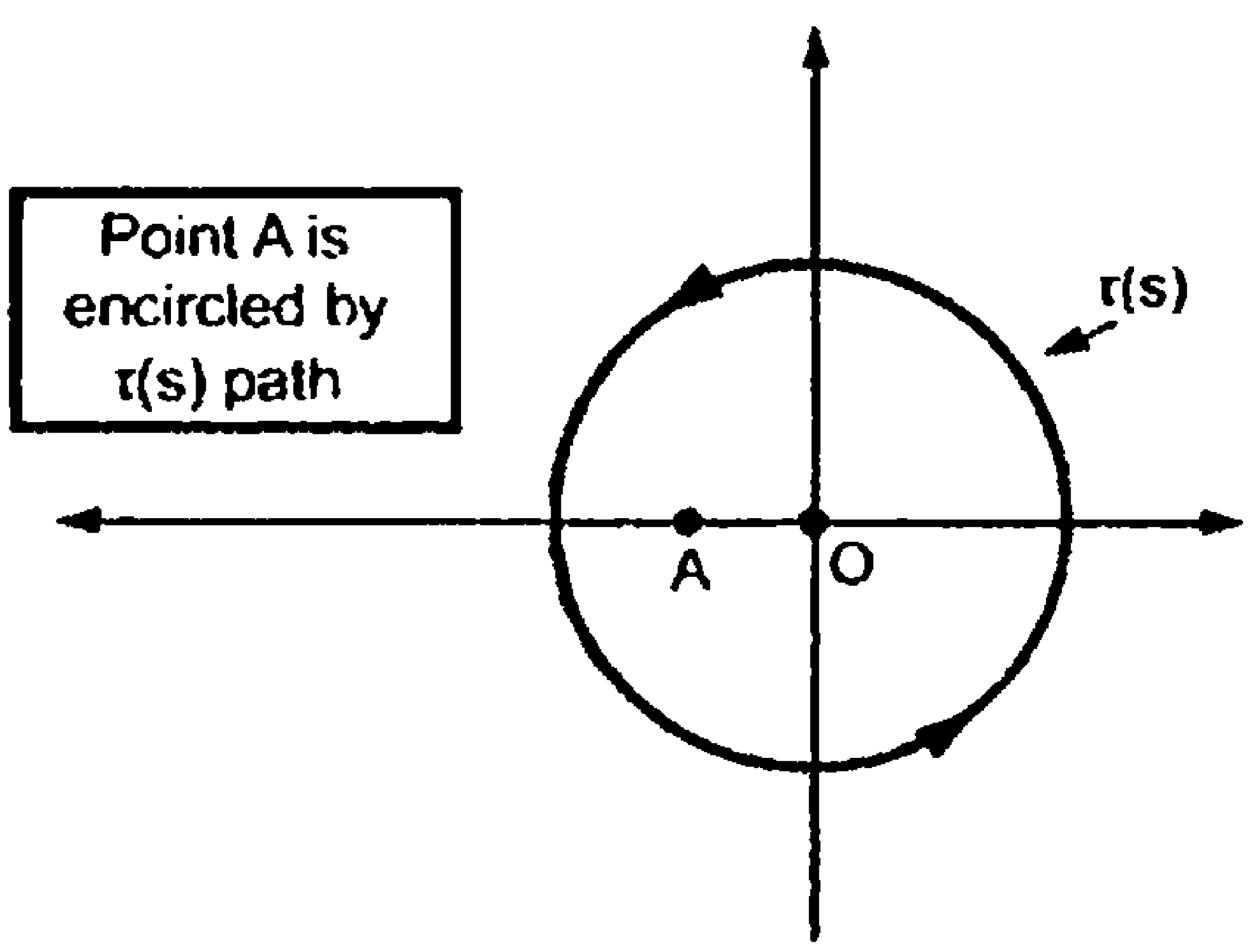


Fig. 12.13

Consider a closed path $\tau(s)$ ir in the Fig. 12.13.

A point is said to be encircled by if it is found to lie inside that closed pa

So in the Fig. 12.13, the points O, A which are inside the path are encircled by that p

This is a simple concept but in some complicated closed paths it is possible that point lies inside the path but actually not encircled by the path. Hence it is always better to count the number of encirclements of a point.

12.7.1 Counting Number of Encirclements

In the Fig. 12.13 it can be easily concluded that the number of encirclements of point A and O is one in anticlockwise direction. But for complicated cases, it is not possible to judge number of encirclements by mere inspection. One simple method can be used to get number of encirclements.

The method includes following steps :

1) Draw a vector from a point whose encirclements are to be determined, in such a way to join any point outside that closed path in any direction. Avoid confusing directions.
2) Identify the number of intersections of this vector with a closed path.
3) Mark these intersections with small arrow on the same vector indicating direction of closed path at the time of intersection.
4) Cancel the oppositely directed encirclements. The remaining arrows gives us the number of encirclements of that point.

Anticlockwise Encirclements are treated as Positive. Clockwise Encirclements treated as Negative.

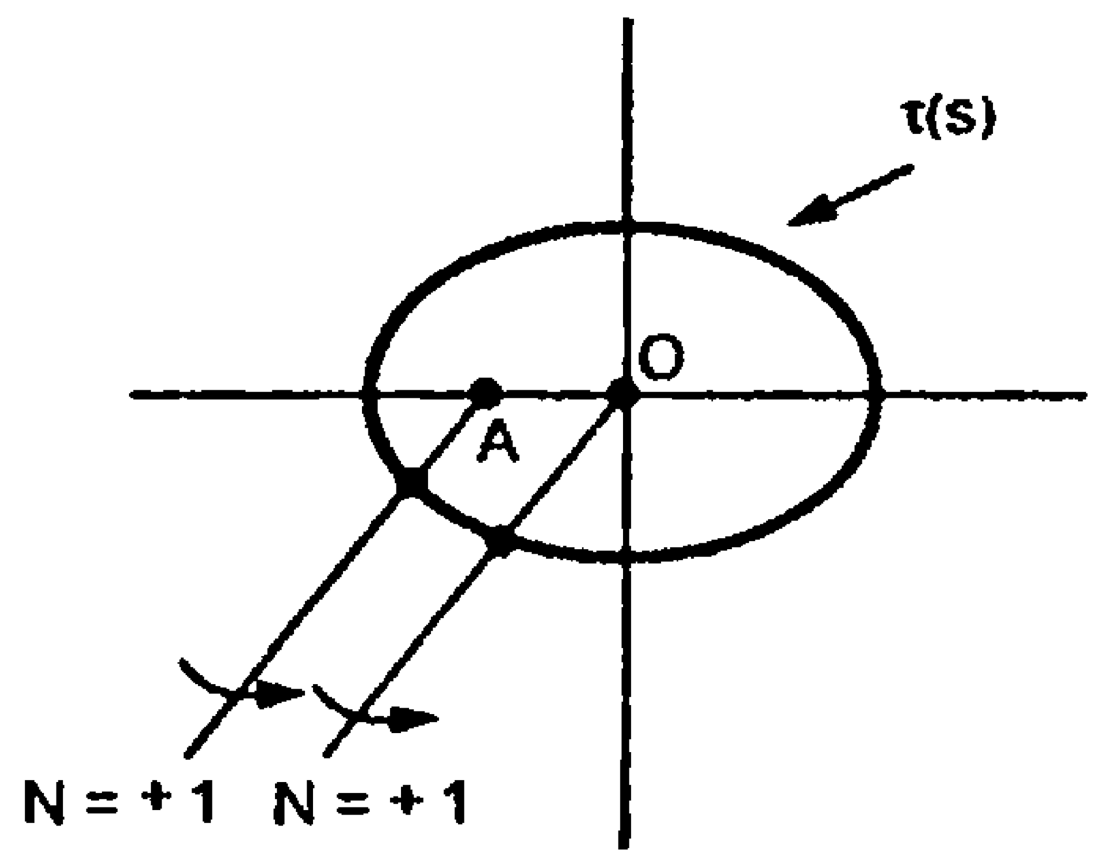


Fig. 12.14

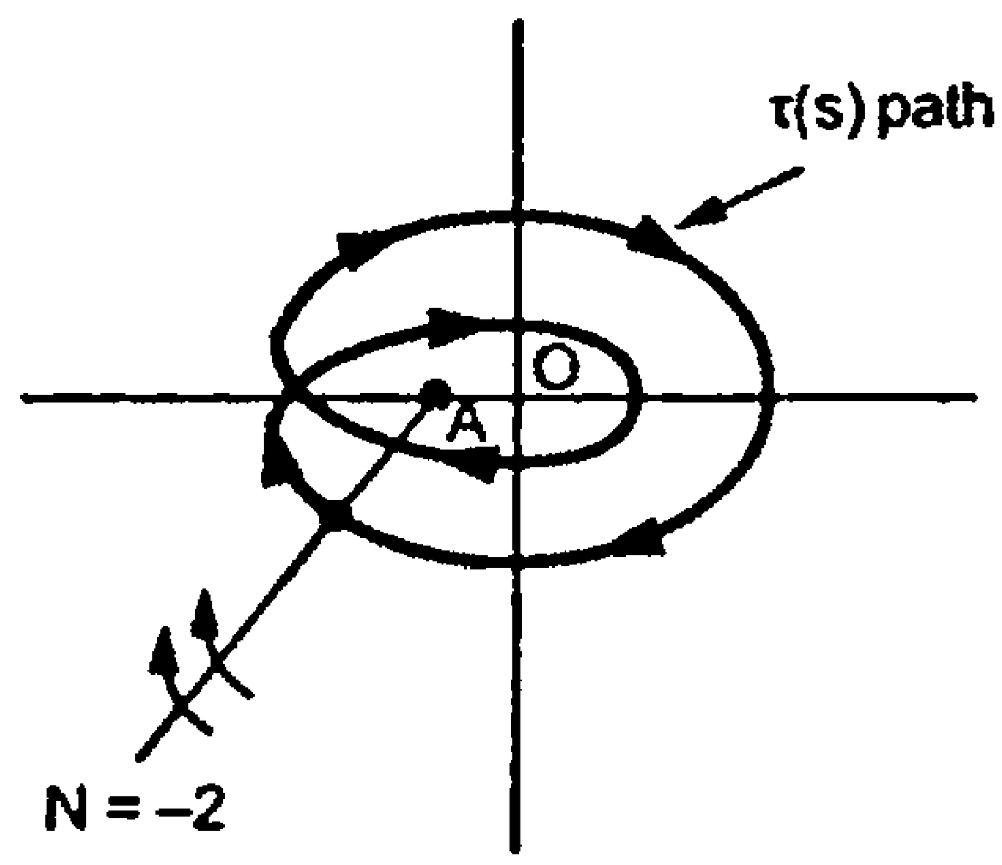


Fig. 12.15

e.g. In the closed path shown in the Fig. 12.14 for both the points O and A we get the direction anticlockwise at the point of intersection of the vector drawn with τ(s) path.

Hence net number of encirclements denoted as N and +1 i.e. one in anticlockwise direction. Consider a closed path as shown in the Fig. 12.15.

The vector drawn from point A, intersects path at two points. At both the points path is travelling in clockwise direction. Hence total number of encirclements are –2 i.e. two in clockwise direction.

Consider another closed path as shown in the Fig. 12.16. In this case the vector drawn from point A intersects the path at two points.

But at these two points, the direction of path is opposite to each other.

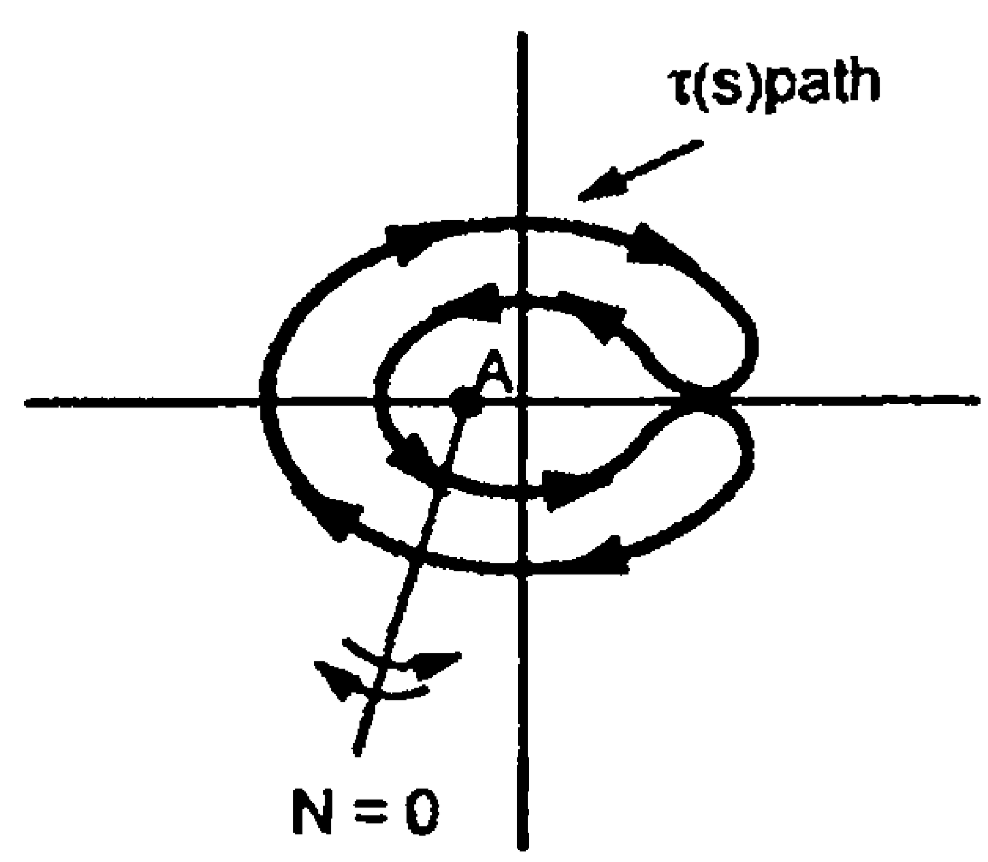


Fig. 12.16

So these two encirclements cancel each other at the time of counting final number of encirclements. Hence number of encirclements of point A are zero (N = 0) though point A is found to lie inside the path τ (s).

12.8 Analytic Function and Singularities

A mathematical function is said to be analytic at a point in a plane if its value and its derivative has finite existence at that point.

If at a point in a plane, the value of function or its derivative is infinite, the function is said to be non analytic at that point and such a point is called singularity of the function. So all such points where given function is not analytic are called singularities of that function.

Consider a function $F(s) = \frac{25}{s(s+1)}$, then it is analytic at all points in s-plane except the point s = 0 and s = –1. This is because the function F(s) has a value ∞ at s = 0 and – 1 which are its poles.

In general, the poles of the function are its singularities. Similarly one more function we will define is single valued function.

A function F(s) who has one and only one value for each separate value of s is said to be single valued.

Consider $F(s) = \sqrt{s}$, for s = 25, F(s) has two values +5 and – 5. Such a function is not single valued. In our control system analysis we will assume that the transfer function of system or the functions G(s) and H(s) are single valued.

12.9 Mapping Theorem or Principle of Argument

The mapping theorem states that, let F(s) be the single valued function, analytic at all points in s - plane except some finite number of points. These are singularities of function F(s) where it is not analytic. Consider an arbitrary closed path τ(s) in s-plane in such a way that the function F(s) is analytic at each and every point on τ(s) path. So the restriction to select a closed path is that it should not pass through the points which are singularities of F(s) i.e. it should not pass through poles of F(s).

Now let P and Z be the number of poles and zeros of F(s) **which are encircled by τ(s) path.** i.e. which are inside the path τ(s). We are not interested in all the poles and zeros of F(s) but only those which are encircled by τ(s) path in s-plane. So hereafter.

P = Number of poles of F(s) encircled by τ(s).

Z = Number of zeros of F(s) encircled by τ(s).

Now according to mapping procedure, the closed path τ(s) in s-plane can be mapped into other plane say F-plane to get a closed path say τ′(s).

Mapping procedure : Consider a function

$$F(s) = \frac{10}{s(s+2)}$$

The mapping theorem is actually related to this mapped locus $\tau'(s)$ in F-plane.

Mapping theorem states that the mapped locus $\tau'(s)$ encircles the new origin of F-plane as many times as the difference between the number of zeros and poles of F(s) which are encircled by $\tau(s)$ path in s - plane mathematically,

$$N = Z - P$$

where

N = Encirclements of origin of F- plane by $\tau'(s)$ path

P = Number of poles of F(s) encircled by $\tau(s)$ path in s-plane

Z = Number of zeros of F(s) encircled by $\tau(s)$ path in s-plane

This statement is also called **Principle of Argument.**

To understand this clearly, consider the following three cases and corresponding mapped loci $\tau'(s)$ in F- plane.

Let $$F(s) = \frac{4(s+1)}{s(s+5)}$$

Case i) P > Z, Let only 1 pole is encircled by $\tau(s)$ path without any zero encircled by $\tau(s)$ path in s-plane.

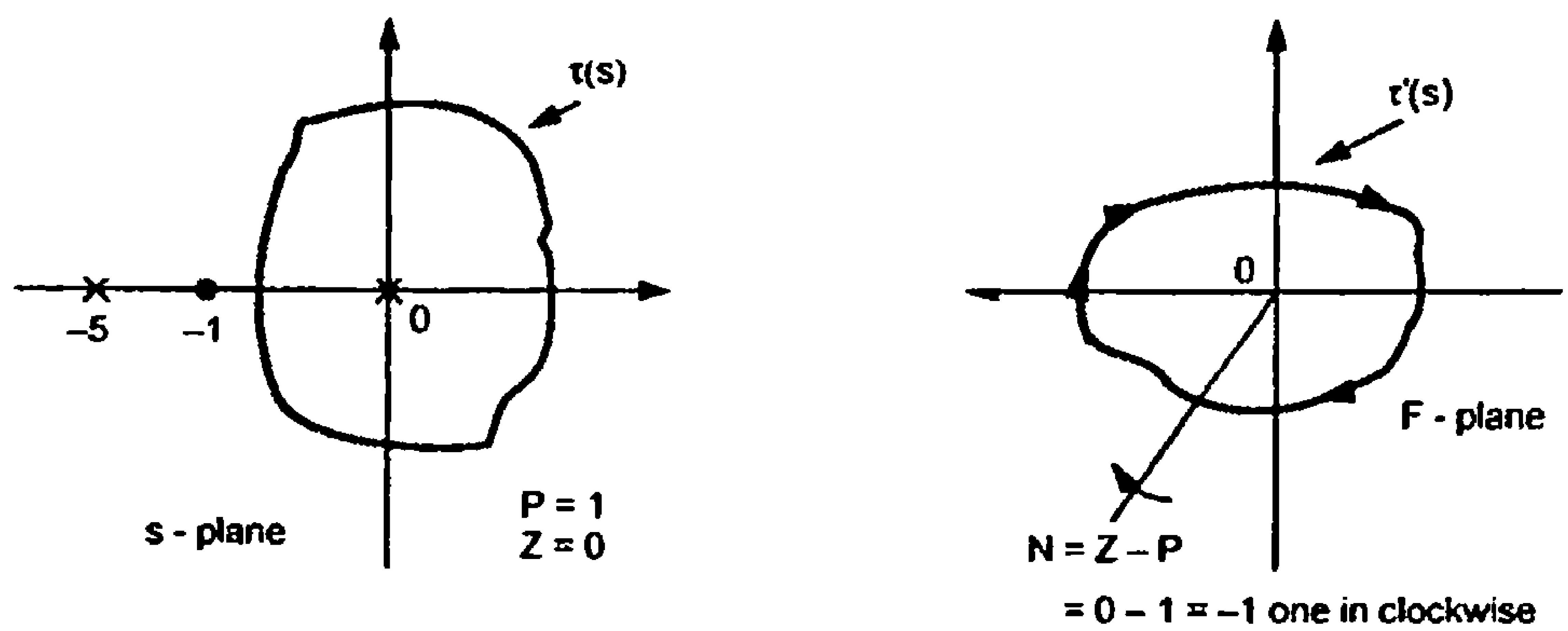


Fig. 12.20

The shape of $\tau'(s)$ is not important but it encircles the origin for 'Z – P' times is important and is the essence of the mapping theorem.

Case ii) P < Z, Let only 1 zero is encircled by $\tau(s)$ path without encircling any pole in s-plane. Then Z = 1, P = 0 hence we will get N = Z – P = + 1 so $\tau'(s)$ path will encircle origin of F-plane once in anticlockwise direction.

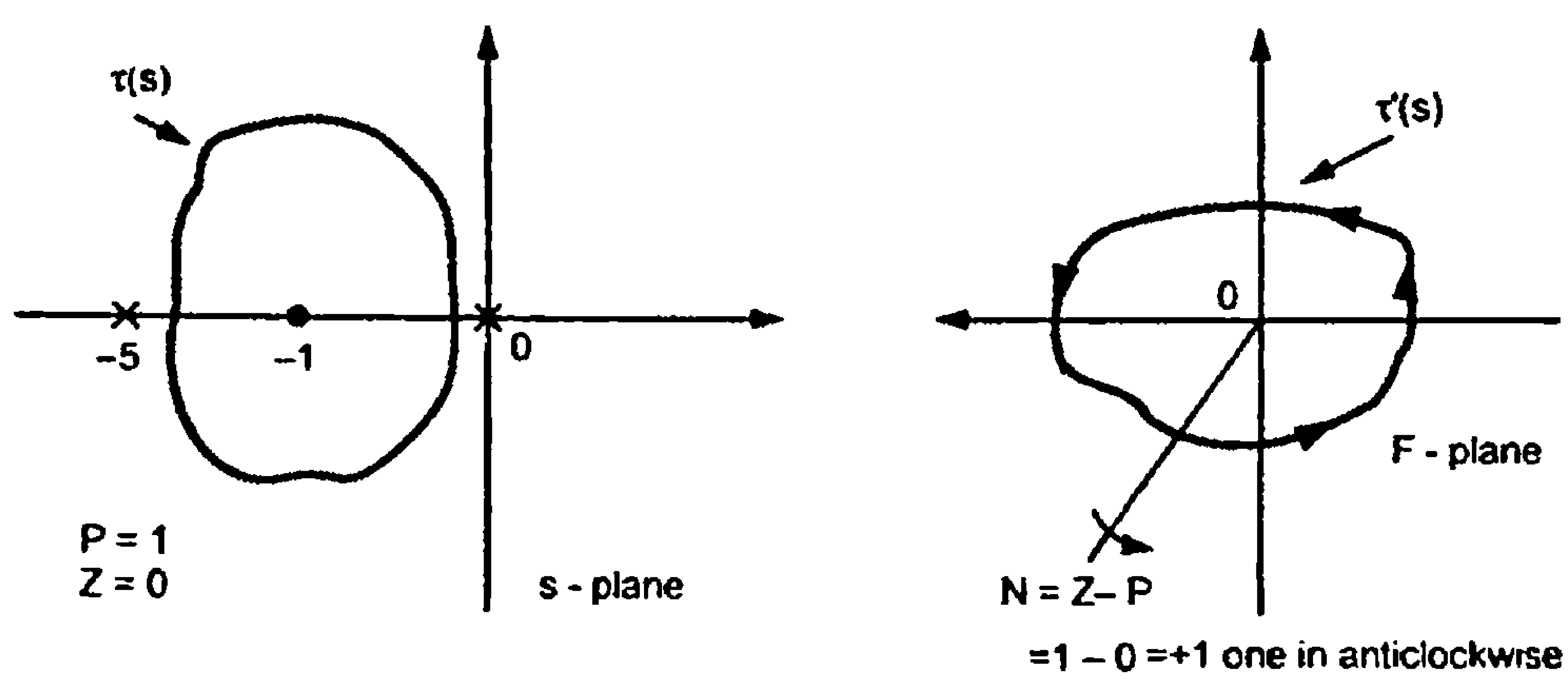


Fig. 12.21

Case iii) P = Z , Let one pole and one zero is encircled by τ(s) path in s - plane. Then Z = P and N= 0. In such a case the τ′(s) path will be in such a way to get zero encirclements of origin of F-plane. This is possible with an origin lying outside the τ′(s) path or encircled by a complicated τ′(s) locus which is giving net encirclements of new origin as zero.

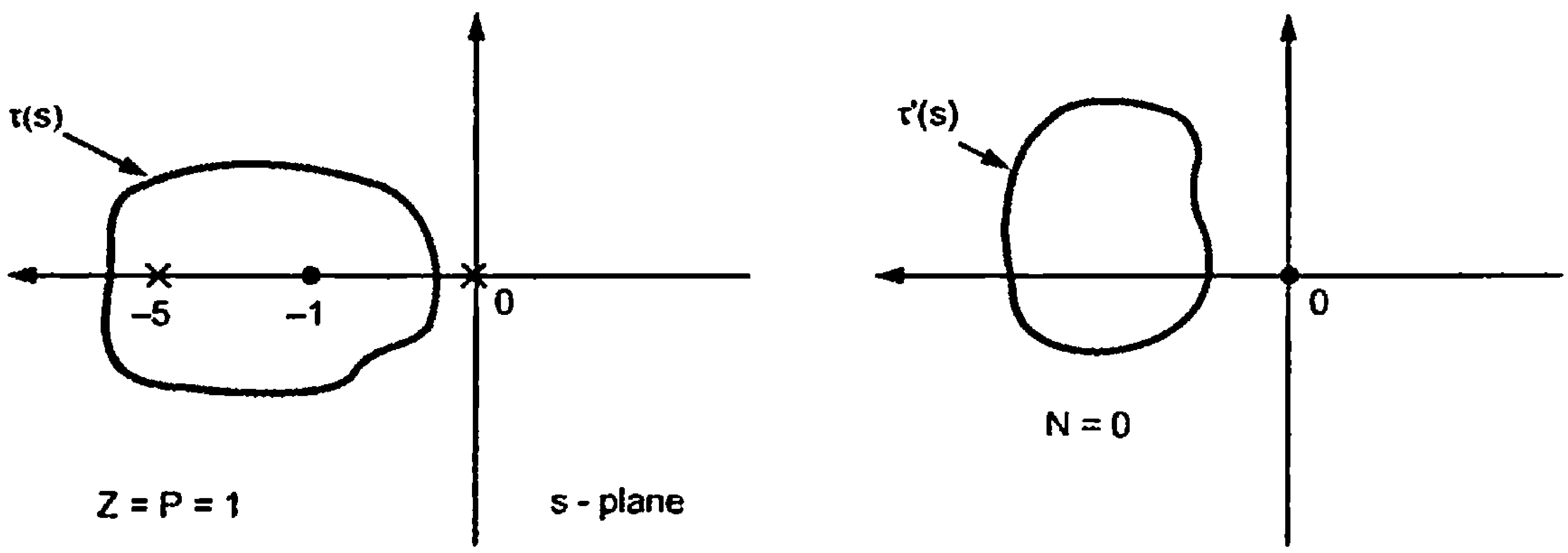


Fig. 12.22

12.10 Nyquist Stability Criterion

Nyquist has used the above theorem effectively to develop a criterion to study the stability of control system in frequency domain.

Nyquist suggested to select a single valued function F(s) as 1+ G(s)H(s) where G(s)H(s) is open loop transfer function of the system.

$$F(s) = 1 + G(s)H(s)$$

Now we have developed the configuration in section 12.7 that

Poles of 1 + G(s)H(s) = Poles of G(s)H(s) = Open loop poles

These are known to us as G(s)H(s) is known to us

But Zeros of 1 + G(s)H(s) = Closed loop poles of the system

For stability, all the zeros of 1 + G(s)H(s) must be in the left half of s-plane, none of the zeros should be in the right half of s-plane. Now the locations of zeros of 1 + G(s)H(s) are unknown to us.

In fact we are trying to study the stability by analyzing where these zeros of 1 + G(s)H(s) are located in s-plane.

Nyquist has suggested that rather than analyzing whether all the zeros are located in left half of s-plane, it is better to examine the presence of any one zero of 1 + G(s)H(s) in right half of s-plane making system unstable. Hence the active region from the stability

$= 0$ as there is no pole of G(s)H(s) in right half of s-plane

$\therefore$ For stability, $N = -P = 0$

i.e. the Nyquist plot obtained by mapping Nyquist path from s-plane to F-plane should not encircle origin of F - plane.

Note : Now for ease of mapping Nyquist path from s-plane to F - plane, instead of considering mapping function as 1+ G(s)H(s), it is considered as G(s)H(s) only.

But due to this, stability criterion remains the same i.e.

$$N = -P$$

where P = Number of poles of G(s)H(s) in right half of s-plane

But the change is,

N = Number of encirclements of a critical point $-1 + j0$ of F-plane by Nyquist plot instead of number of encirclements of an origin.

So in all the problems solved hereafter, N is the number of encirclements of a critical point $-1 + j0$ and not the encirclements of origin as mapping function used is G(s)H(s) and not the function 1 + G(s)H(s).

12.11 Generalized Nyquist Path and its Mapping

If the function has poles at origin or poles on the imaginary axis, Nyquist path cannot be selected along imaginary axis passing through origin. This is because mapping theorem states that at every point on Nyquist path, function must be analytic. But at its poles it can not be analytic. In such a case Nyquist path is modified in such a way to bypass these poles by selecting semicircles of radius tending to zero around them but still encircling entire right half of s-plane.

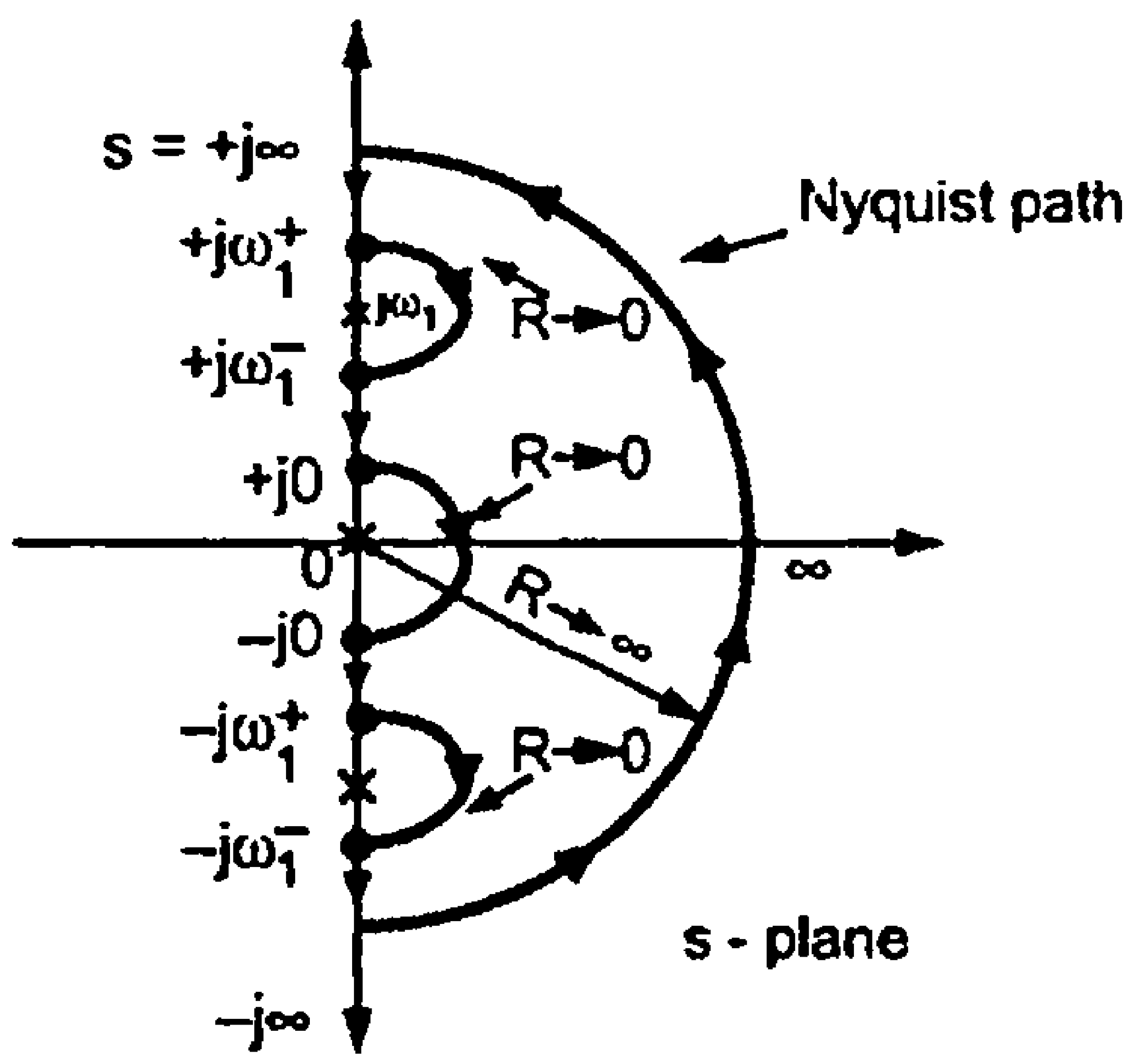


Fig. 12.24 Generalized Nyquist Contour

Depending upon the situation of poles of G(s)H(s), Nyquist path should be selected. The guidelines for selection of Nyquist path are given here.

Let F(s) has two poles on imaginary axis at $\pm j\omega_1$ while one pole at origin. Then Nyquist path should be selected as shown in the Fig.12.24.

The points by which path is modified are :

$+j\omega_1^+ \rightarrow$ A point just above $+j\omega_1$, very very close to $+j\omega_1$.

$+j\omega_1^-$ → A point which is very very close to $+j\omega_1$, but just below it.

$-j\omega_1^+$ → A point which is very very close to $-j\omega_1$, but just above it.

$-j\omega_1^-$ → A point which is very very close to $-j\omega_1$ but just below it.

$+j0$ → A point which is very very close to origin but just above it on positive imaginary axis hence denoted as $+j0$.

$-j0$ → A point which is very very close to origin but just below it on negative imaginary axis hence denoted as $-j0$.

The various sections of Nyquist path are,

Section	Start	End	Comment
I	$s = +j\infty$	$s = +j\omega_1^+$	Along the imaginary axis.
II	$s = +j\omega_1^+$	$s = +j\omega_1^-$	Along a semicircle of radius tending to zero
III	$s = +j\omega_1^-$	$s = +j0$	Along the imaginary axis.
IV	$s = +j0$	$s = -j0$	Along semicircle of radius tending to zero.
V	$s = -j0$	$s = -j\omega_1^+$	Along the imaginary axis.
VI	$s = -j\omega_1^+$	$s = -j\omega_1^-$	Along semicircle of radius tending to zero.
VII	$s = -j\omega_1^-$	$s = -j\infty$	Along the imaginary axis.
VIII	$s = -j\infty$	$s = +j\infty$	Along the semicircle of $R \to \infty$ encircling entire right half.

Now mapping of these sections in F-plane can be achieved by drawing the polar plots for various sections by shortcut method.

We have $F(s) = G(s)H(s)$

$\therefore$ In frequency domain $= G(j\omega)H(j\omega)$

Now for section I,

Starting point is $s = +j\infty$ i.e. $\omega = \infty$

Terminating point is $s = +j\omega_1^+$ i.e. $\omega = \omega_1^+$

Substitute to get magnitude and phase angle of $G(j\omega)H(j\omega)$ at $\omega = \infty$ and $\omega = \omega_1^+$. Get the rotation of plot and then rotate the starting point through angle of rotation to reach to terminating point. This rough curve gives us mapping of section I in F-plane.

Now for section II, the terminating point of section I becomes starting and using same procedure obtain the mapping of section II.

After mapping all sections, we will get a closed plot called **Nyquist plot.**

Now count number of encirclements of –1 +j0 and check whether it satisfies N = – P.

Key Point: *The section V, VI and VII are mirror images of section III, II and I respectively, Hence their mapping in F-plane will be also mirror images **about real axis.***

Similarly for last section i.e. $s = -j\infty$ to $s = +j\infty$ i.e. $\omega \to -\infty$ to $\omega \to +\infty$. Here $|G(j\omega)H(j\omega)|$ becomes zero for both starting and terminating points. Hence the two points coincide at origin. Actually there exists some rotation about origin with $R \to 0$ before plot terminates at origin. But **these rotations does not affect the encirclements of –1 +j0 as for these encirclements R is almost zero.** In our analysis hence we are going to show mapping of last section as a single point i.e. origin . There is no need of the analysis of the last section.

So joining polar plots of all the sections one after the other, the mapped locus called Nyquist plot is obtained. From the encirclements of –1+j0 by Nyquist plot, stability of the system can be predicted. **Calculation of G.M. and P.M. from Nyquist plot is exactly similar to the calculation of G.M. and P.M. from the polar plot** as discussed earlier in section 12.3 and 12.4.

12.12 Steps to Solve Problems by Nyquist Criterion

Step 1 : Count how many number of poles of G(s)H(s) are in the right half of s-plane i.e. with positive real part. This is the value of P.

Step 2 : Decide the stability criterion as N = – P i.e. how many times Nyquist plot should encircle '–1 +j0' point for absolute stability.

Step 3 : Select Nyquist path as per the function G(s)H(s).

Step 4 : Analyse the sections as starting point and terminating point of plot. Last section analysis not required.

Step 5 : Mathematically find out ω_{pc} and intersection of Nyquist plot with negative real axis by rationalizing $G(j\omega)H(j\omega)$.

Step 6 : With the knowledge of step 4 and 5, sketch the Nyquist plot.

Step 7 : Count the number of encirclements N of –1 + j0 by Nyquist plot. If this matches with the criterion decided in step 2 system is stable, otherwise unstable.

$$\text{G.M.} = \frac{1}{|OQ|} \quad \text{where}$$

Q = Intersection point of Nyquist plot with negative real axis obtained in step 5.

i.e. $$\text{G.M.} = 20 \log_{10} \frac{1}{|OQ|} \text{ dB}$$

$$\therefore \quad \left|-\frac{K}{240}\right| < 1$$

$$\therefore \quad K < 240$$

So range of values of K for stability is

$$\boxed{0 < K < 240}$$

Exercise :

Students are expected to solve the following problems. The Nyquist plot of them is same as obtained in Ex. 12.6, only the point Q has different co-ordinates.

1) $G(s)H(s) = \dfrac{1}{s(1+s)(1+2s)}$

Solution : $\omega_{pc} = \dfrac{1}{\sqrt{2}}$, $Q = -0.66 + j0$ N = 0, stable.

2) $G(s)H(s) = \dfrac{K}{s(s+4)(s+8)}$

Solution : $\omega_{pc} = \sqrt{32}$, $Q = -\dfrac{K}{384}$ for stability , $0 < K < 384$

3) $G(s)H(s) = \dfrac{K}{s(s+3)(s+10)}$

Solution : $\omega_{pc} = \sqrt{30}$, $Q = -\dfrac{K}{390}$ for stability, $0 < K < 390$

4) $G(s)H(s) = \dfrac{12}{s(s+3)(s+6)}$

Solution : $\omega_{pc} = \sqrt{18}$, $Q = -0.074 + j0$ N = 0 stable.

Example 12.7 : *For a feedback control system ,*

$$G(s)H(s) = \frac{40}{(s+4)(s^2+2s+2)}$$

Find Gain Margin and stability from Nyquist plot.

Solution : Step 1 : P = 0

Step 2 : N = − P = 0

Step 3 : No pole at origin or purely on imaginary axis. Therefore Nyquist path is as shown in the Fig. 12.27.

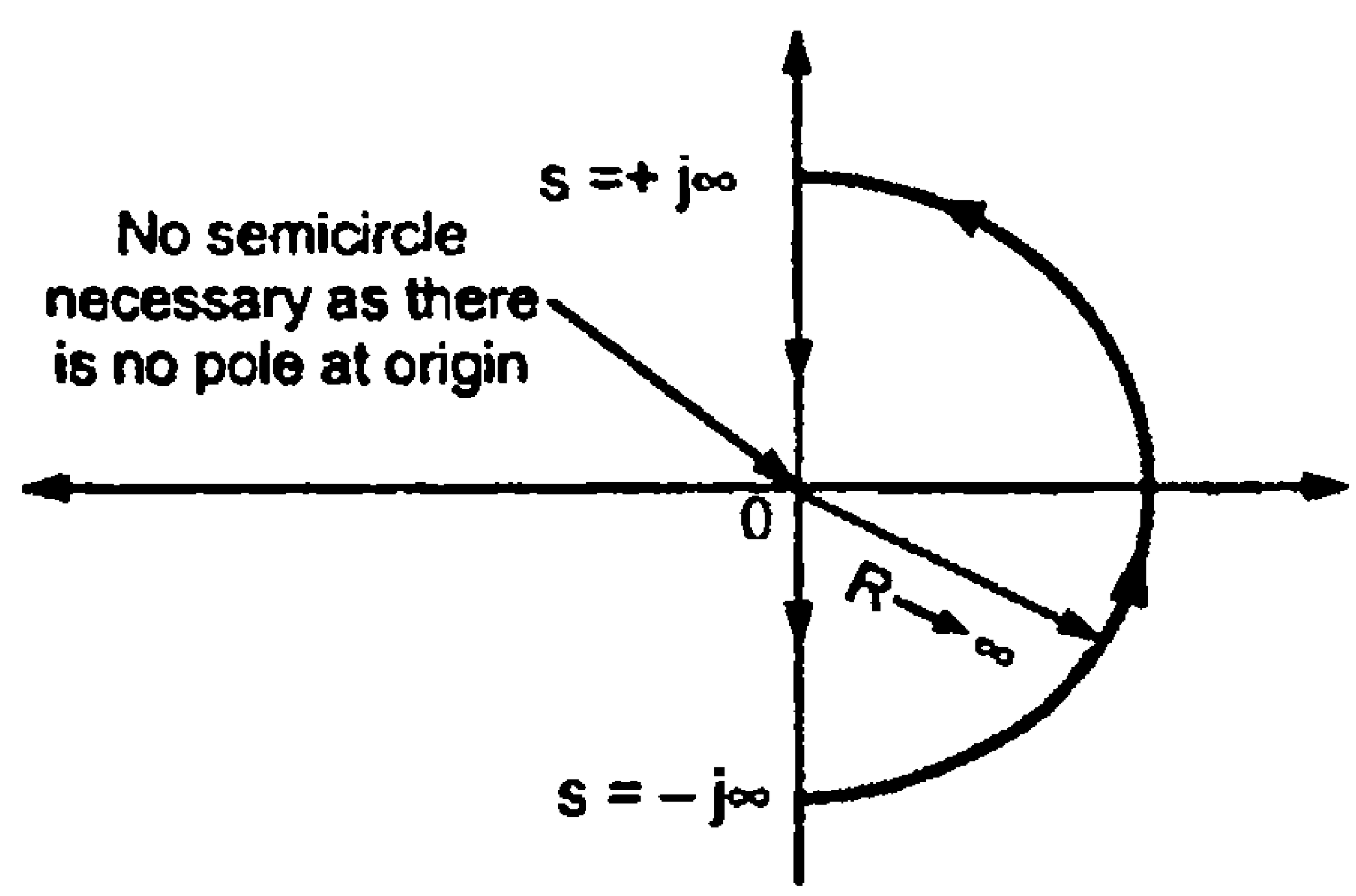


Fig. 12.27

Actually Nyquist path is divided into two sections. But for simplicity we will analyse it in three sections as,

Section I s = + j∞ to s = 0

Step 6 : Nyquist plot is

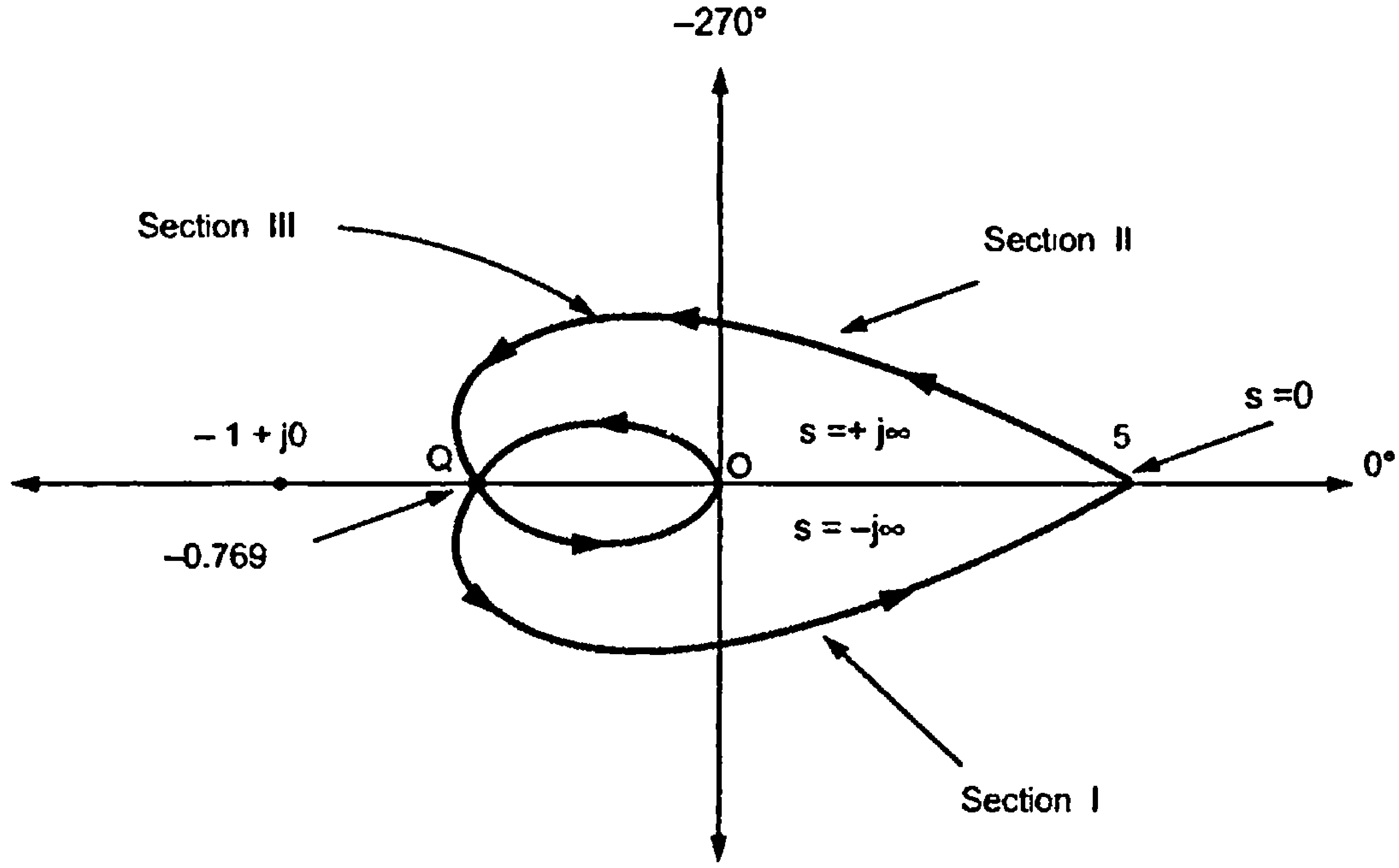


Fig. 12.28

Step 7 : Critical point $-1 + j0$ is outside the plot

$\therefore \quad N = 0$

This matches with the criterion of stability, hence the system is stable in nature.

$$\text{G.M.} = \frac{1}{l(\text{OQ})} = \frac{1}{0.769} = 1.3$$

or

$$\text{G.M.} = 20 \text{ Log} \frac{1}{|\text{OQ}|} = 20 \text{ Log } 1.3 = +\ 2.27 \text{ dB}$$

Exercise

Solve similarly the following problems whose Nyquist plot shape will be same as obtained in Ex. 12.7 above, only the co-ordintates of point Q are different.

5) $G(s)H(s) = \dfrac{10}{(s+5)(s^2+2s+2)}$

Solution : $\omega_{pc} = \sqrt{12}$, Q = – 0.3988, Stable , G.M. = + 8 dB

6) $G(s)H(s) = \dfrac{13}{(s+2)(s^2+2s+5)}$

Solution : $\omega_{pc} = 3$, Q = – 0.15 stable

7) $G(s)H(s) = \dfrac{30}{(s+2)(s^2+2s+5)}$

Solution : $\omega_{pc} = 3$, Q = – 1.1538, Unstable , G.M. = – 1.24 dB

Step 5 : $G(j\omega)H(j\omega) = \dfrac{(1+0.5\,j\omega)}{(-\omega^2)(1+0.1\,j\omega)(1+0.02\,j\omega)}$

Rationalizing G(jω)H(jω) and separating real and imaginary part we get,

$$G(j\omega)H(j\omega) = \frac{(1+0.5\,j\omega)[1-0.12\,j\omega-0.002\,\omega^2]}{(-\omega^2)(1+0.01\,\omega^2)(1+0.0004\,\omega^2)}$$

$$= \frac{(1+0.058\,\omega^2)}{D} + \frac{j\omega[0.38-0.001\,\omega^2]}{D}$$

Equating imaginary part to zero,

$$\omega(0.38-0.001\,\omega^2) = 0 \quad \therefore \quad \omega^2 = \frac{0.38}{0.001} = 380 \quad \therefore \omega_{pc} = 19.4935 \text{ rad/sec}$$

Substituting in real part,

$$\text{Point Q} = \frac{(1+0.058\times 380)}{(-380)(1+0.01\times 380)(1+0.0004\times 380)} = \mathbf{-0.0109}$$

Step 6 : The section I starts from origin tangential to – 270° and crossing negative real axis at point Q = – 0.0109 and is terminating at $\infty\angle-180°$ i.e. mapping of s = + j0. So rotation of plot in section I is 90° anticlockwise but it is crossing negative real axis while doing so.

∴ Nyquist plot is,

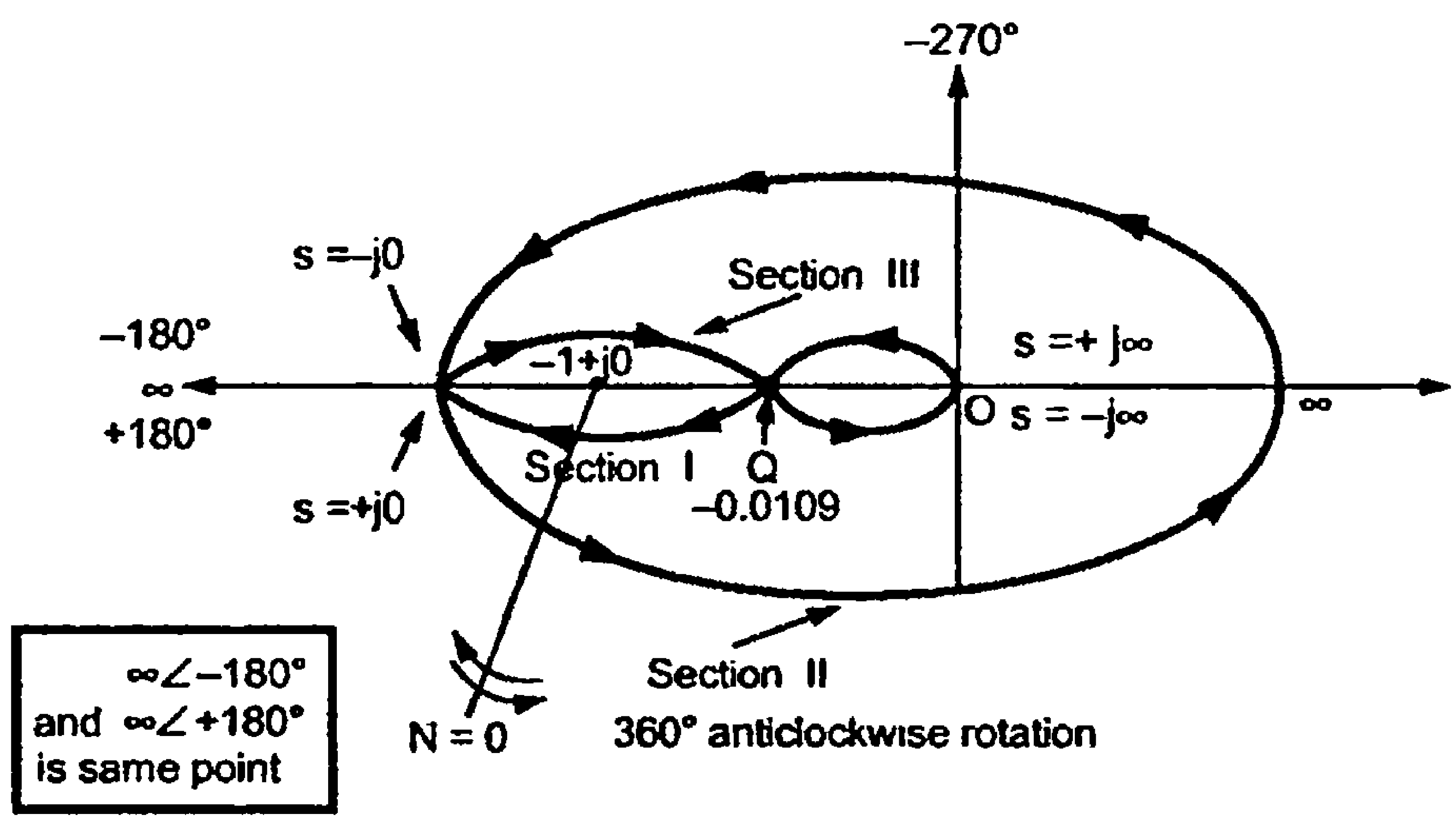


Fig. 12.30

Step 7 : Critical point –1 + j0 is getting encircled once in clockwise and once in anticlockwise.

∴ **N = 0** This satisfies the stability criterion.

∴ **System is stable.**

$$G.M. = 20 \text{ Log} \frac{1}{|OQ|} = 20 \text{ Log} \frac{1}{|0.0109|} \text{ dB} = +39.19 \text{ dB}$$

Exercise

Solve the following whose Nyquist plots same as in Ex. 12.8 but co-ordinates of point Q are different.

$$G(s)H(s) = \frac{(1+4s)}{s^2(1+s)(1+2s)}$$

Solution : $\omega_{pc} = \frac{1}{\sqrt{8}}$, Q = –10.66 N = +2, unstable.

Example 12.9 : *Sketch the Nyquist plot and comment on closed loop stability of a system whose open loop transfer function is*

$$G(s)H(s) = \frac{10}{s^2(s+2)}$$

Solution : Step 1 : P = 0

Step 2 : N = – P = 0 for stability

Step 3 : As 2 poles at origin, Nyquist path is as shown in the Fig. 12.31.

Step 4 : $G(j\omega)H(j\omega) = \frac{10}{j\omega \cdot j\omega \cdot (2+j\omega)}$

Section I : s = + j∞ to s = + j0

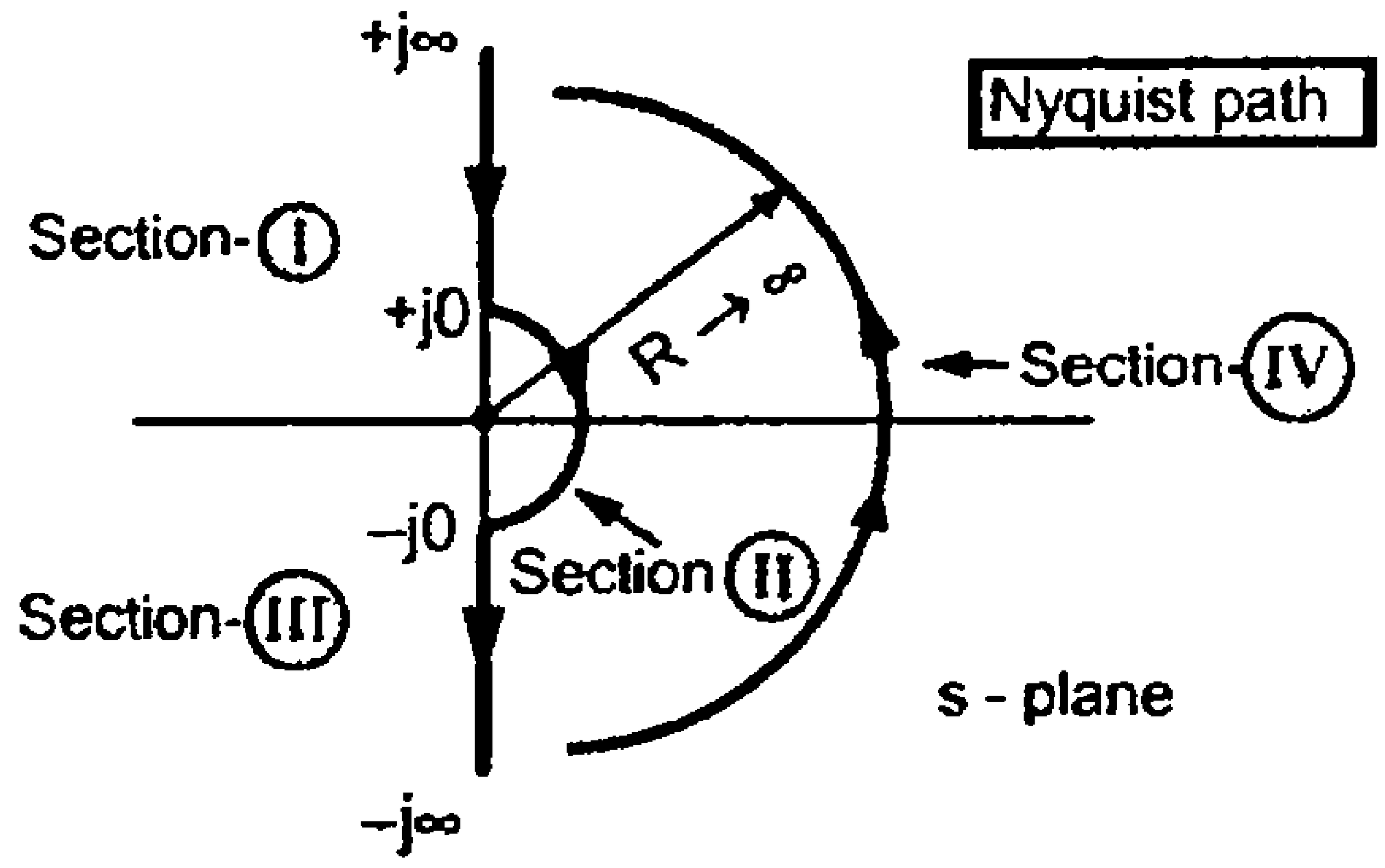


Fig. 12.31

Starting point	ω = + ∞	$0\angle\frac{0°}{90°\cdot 90°\cdot 90°} = 0\angle -270°$	– 180 ° – (–270°) = + 90° Anticlockwise rotation
Terminating point	ω → + 0	$\infty\angle\frac{0°}{90°\cdot 90°\cdot 0°} = \infty\angle -180°$	

Section II : s = +j0 to s = – j0

Starting point	ω = + 0	$\infty\angle -180°$	180 – (– 180°) = + 360° Anticlockwise rotation
Terminating point	ω → – 0	$\infty\angle +180°$	

Section III : Mirror image of section I.

Section IV: Not required.

Step 3 : As 2 poles at origin, the Nyquist path is

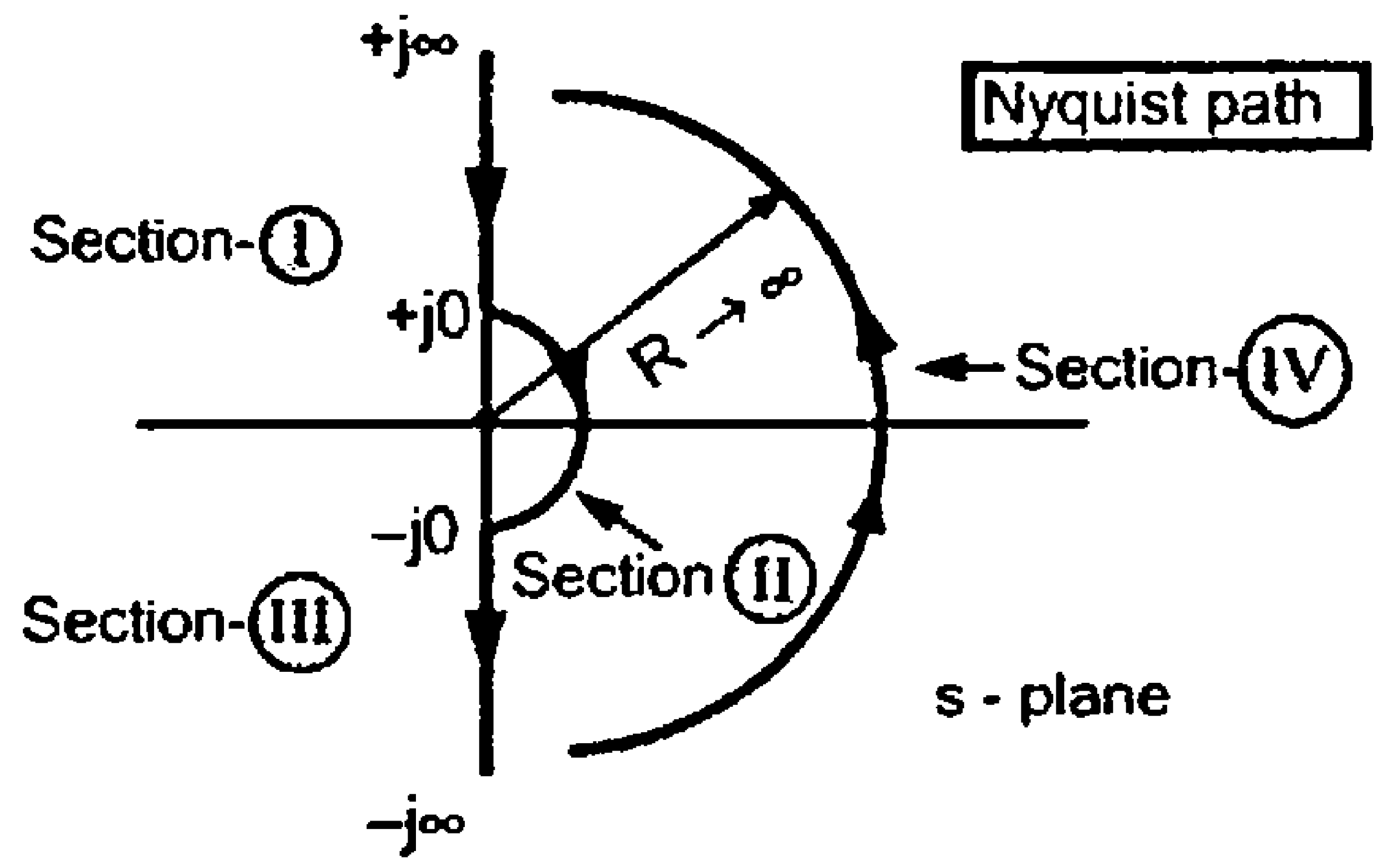


Fig. 12.29

Step 4 : $G(j\omega)H(j\omega) = \dfrac{10(1+T_1 j\omega)}{j\omega \cdot j\omega(1 + T_2 j\omega)}$

Section I : $s = +j\infty$ to $s = +j0$

Starting point	$\omega \to +\infty$	$0\angle\dfrac{0°\ 90°}{90°\ 90°\ 90°} = 0\angle -180°$	$-180° - (-180°) = +0°$ Anticlockwise rotation
Terminating point	$\omega \to +0$	$\infty\angle\dfrac{0°\ 0°}{90°\ 90°\ 0°} = \infty\angle -180°$	

Now 0° rotation is not giving exact indication of the nature of Nyquist plot for section I. So examine the variations in $G(j\omega)H(j\omega)$ for frequencies from $\omega \to \infty$ to $\omega \to +0$.

$$\angle G(j\omega)H(j\omega) = -180° + \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

As angle contribution by 2 poles at origin is – 180°.

Case i) $T_1 > T_2$

when $T_1 > T_2$, $\tan^{-1}\omega T_1 > \tan^{-1}\omega T_2$ for any value of ω

$\therefore$ Net $\angle G(j\omega)H(j\omega) = -180° + \alpha$

Where $\alpha = +\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 =$ small positive for $T_1 > T_2$.

$\therefore$ $\angle G(j\omega)H(j\omega)$ will take the values as $-175°, -176°, -178°, -179°, -179.9°$ and finally $-180°$ as ω varies from $+\infty$ to $+0$ without crossing negative real axis.

Hence for $T_1 > T_2$., Nyquist plot for section I will remain in third quadrant, finally meeting at $\infty \angle -180°$, without intersecting negative real axis.

Case ii) $T_1 < T_2$

In this case, $\tan^{-1}\omega T_2 > \tan^{-1}\omega T_1$

Case ii) $T_1 < T_2$

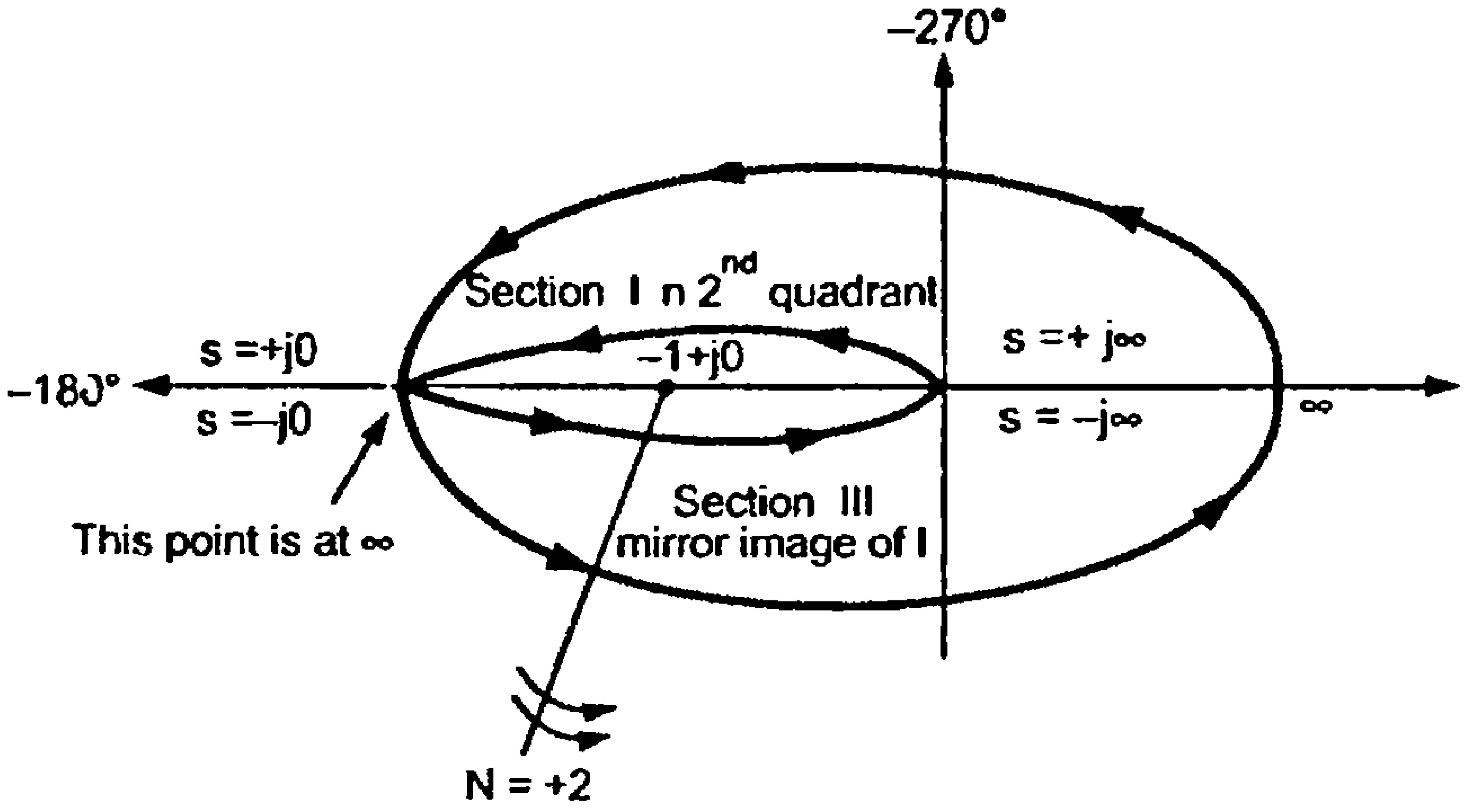


Fig. 12.35 $T_1 < T_2$

Case iii) $T_1 = T_2$

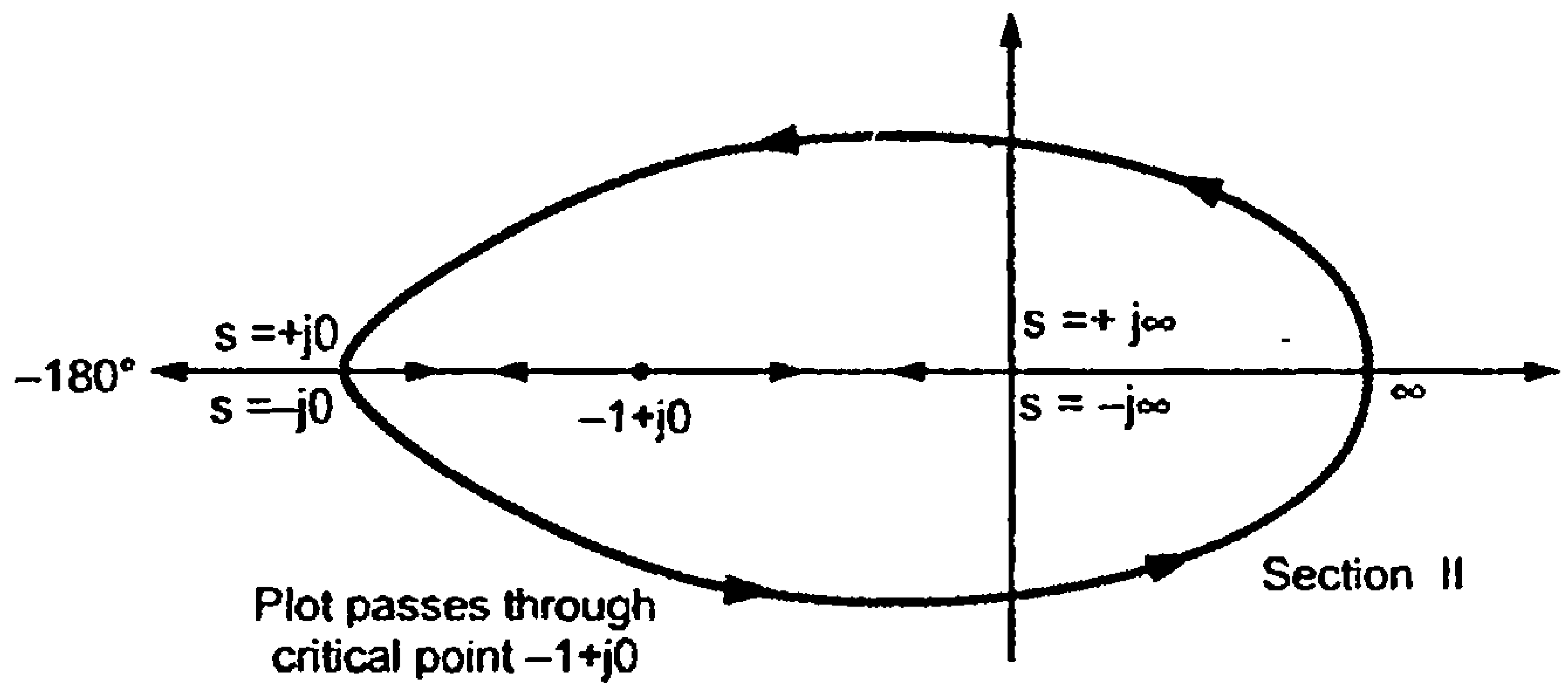


Fig. 12.36 $T_1 = T_2$

Step 7 : Case i) N = 0, satisfying the criterion ∴ System is **stable.**

Case ii) N = +2, not satisfying the criterion ∴ System is **unstable.**

Case iii) As plot actually passes through $-1+j0$, system is **marginally stable.**

12.13 Behaviour of Right Half Pole

Consider a factor in G(s)H(s) as $\frac{1}{(s-3)}$ i.e. pole in the right half of s-plane.

$$\therefore \quad G(j\omega)H(j\omega) = \frac{1}{-3+j\omega}$$

For $\omega \rightarrow \infty$, $-3+j\omega$ will approach to point P as shown in the Fig. 12.37.

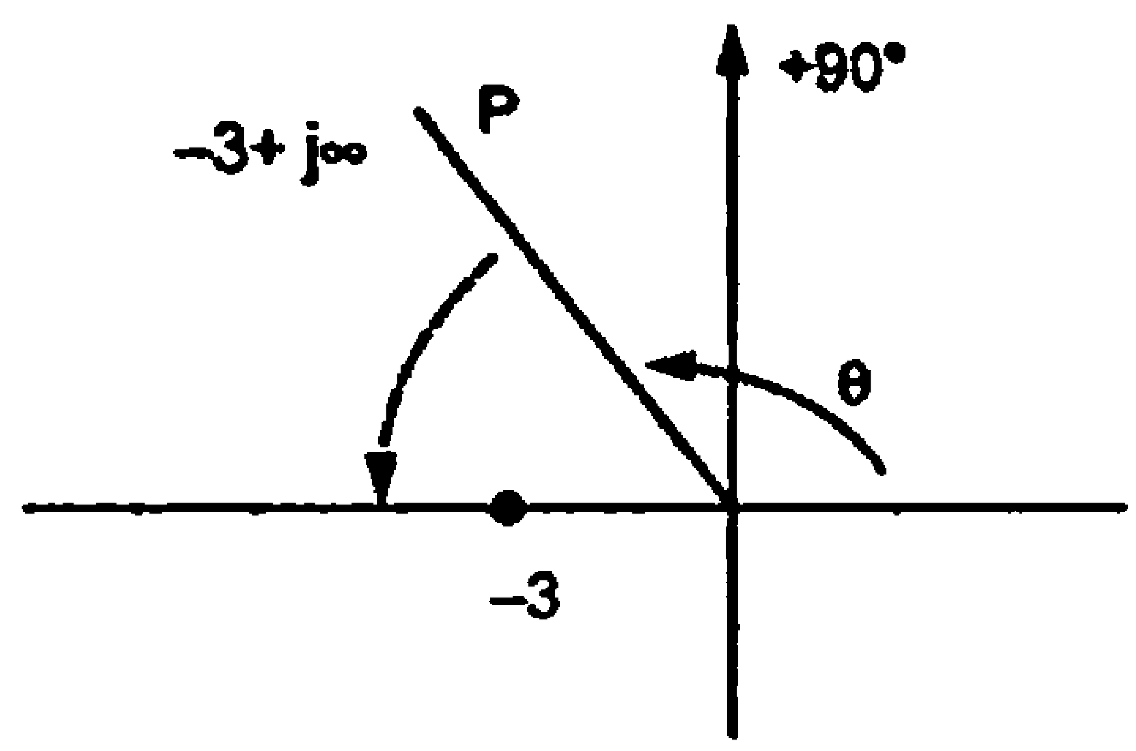


Fig.12.37

In limiting case θ ≈ 90° so this factor will contribute +90° in the denominator. i.e. as $\omega \to \infty$, $\angle G(j\omega)H(j\omega) = 0° / +90° = -90°$. When $\omega \to +0$, $-3+j\omega$ approaches to a point $-3+j0$ on the negative real axis whose angle is +180°. This should be considered as +180° as we are travelling in anticlockwise direction. When $\omega \to -0$, $-3+j\omega$ remains at the same position of $-3-j0$. i.e. on the negative real axis contributing angle + 180° again. + j0 and − j0 are negligible compared to − 3 so $-3 \pm j0$ is same point.

$\therefore$ As $\omega \to +0$, or -0 $\angle G(j\omega)H(j\omega) = \dfrac{0°}{+180°} = -180°$

Factor	$\omega \to + \infty$	$\omega \to +0$ or -0
(s − a)	+ 90°	+ 180°

If **factor is pole**, these angles will be in the denominator, effectively will become −90° and −180°. **If factor is zero**, these angle will contribute as + 90° and 180° as it is.

Example 12.11 : *Sketch the Nyquist Plot for a system with*

$$G(s)H(s) = \frac{10\,(s+3)}{s(s-1)}.$$ *Comment on the closed loop stability.*

Solution : Step 1 : P = 1 as 1 pole in right half of s-plane.

Step 2 : N = − P = − 1 i.e. Nyquist plot must encircle −1 + j0 point once in clockwise for stability.

Step 3 : One pole at origin so Nyquist path is,

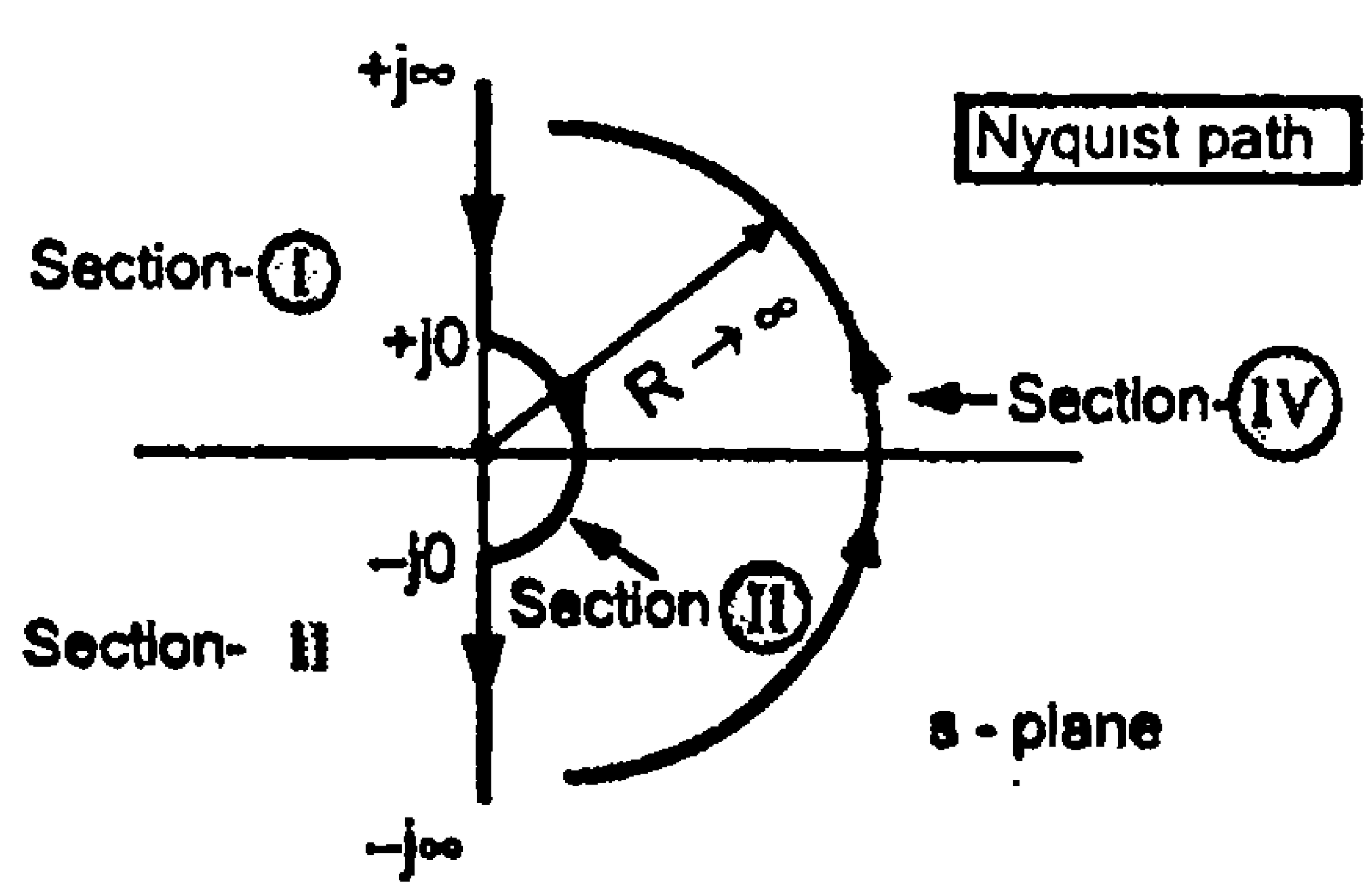


Fig. 12.29

Step 4 : $G(j\omega)H(j\omega) = \dfrac{10\,(3+j\omega)}{j\omega(+j\omega-1)}$

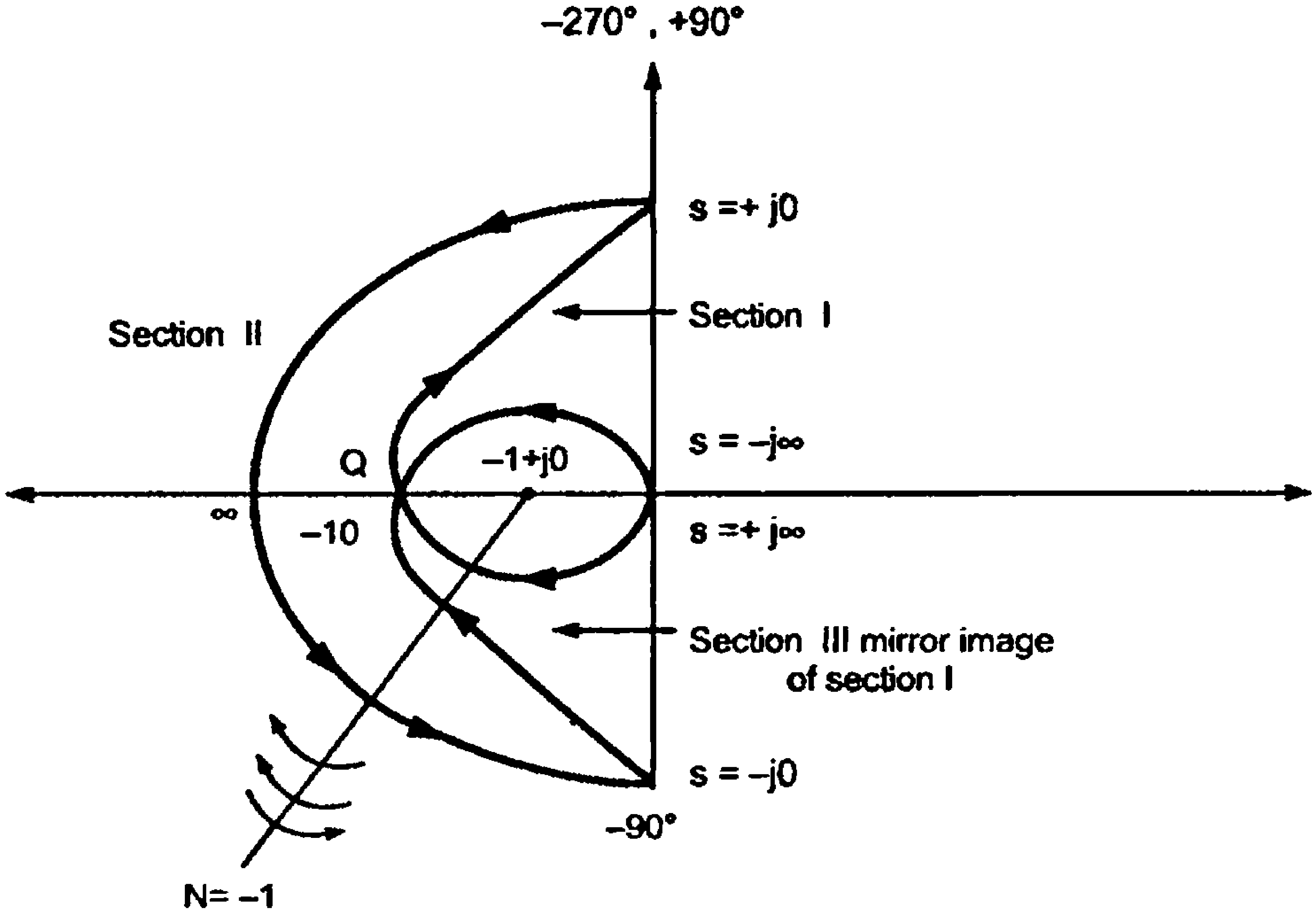


Fig 12.39

Example 12.12 : *Sketch the Nyquist plot for a system with the open loop transfer function,*

$$G(s)H(s) = \frac{K(1+0.5s)(1+s)}{(1+10s)(s-1)}$$

Determine the range of values of K for which the system is stable.

Solution : Step 1 : There is one open loop pole in the right half of s-plane so,

$$P = 1$$

Step 2 : Nyquist criterion for the stability is N = – P = – 1.

Thus for stability, Nyquist plot must encircle critical point – 1 + j0 once in clockwise direction.

Step 3 : Nyquist path is as shown in the Fig.12.40.

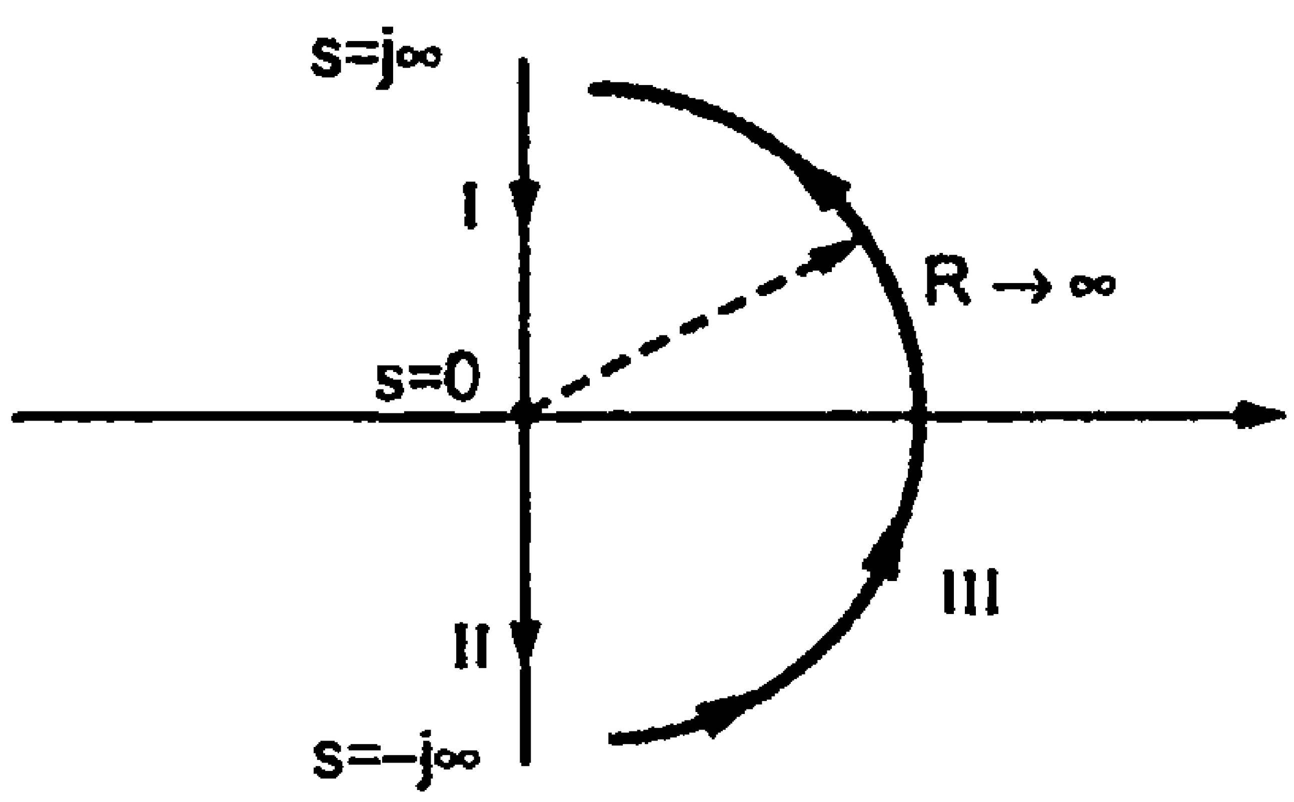


Fig. 12.40

Equating imaginary part to zero for intersection with negative real axis,

$$\omega[7.5-19.5\omega^2] = 0$$

Now ω_{pc} cannot be zero hence,

$$\omega_{pc}^2 = \frac{7.5}{19.5}$$

$$\therefore \qquad \omega_{pc} = \mathbf{0.620173\ rad/sec.}$$

At this ω_{pc}, the $G(j\omega)H(j\omega)$ value is,

$$G(j\omega)H(j\omega) = \frac{K(-9.1065)}{(54.63905)}$$

$$= \mathbf{-0.1667\ K}$$

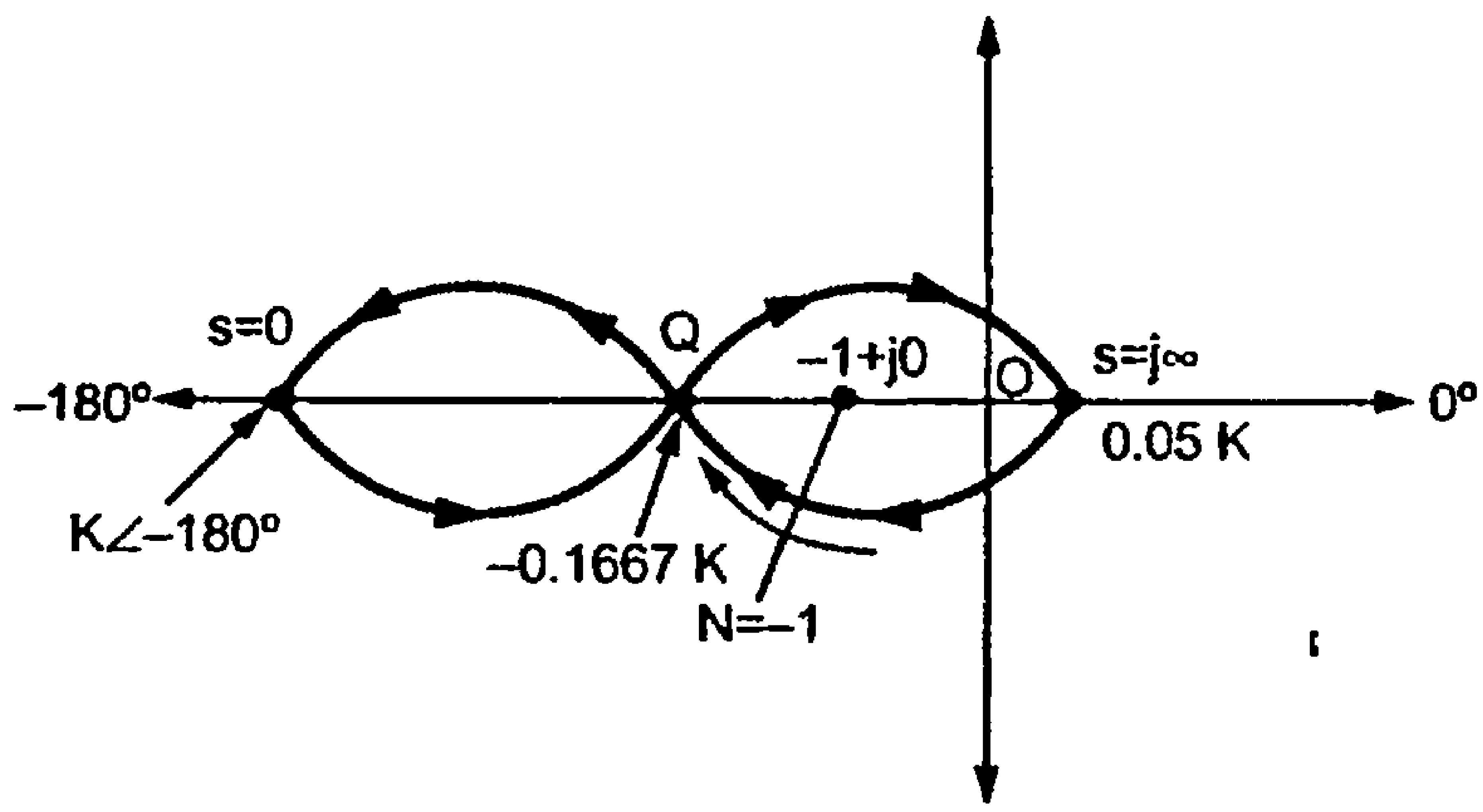


Fig. 12.41

Thus point of intersection of Nyquist plot with negative real axis is – 0.1667 K.

Step 6 : Using the intersection point Q, the Nyquist plot for the system is as shown in the Fig. 12.41.

Step 7 : For stability, N = –1

This gets satisfied only when point Q is to the left of critical point – 1 + j

$$\therefore \qquad |OQ| > 1$$

$$\therefore \qquad 0.1667\ K > 1$$

$$\therefore \qquad K > \frac{1}{0.1667} > 6$$

Thus **6 < K < ∞** is the range of values of K for which the system is stable.

12.14 Advantages of Nyquist Plot

1) It gives same information about absolute stability as provided by Routh's criterion.
2) Useful for determining the stability of the closed loop system from open loop transfer function without knowing the roots of characteristic equation.
3) It also indicates relative stability giving the values of G.M. and P.M.
4) It indicates reality, the manner in which system should be compensated to yield desired response.
5) Information regarding frequency response can be obtained.
6) Very useful for analysing conditionally stable systems.

12.15 Log Magnitude - Phase Plots

We know that in frequency response analysis, the magnitude and phase angle of open loop transfer function are plotted against the frequency. There is one more method similar to the polar plot in which magnitude is plotted against the phase angle, for various values of ω to obtain the frequency response. The difference between polar plot and this method is that in polar plot magnitude M is plotted against angle ϕ directly as per polar co-ordinate system while in this method the magnitude M is expressed in dB and plotted against phase angle ϕ in degrees on ordinary graph paper. The plots obtained by plotting M in dB against ϕ in degrees for various values of ω, on ordinary graph paper are called **magnitude-phase plots, gain-phase plots** or **Nichols plot**.

The steps to obtain magnitude-phase plot of a given system are,

1. Consider open loop transfer function $G(j\omega)H(j\omega)$ of the given system.
2. Obtain the expression for $|G(j\omega)H(j\omega)|$ in terms of ω i.e. $M(\omega)$.
3. Obtain the expression for $\angle G(j\omega)H(j\omega)$ in terms of ω i.e. $\phi(\omega)$.
4. Tabulate the values of $M(\omega)$ expressed in dB and $\phi(\omega)$ in degrees for various values of ω.
5. Select suitable scale on an ordinary graph paper with y-axis representing magnitude in dB and X-axis representing phase angle in degrees.
6. Plot all the points tabulated on the graph paper.
7. The smooth curve obtained by joining all such plotted points represents magnitude-phase plot of a given system.

The data for magnitude-phase plot can also be obtained from the Bode plot of the system. One such typical magnitude-phase plot is shown in the Fig. 12.42.

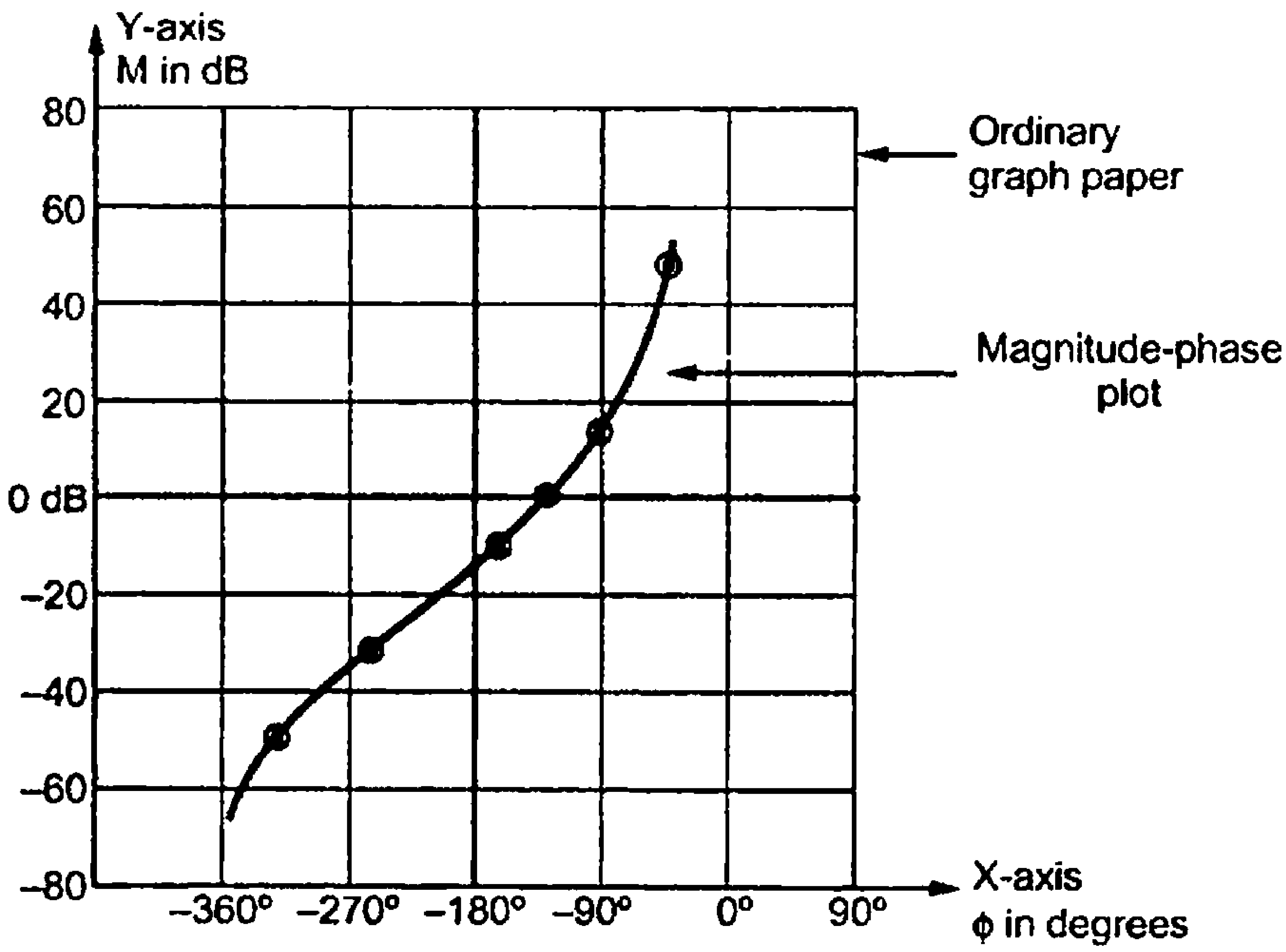


Fig. 12.42 Typical magnitude-phase plot

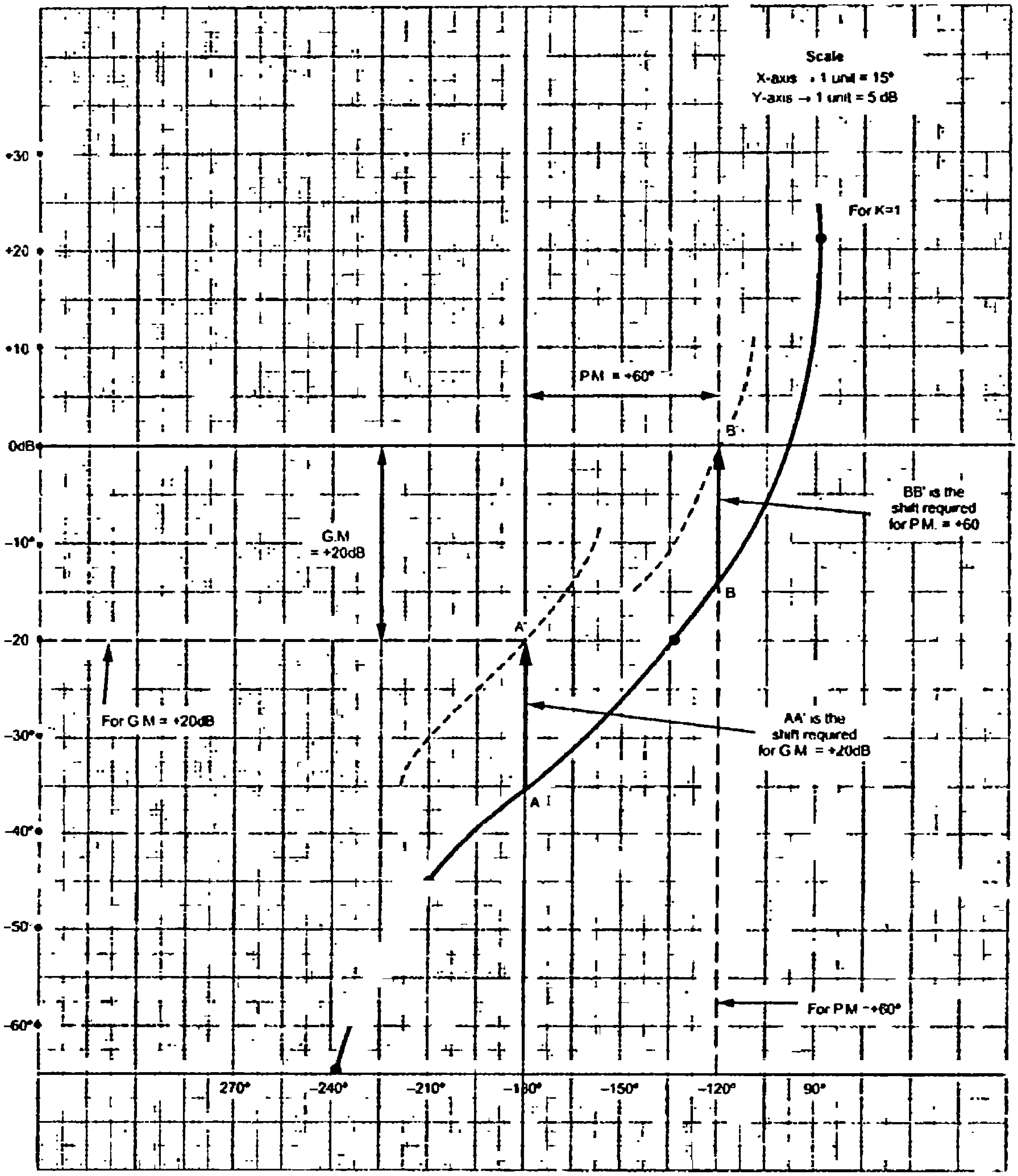


Fig. 12.44

From the plot obtained, the point A should be at A′ on –180° line to get G.M. = +20 dB. So required shift is AA′ = +16 dB as per scale. Upward shift hence positive.

$$20 \text{ Log } K = 16$$

$$\therefore \quad K = 6.31 \text{ for G.M.} = +20 \text{ dB}$$

And point B should be at B′ for P.M. = +60°. So required shift is BB′=+14 dB as per scale. Upward shift hence positive.

$$\therefore \quad 20 \text{ Log } K = 14$$

$$\therefore \quad K = 5.01 \text{ for P.M.} = +60°$$

Examples with Solutions

Example 12.14 : *Access the stability of a system with*

$$G(s)H(s) = \frac{100(1+5s)}{s^4(1+s)}.$$

Use Nyquist criterion.

Solution : Step 1 : No pole in right half of s-plane hence P = 0.

Step 2 : According to Nyquist criterion, N = – P = 0

So Nyquist plot should not encircle point – 1 + j0, for stability.

Step 3 : Four poles at origin, so Nyquist Path is

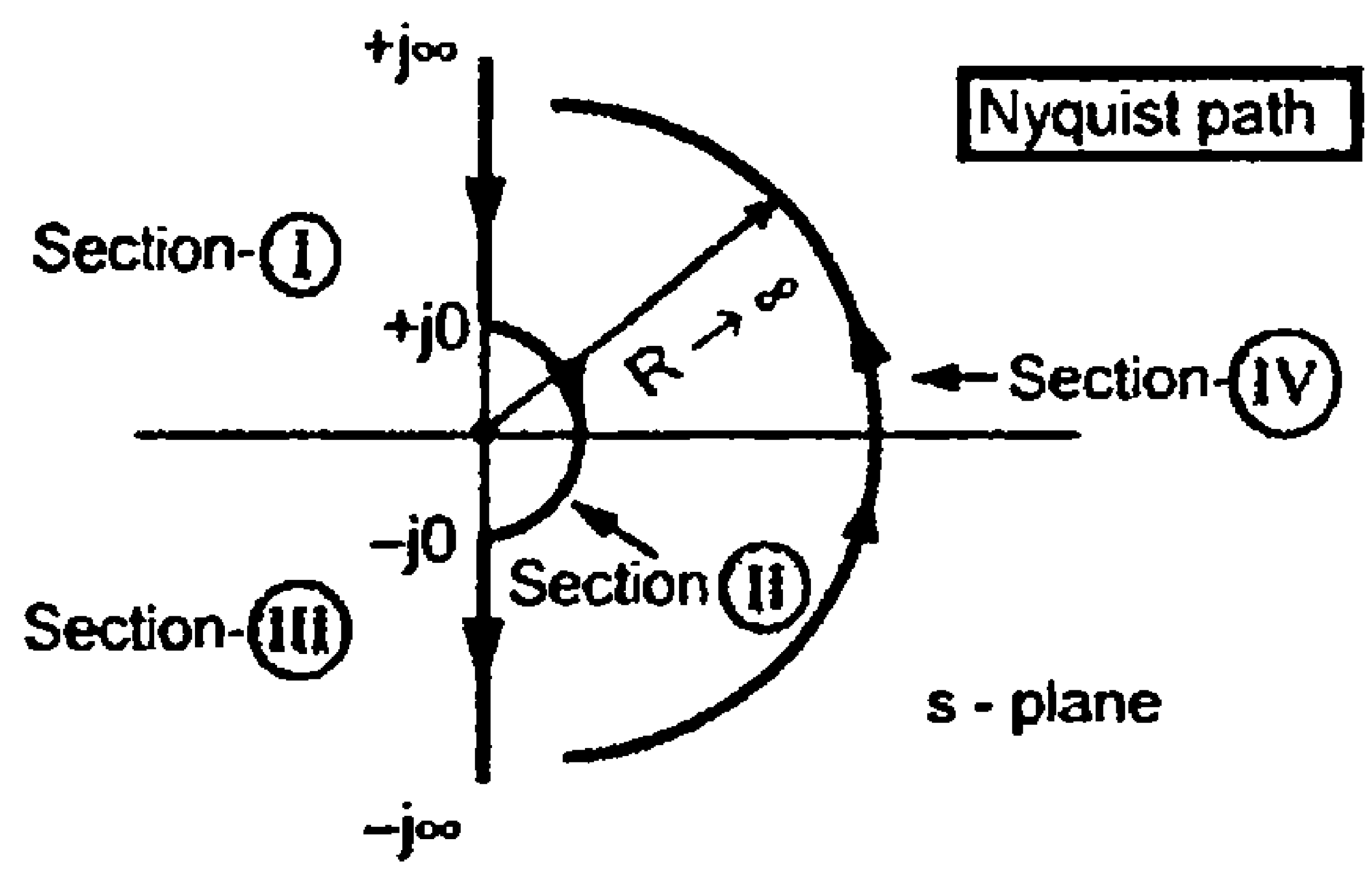


Fig. 12.45

Step 4 : Analysis of sections :

Section I : s = + j∞ to s = + j0

Starting point	$\omega \to +\infty$	$0 \angle \frac{90°}{360°\cdot 90°} = 0 \angle -360°$	$-360° - (-360°) = 0°$ No rotation of plot
Terminating point	$\omega \to +0$	$\infty \angle \frac{0°}{360°\cdot 0°} = \infty \angle -360°$	

Now angle contribution by 1 + 5s is, $\tan^{-1}\left(\frac{5\omega}{1}\right)$ while the angle contribution by

$\frac{1}{1+s}$ is, $-\tan^{-1}\left(\frac{\omega}{1}\right)$

Now $$\left|\tan^{-1}\left(\frac{5\omega}{1}\right)\right| > \left|\tan^{-1}\left(\frac{\omega}{1}\right)\right| \quad ...(1)$$

So net angle, $\phi_R = -360° + \tan^{-1}\left(\frac{5\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{1}\right)$ ... (2)

And from (1), $\phi_R > -360°$ i.e. $-359°, -356°, -350°$

Hence section I will be always in first quadrant.

Section II : $s = +j0$ to $s = -j0$

Starting point	$\omega \to +0$	$\infty \angle -360°$	$360° - (-360°) = +720°$ Anticlockwise rotation
Terminating point	$\omega \to -0$	$\infty \angle +360°$	

Section III : Mirror image of section I, i.e. in third quadrant.

Section IV : Not required.

Step 5 : No finite intersection with negative real axis.

Step 6 : Nyquist plot is shown below.

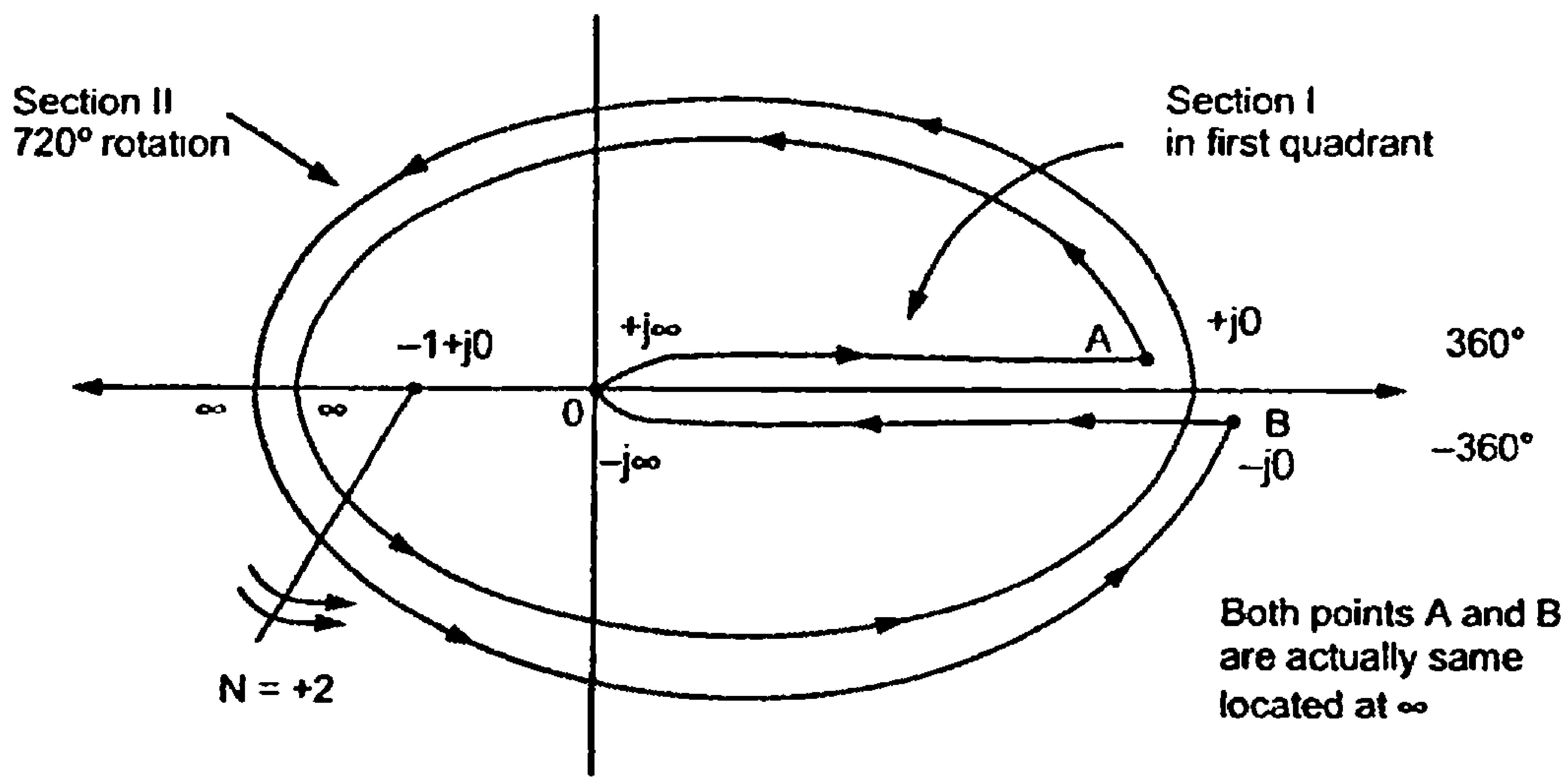


Fig. 12.46

Step 7 : $N = +2$ from the plot

But N should be zero for stability, as per step 1.

There are two roots of characteristic equation in right half of s-plane.

$$Z = 2$$

Hence the given system is **Unstable**.

➡ **Example 12.15 :** *Discuss the stability of the system using Nyquist criterion for a system with open loop transfer function.*

$$G(s)H(s) = \frac{10}{s^2(1+0.25s)(1+0.5s)}$$

Solution : Step 1 : No pole in right half of s-plane hence P = 0.

Step 2 : N = – P = 0, for stability Nyquist plot should not encircle – 1 + j 0 point.

Step 3 : As 2 poles at origin the Nyquist path is as shown in the Fig. 12.47.

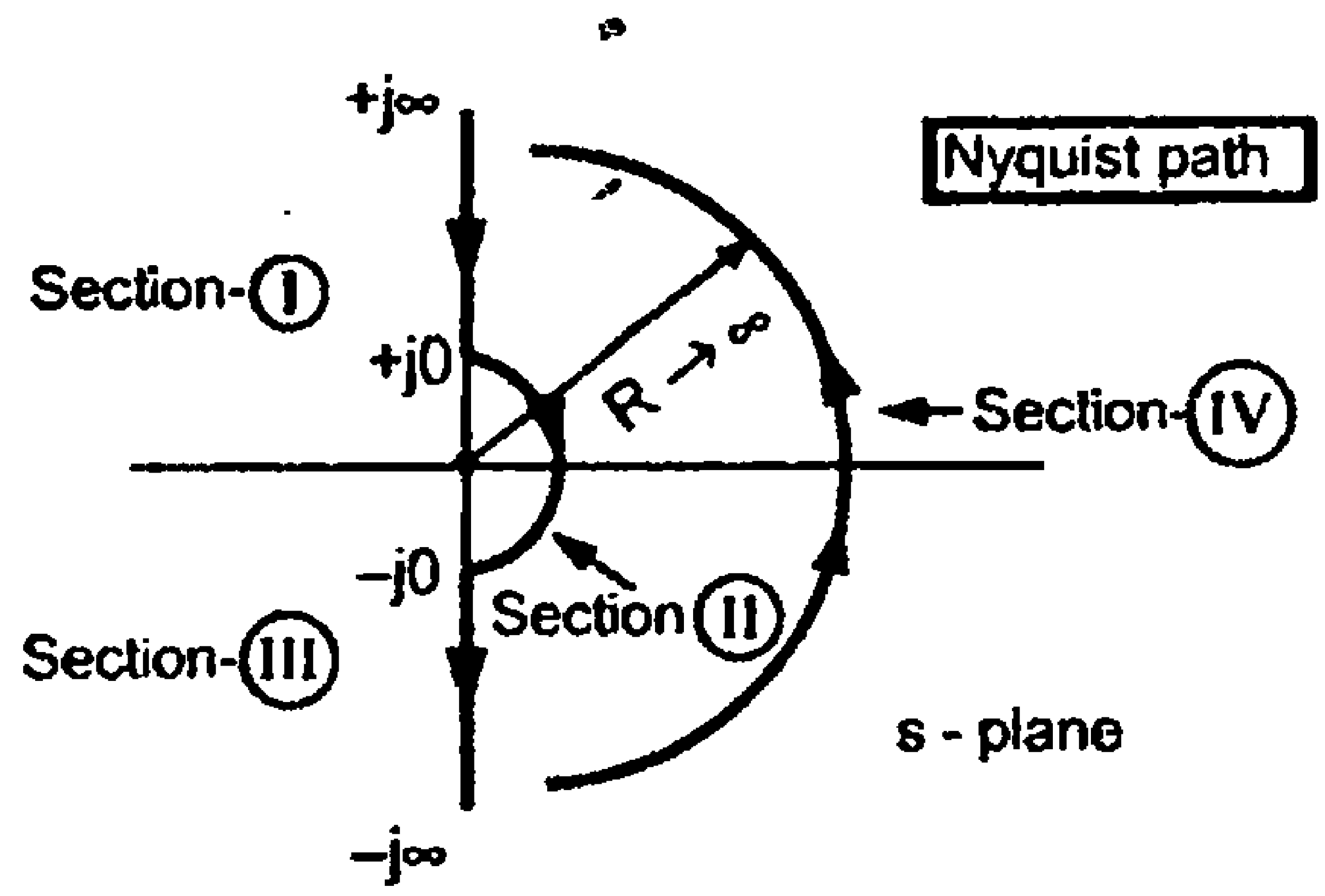


Fig. 12.47

Step 4 : Analysis of sections, replace s by j ω.

$$\therefore \quad G(j\omega)H(j\omega) = \frac{10}{(j\omega)^2(1+0.25\,j\omega)(1+0.5\,j\omega)}$$

Section I : s = + j ∞ to s = + j 0

Starting point	$\omega \to +\infty$	$0\angle\frac{0^\circ}{180^\circ \cdot 90^\circ \cdot 90^\circ} = 0 \angle -360^\circ$	– 180° – (–360°) = + 180° Anticlockwise rotation
Terminating point	$\omega \to +0$	$\infty\angle\frac{0^\circ}{180^\circ \cdot 0^\circ \cdot 0^\circ} = \infty\angle -180^\circ$	

Section II : s = + j 0 to s = – j 0

Starting point	$\omega \to +0$	$\infty\angle -180^\circ$	180° – (–180°) = + 360° Anticlockwise rotation
Terminating point	$\omega \to -0$	$\infty\angle\frac{0^\circ}{-180^\circ \cdot 0^\circ \cdot 0^\circ} = \infty\angle +180^\circ$	

Section III : Mirror image of section I

Section IV : Not required.

Step 5 : Intersection with negative real axis

Rationalizing G(jω)H(jω),

$$G(j\omega)H(j\omega) = \frac{10(1-0.25j\omega)(1-0.5j\omega)}{-\omega^2(1+0.0625\omega^2)(1+0.25\omega^2)} = \frac{10[1-0.75\,j\omega-0.125\,\omega^2]}{-\omega^2(1+0.0625\,\omega^2)(1+0.25\,\omega^2)}$$

Example 12.18 : *The open loop transfer function is* $\frac{(s+2)(s+8)}{s^3}$. *Is the closed loop system stable ? If not, deduce the number of unstable poles. Use the Nyquist criterion to arrive at your answers.*

Solution :

$$G(j\omega)H(j\omega) = \frac{(2+j\omega)(8+j\omega)}{(j\omega)^3}$$

Step 1 : No pole in r.h.s. of s-plane, P = 0

Step 2 : For stability, N = – P = 0

Step 3 : Nyquist path is,

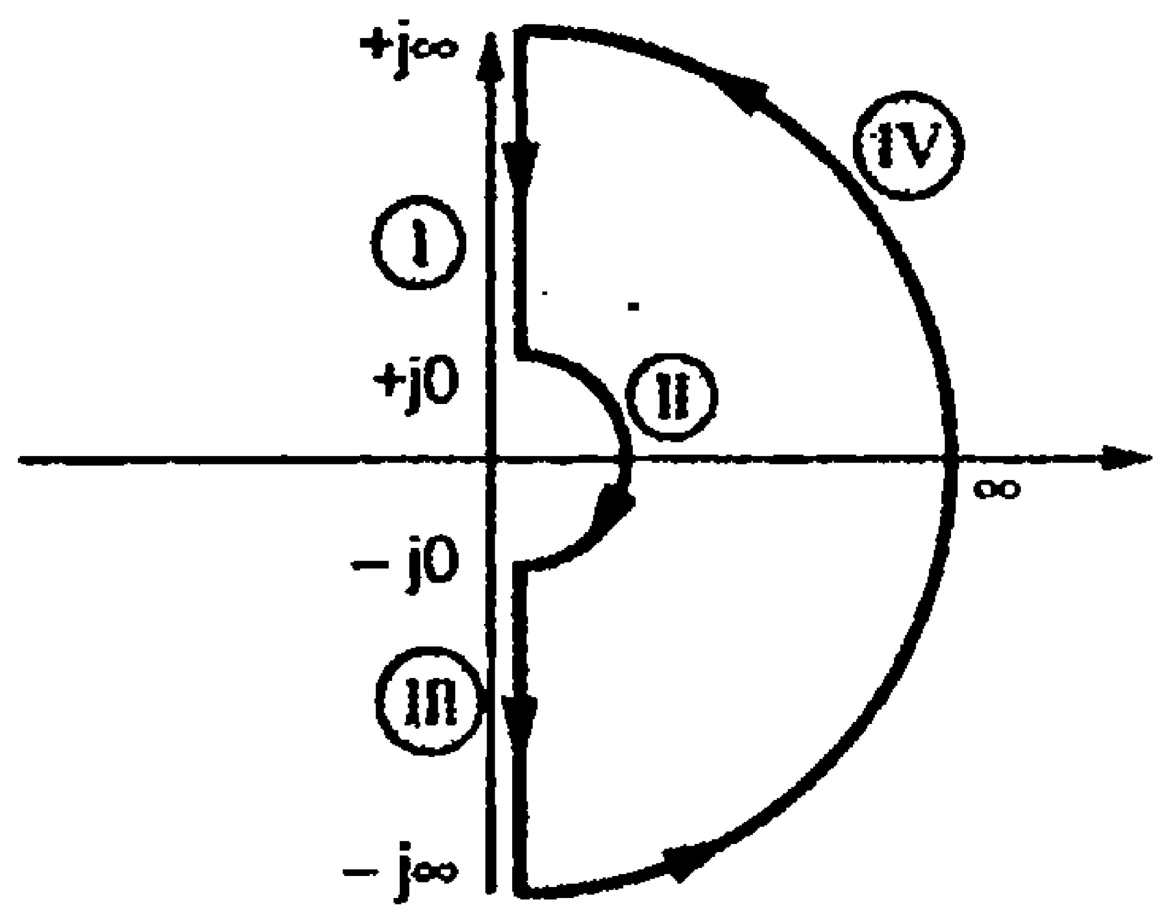


Fig. 12.52(a)

Step 4 : Section I :

s = + j∞	$0\angle\frac{90^\circ 90^\circ}{270^\circ} = 0\angle-90^\circ$	$-270^\circ - (-90^\circ) = -180^\circ$ Clockwise
s = + j0	$\infty\angle\frac{0^\circ 0^\circ}{270^\circ} = \infty\angle-270^\circ$	

Section - II :

s = + j0 i.e. ω = + 0	$\infty\angle-270^\circ$	$270^\circ - (-270^\circ) = 540^\circ$ Anticlockwise
s = – j0 i.e. ω = – 0	$\infty\angle+270^\circ$	

Section - III is mirror image of section - I, about real axis.

Section - IV not required.

Step 5 : Calculate point Q which is intersection of Nyquist plot with negative real axis.

$$G(j\omega)H(j\omega) = \frac{(2+j\omega)(8+j\omega)}{(-j\omega)^3} \quad \ldots \frac{1}{j} = -j$$

$$= \frac{j[16+10j\omega-\omega^2]}{\omega^3} = \frac{-10}{\omega^2} + \frac{j}{\omega^3}(16-\omega^2)$$

$\therefore \quad 16-\omega^2 = 0$ ... for making imaginary part zero.

$\therefore \quad \omega_{pc} = 4$

$\therefore \quad \text{Point Q} = \frac{-10}{16} = -0.625$

Step 6 : The Nyquist plot is shown in the Fig. 12.52(b).

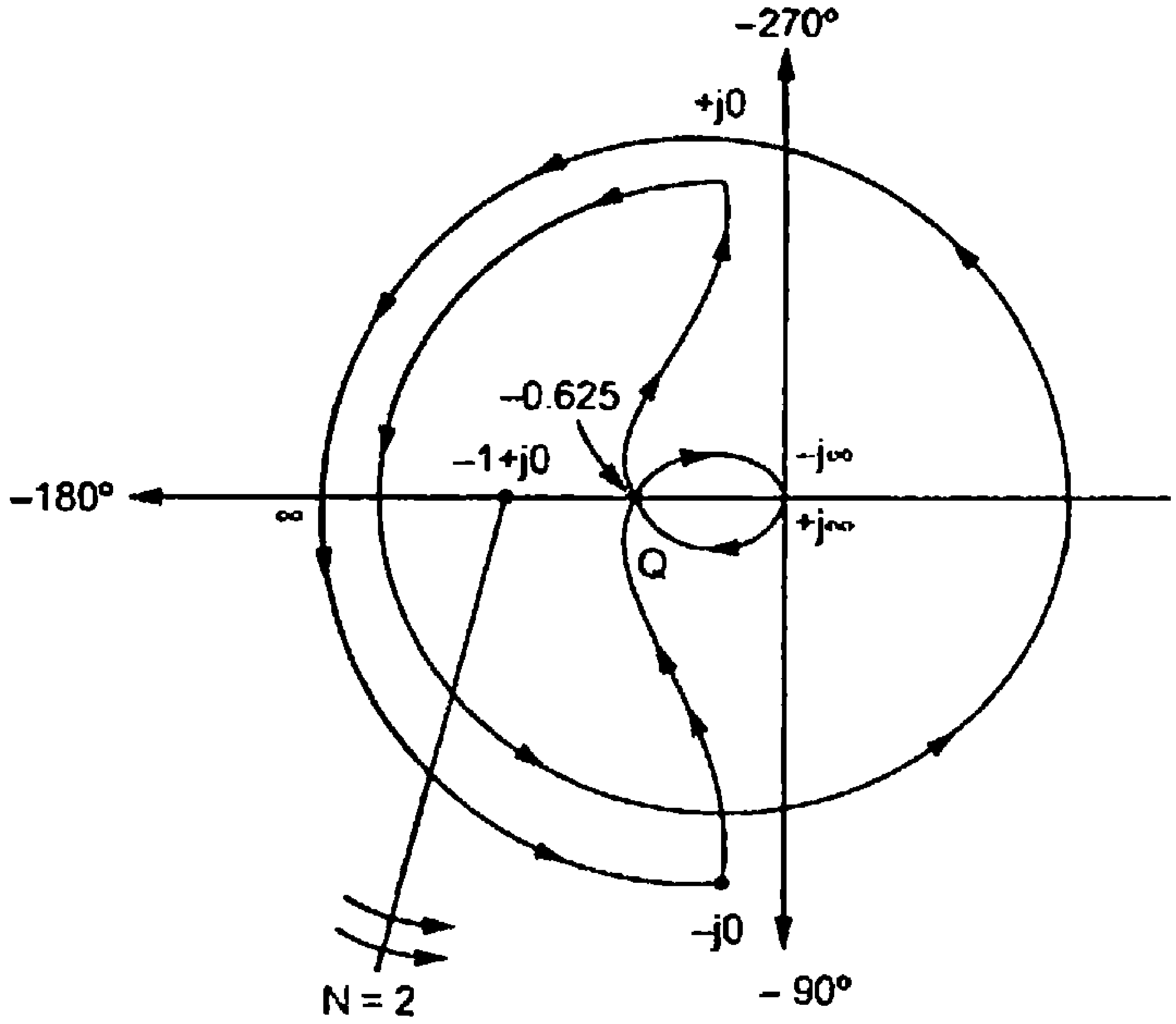


Fig. 12.52(b)

Step 7 : N = 2 from the Nyquist plot and must be zero for stability.

Thus there are 2 closed loop poles located in right half of s-plane, making system **unstable.**

Example 12.19 : *For a unity feedback system* $G(s) = \frac{K(1+s)^2}{s^3}$, *determine the range of K for the system to be stable using Nyquist criterion.* **(M.U. : Dec.-1998)**

Solution :

Step 1 : No pole in R.H.S. of s-plane hence P = 0.

Step 2 : According to Nyquist criteria, N = – P = 0. So Nyquist plot should not encircle critical point.

Step 3 : Nyquist path is as shown, divided into four sections.

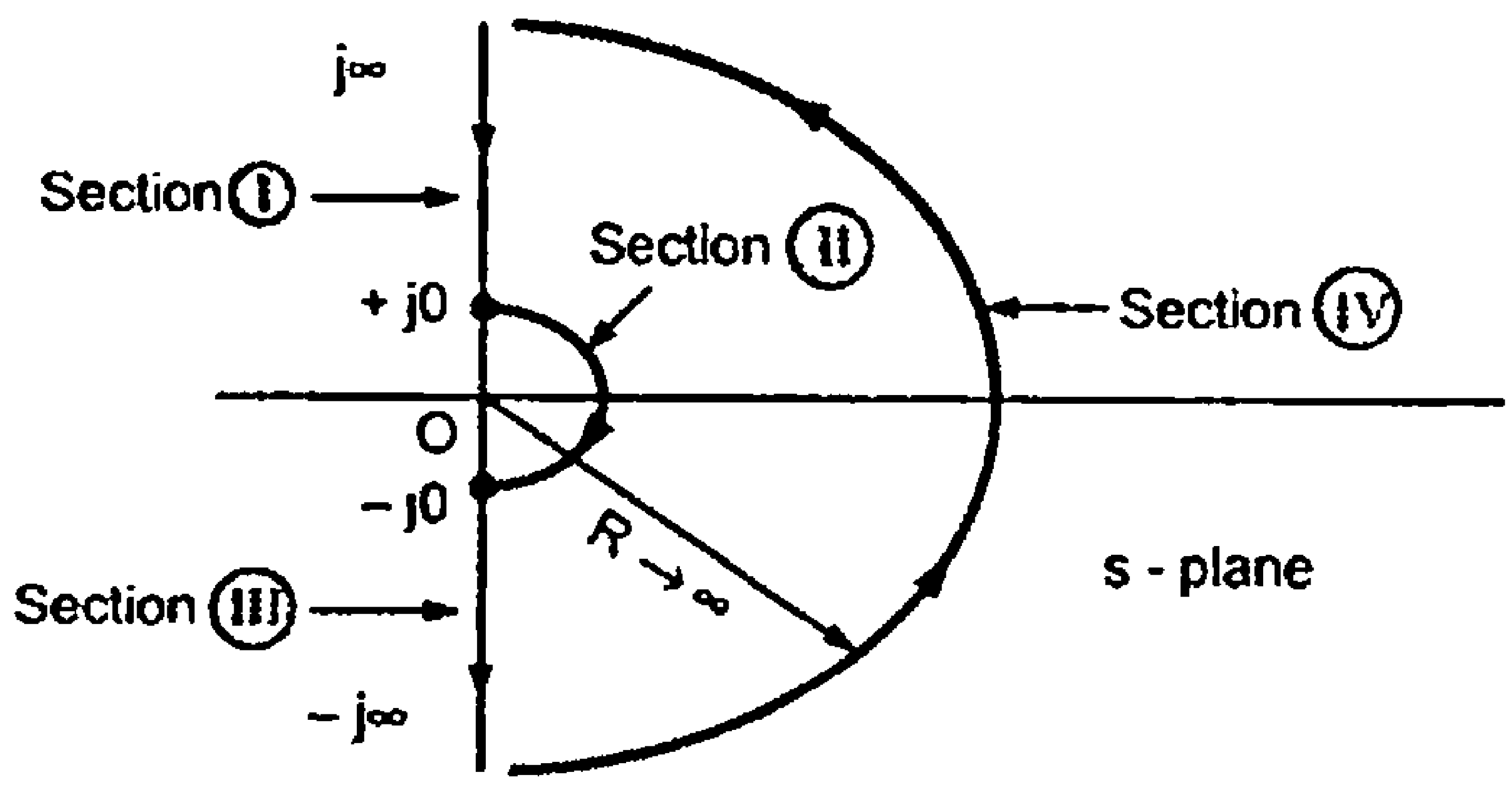


Fig. 12.53

Step 4 : Analysis of sections.

Section I : $s = +j\infty$ to $s = j0$

Starting point $\omega \to +\infty$	$0 \angle \frac{90°90°}{90°90°90°} = 0 \angle -90°$	$-270° - (-90°) = -180°$ Clockwise rotation
Terminating point $\omega \to +0$	$\infty \angle \frac{0°}{270°} = \infty \angle -270°$	

Section II : $s = +j0$ to $s = -j0$

Starting point $\omega \to +0$	$\infty \angle -270°$	$+270° - (-270°) = +540°$ Anticlockwise rotation
Terminating point $\omega \to -0$	$\infty \angle +270°$	

Section III : Mirror image of section-I about real axis.

Section IV : Not required.

Step 5 : Intersection with real axis.

$$G(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3} = \frac{K(1+j\omega)^2}{-j\omega^3} = \frac{jK(1+j\omega)^2}{\omega^3} \qquad \ldots \frac{1}{j} = -j$$

$$= \frac{jK}{\omega^3}[1+2j\omega-\omega^2] = \frac{-2K}{\omega^2}+\frac{jK}{\omega^3}(1-\omega^2)$$

For imaginary part to be zero,

$1-\omega^2 = 0$ i.e. $\omega^2 = 1$ i.e. $\omega_{pc} = 1$

Using in real part,

$$Q = \frac{-2K}{1} = -2K$$

Step 6 : Nyquist plot is as shown,

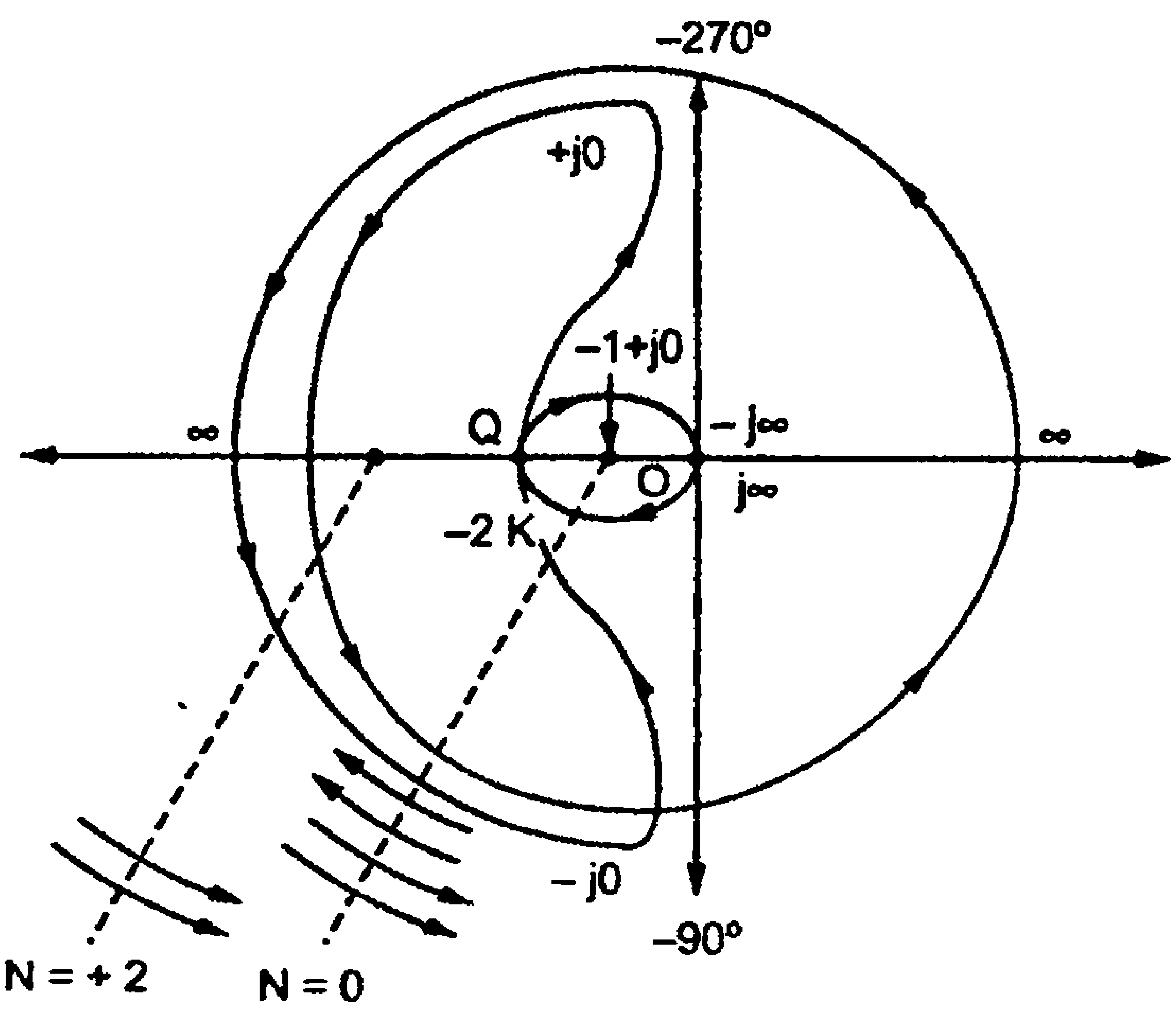


Fig. 12.54

Step 7 : Critical point must be between OQ to have N = 0.

$\therefore \quad |OQ| > 1$

$\therefore \quad |-2K| > 1$

$$\therefore \quad \boxed{K > \frac{1}{2}}$$

This is the range of K for stability of the system.

Example 12.20 : *For the polar plot shown in the Fig. 12.55,*

i) determine the gain margins in dB and the phase margins if OA = – 0.5, OB = – 1, OC = – 2, OD = – 2.5, OE = – 0.866 + j 0.5 and OF = – 0.643 – j 0.766.

ii) complete the Nyquist plot and determine whether the system is stable, if all poles are in LH of s plane.

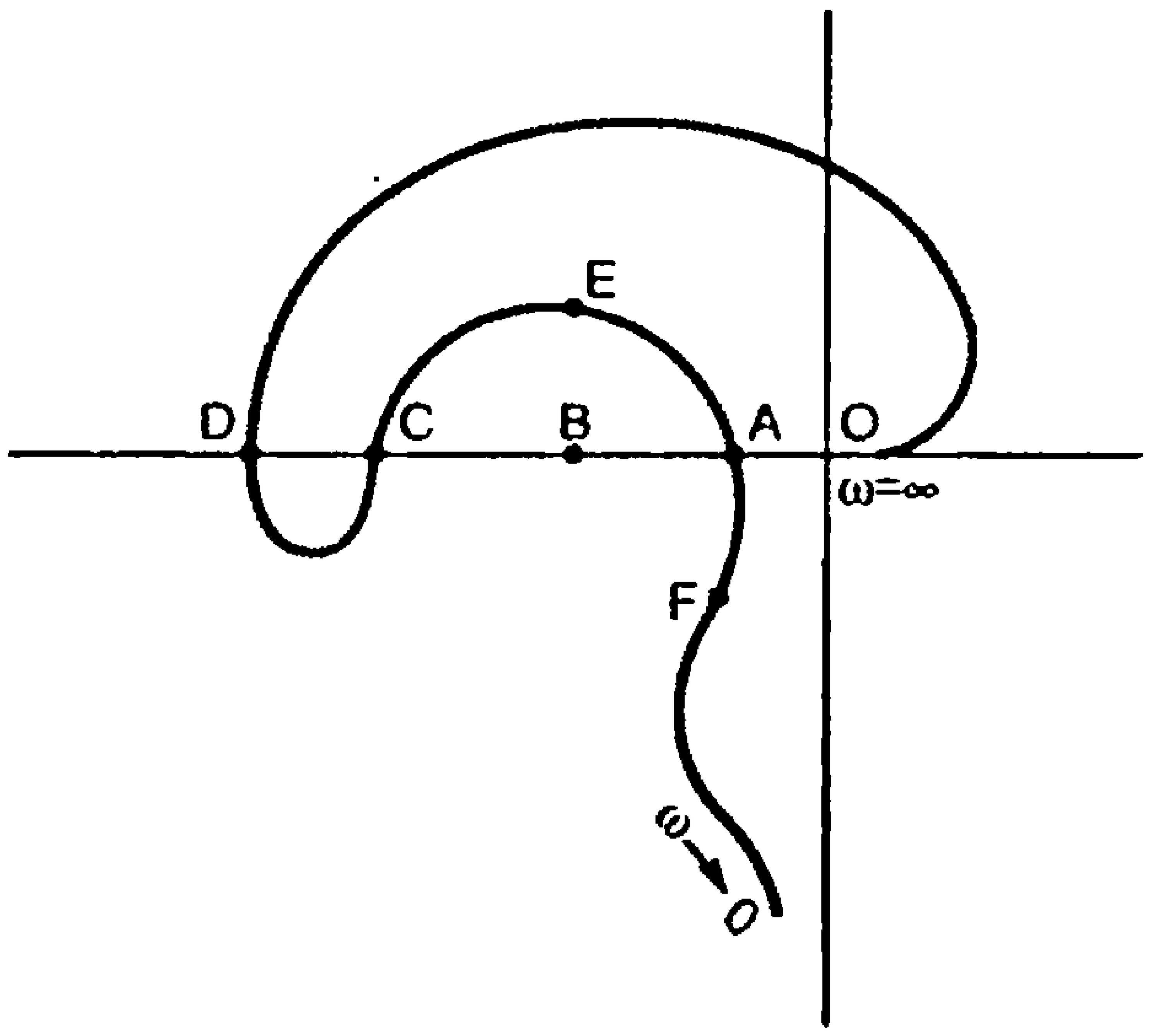


Fig. 12.55

Solution : i) The gain margin is,

$$\text{G.M.} = 20 \log \frac{1}{|OQ|} = -20 \log |OQ| \, dB$$

Where $\quad Q$ = Intersection of Nyquist plot with negative real axis.

So for given plot gain margins are,

$$GM_1 = -20 \log |OA| = -20 \log 0.5 = +\mathbf{6.02\ dB}$$

$$GM_2 = -20 \log |OC| = -20 \log 2 = -\mathbf{6.02\ dB}$$

$$GM_3 = -20 \log |OD| = -20 \log 2.5 = -\mathbf{7.958\ dB}$$

As no pole in RHS of s-plane, P = 0.

and $\quad N = -P = 0$ hence **system is stable.**

Example 12.21 : *Given* $G(s)H(s) = \frac{12}{s(s+1)(s+2)}$. *Draw the polar plot and hence determine if system is stable and its gain margin and phase margin.*

Solution : In the frequency domain,

$$G(j\omega)H(j\omega) = \frac{12}{j\omega(1+j\omega)(2+j\omega)}$$

$$M = \frac{12}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}, \quad \phi = -90^\circ - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{\omega}{2}$$

For polar plot ω varies from 0 to ∞.

Start $\omega = 0$	$M = \infty$	$\phi = -90^\circ$	$-270^\circ - (-90^\circ) = -180^\circ$ Clockwise
End $\omega = \infty$	$M = 0$	$\phi = -270^\circ$	

For intersections with negative real axis, rationalize G(jω)H(jω),

$$G(j\omega)H(j\omega) = \frac{12(-j\omega)(1-j\omega)(2-j\omega)}{(j\omega)(-j\omega)(1+j\omega)(1-j\omega)(2+j\omega)(2-j\omega)}$$

$$= \frac{-12j\omega[2-3j\omega-\omega^2]}{\omega^2(1+\omega^2)(4+\omega^2)} = \frac{-36}{(1+\omega^2)(4+\omega^2)} - \frac{12j\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

Equation imaginary part to zero, $2-\omega^2 = 0$

$\therefore \quad \omega_{pc} = \sqrt{2} = 1.414$ rad/sec

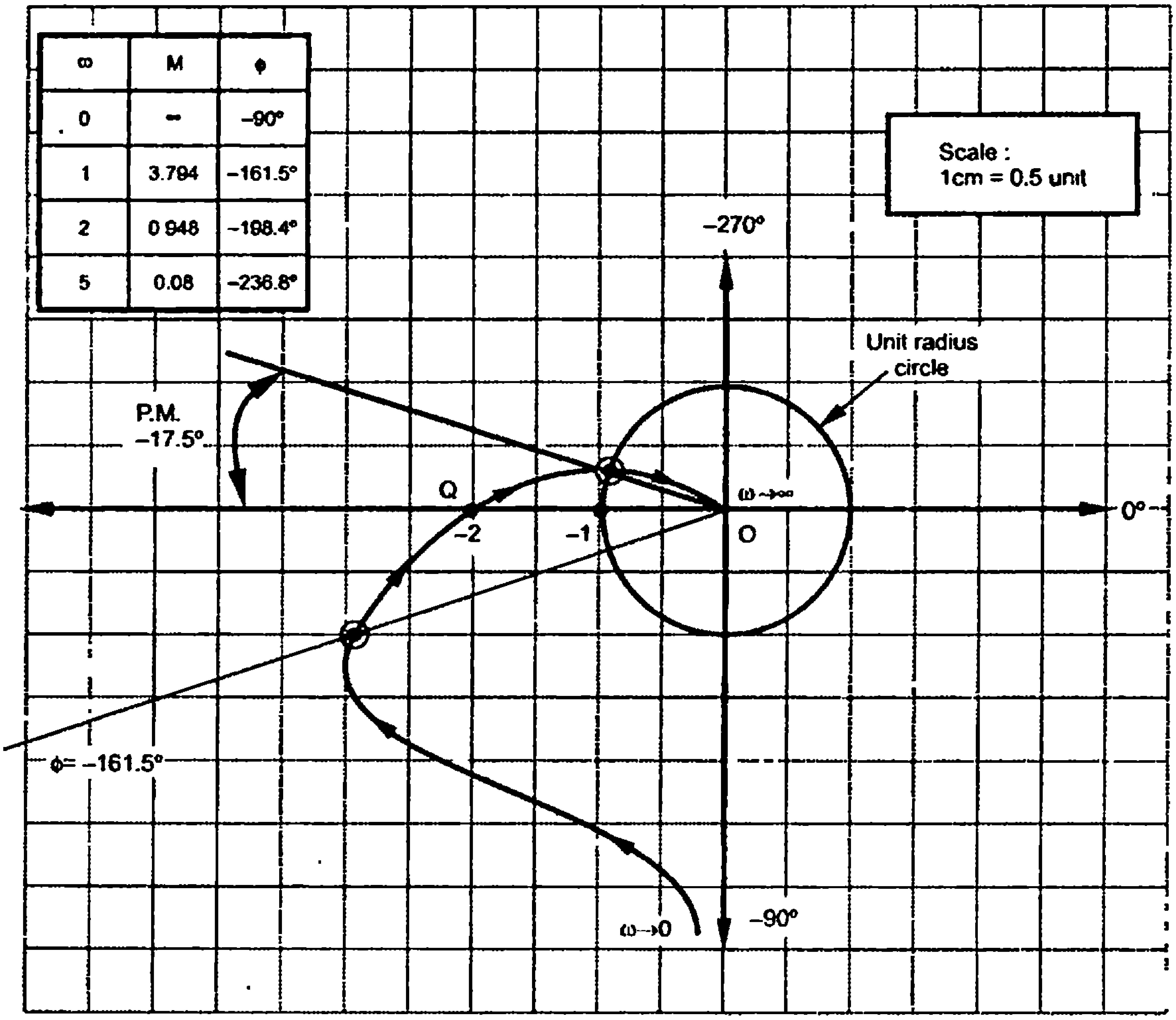
Using in real part, $\quad Q = \frac{-36}{(1+2)(4+2)} = -2$

As phase margin is required, it is necessary to draw polar plot to the scale, for which it is necessary to obtain few points.

$$G.M. = 20 \text{ Log} \frac{1}{|OQ|} = 20 \text{ Log} \frac{1}{2} = -6.02 \text{ dB}$$

$$P.M. = -17.5^\circ$$

As both G.M. And P.M. are negative, system is **unstable.**



ω	M	ϕ
0	∞	-90°
1	3.794	-161.5°
2	0.948	-198.4°
5	0.08	-236.8°

Fig. 12.58

Example 12.22 : *Use Nyquist analysis to determine the stability range of K if*

$$G(s)H(s) = \frac{K(s+5)}{s(s-2)}$$

(M.U. : May-1998)

Solution : Step 1 : P =1 as 1 open loop pole is in the right half of s-plane.

Step 2 : N = – P = –1 is the criterion for stability i.e. Nyquist plot must encircle – 1 + j 0 point once in clockwise direction, for stability.

Step 3 : One pole at origin, hence Nyquist path is as shown in the Fig. 12.59

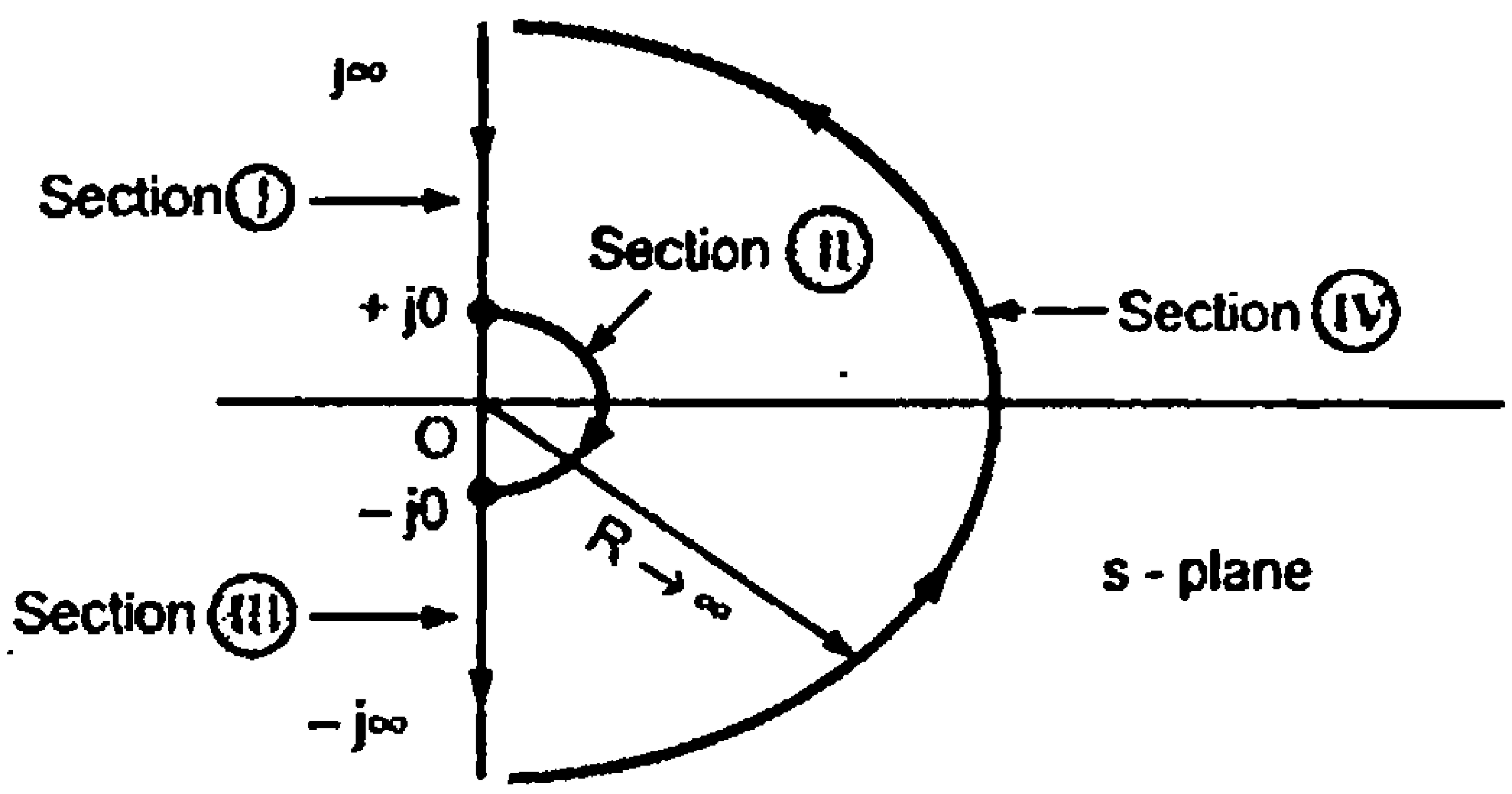


Fig. 12.59

$$G(j\omega)H(j\omega) = \frac{(1+j\omega)(j\omega+4)}{-\omega^2(j\omega+4)(j\omega-4)} = \frac{j\omega-\omega^2+4+4j\omega}{(-\omega^2)(-\omega^2-16)}$$

$$= \frac{(4-\omega^2)+5j\omega}{\omega^2(\omega^2+16)}$$

The imaginary part is zero only when $\omega = 0$. But $\omega = 0$ is not existing on path. Hence **there is no finite intersection of Nyquist plot with imaginary axis.**

Step 5 : The plot is,

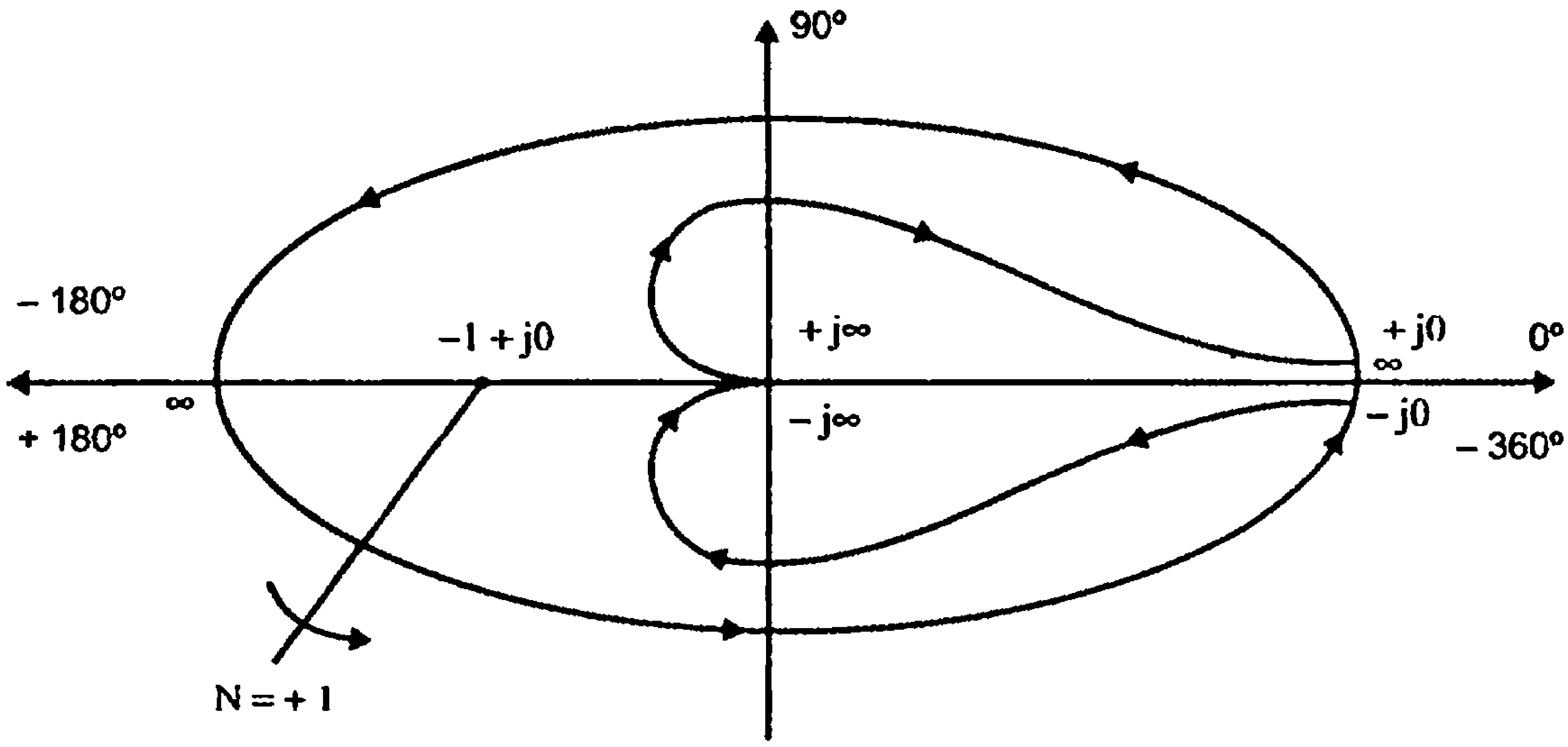


Fig. 12.61(b)

Step 6 : As – 1 + j0 is encircled once in anticlockwise direction,

$$N = +1$$

But N must be –1 for stability.

Now $\quad N = 1$ and $P = 1$

$$N = Z - P$$

$\therefore \quad 1 = Z - 1$

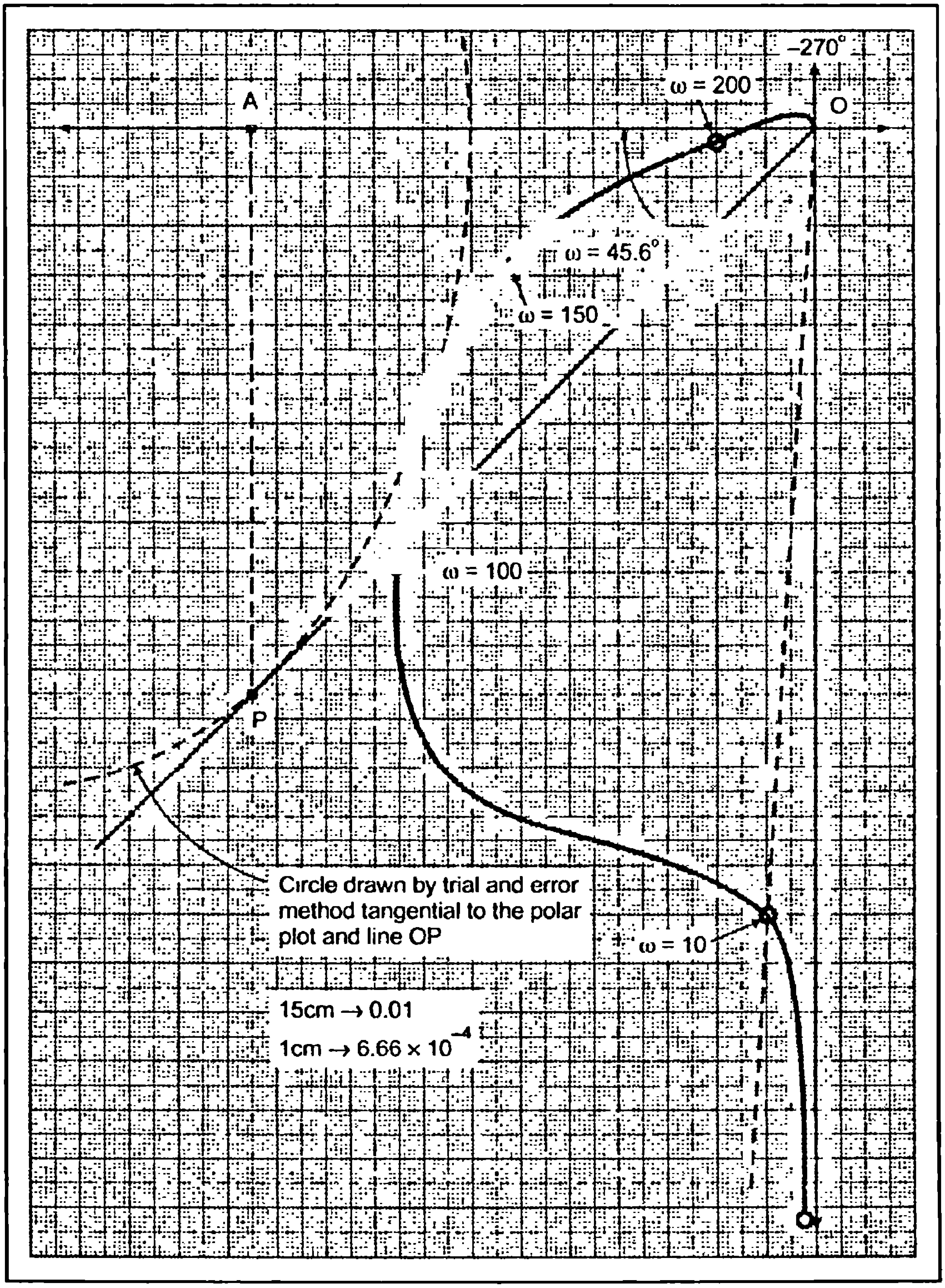
$\therefore \quad Z = 2$

There are two roots in right half of s-plane. **The closed loop system is unstable in nature.**

Example 12.24 : *Find a value of* K_B *for which the system whose open loop transfer function is* **(M.U. : Dec.-1997)**

$$G(s)H(s) = \frac{K_B}{s\left(1+\frac{s}{200}\right)\left(1+\frac{s}{250}\right)}$$

has a resonant peak M_p *of 1.4 dB.*



-270°
ω = 200
O
A
ω = 45.6°
ω = 150
ω = 100
P
Circle drawn by trial and error method tangential to the polar plot and line OP
ω = 10
15cm → 0.01
1cm → 6.66 × 10^-4

Fig. 12.62 (a)

$$\therefore \quad \sin \omega T = -\omega T \cos \omega T$$

$$\therefore \quad \tan \omega T = -\omega T$$

$$\therefore \quad \omega T = \tan^{-1}(-\omega T)$$

Now $$\tan^{-1}(-\omega T) = (-\omega T) - \frac{(-\omega T)^3}{3}$$

Considering first two terms we get

$$\omega T = -(-\omega T) - \frac{(-\omega T)^3}{3}$$

$$\therefore \quad 2\omega T = \frac{(\omega T)^3}{3}$$

$$\therefore \quad (\omega T)^2 = 6$$

$$\therefore \quad \omega^2 = \frac{6}{T^2}$$

$$\therefore \quad \omega_{pc} = \frac{\sqrt{6}}{T}$$

$$\therefore \quad \text{Intersection point} = \frac{(\cos\sqrt{6} - \sqrt{6}\sin\sqrt{6})}{(1+6)}$$

$$= -0.333.$$

For higher values of ω, the plot spirals round the origin.

Hence polar plot begins at 1 ∠0° and then revolves repeately about the origin between 0° and − 360° as ω increases. So polar plot is as shown in the Fig. 12.63.

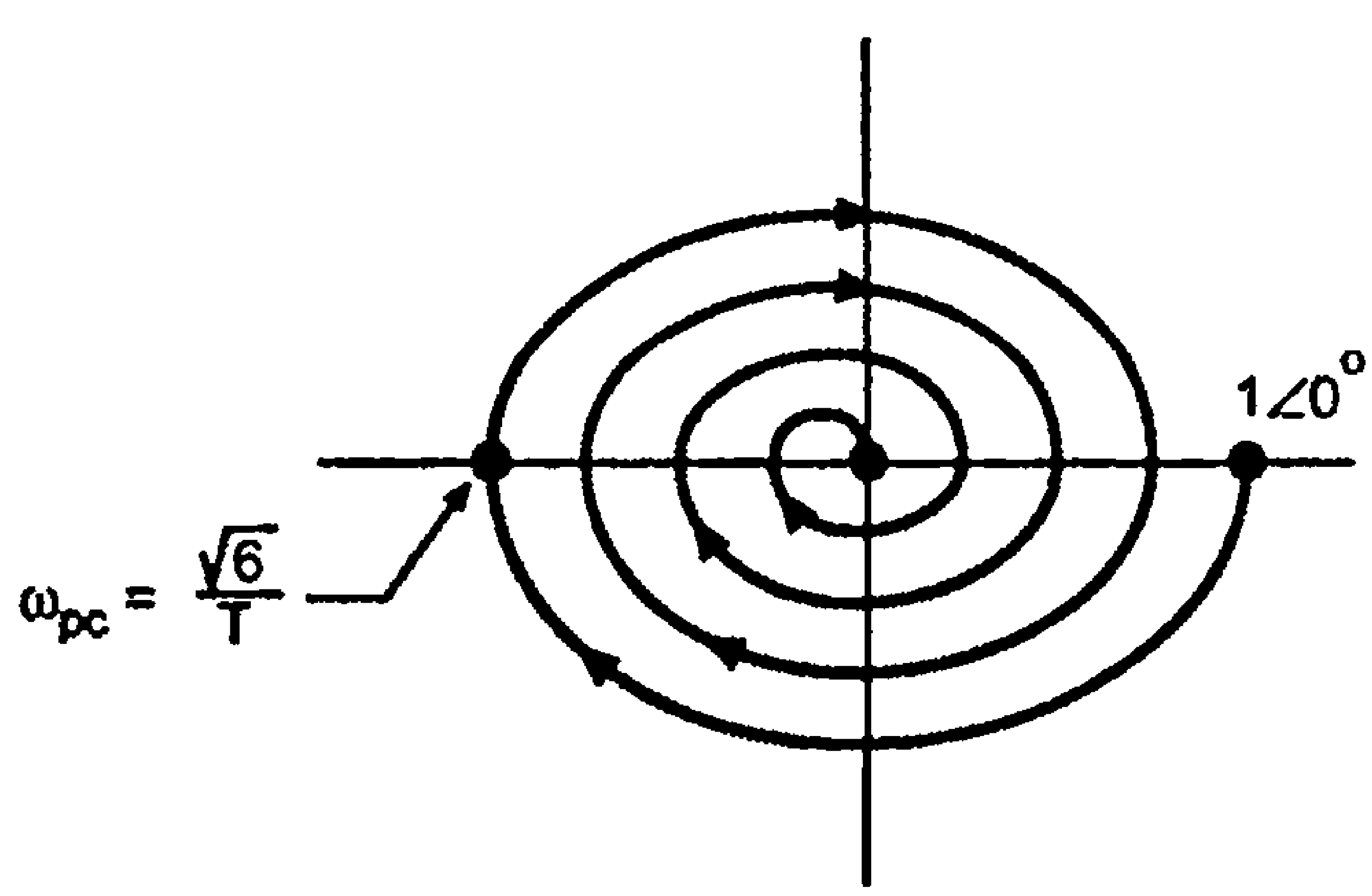


Fig. 12.63

Example 12.26 : *The open loop transfer function of a feedback control system is given by -*

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}.$$

Plot polar plot for the above transfer function and from polar plot determine the value of K for stable operation. **(M.U. : May-2003)**

Solution : In frequency domain,

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(1+j\omega)(2+j\omega)}$$

For polar plot, vary ω from 0 to ∞.

For $\omega = 0$, $G(j\omega)H(j\omega) = \infty\angle -90°$

For $\omega = \infty$, $G(j\omega)H(j\omega) = 0\angle -270°$

$\therefore$ Rotation $= -270° - (-90°) = -180°$ clockwise

To find intersection with negative real axis, rationalize $G(j\omega)H(j\omega)$

$$G(j\omega)H(j\omega) = \frac{K(-j\omega)(1-j\omega)(2-j\omega)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

$$= \frac{-Kj\omega(2-3j\omega-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

$$= \frac{-3K\omega^2}{\omega^2(1+\omega^2)(4+\omega^2)} - \frac{Kj\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

$\therefore$ $2-\omega^2 = 0$

$\therefore$ $\omega = \sqrt{2}$

$\therefore$ Intersection point $= \frac{-3K}{3\times 6} = -\frac{-K}{6}$

So polar plot is as shown in the Fig. 12.64..

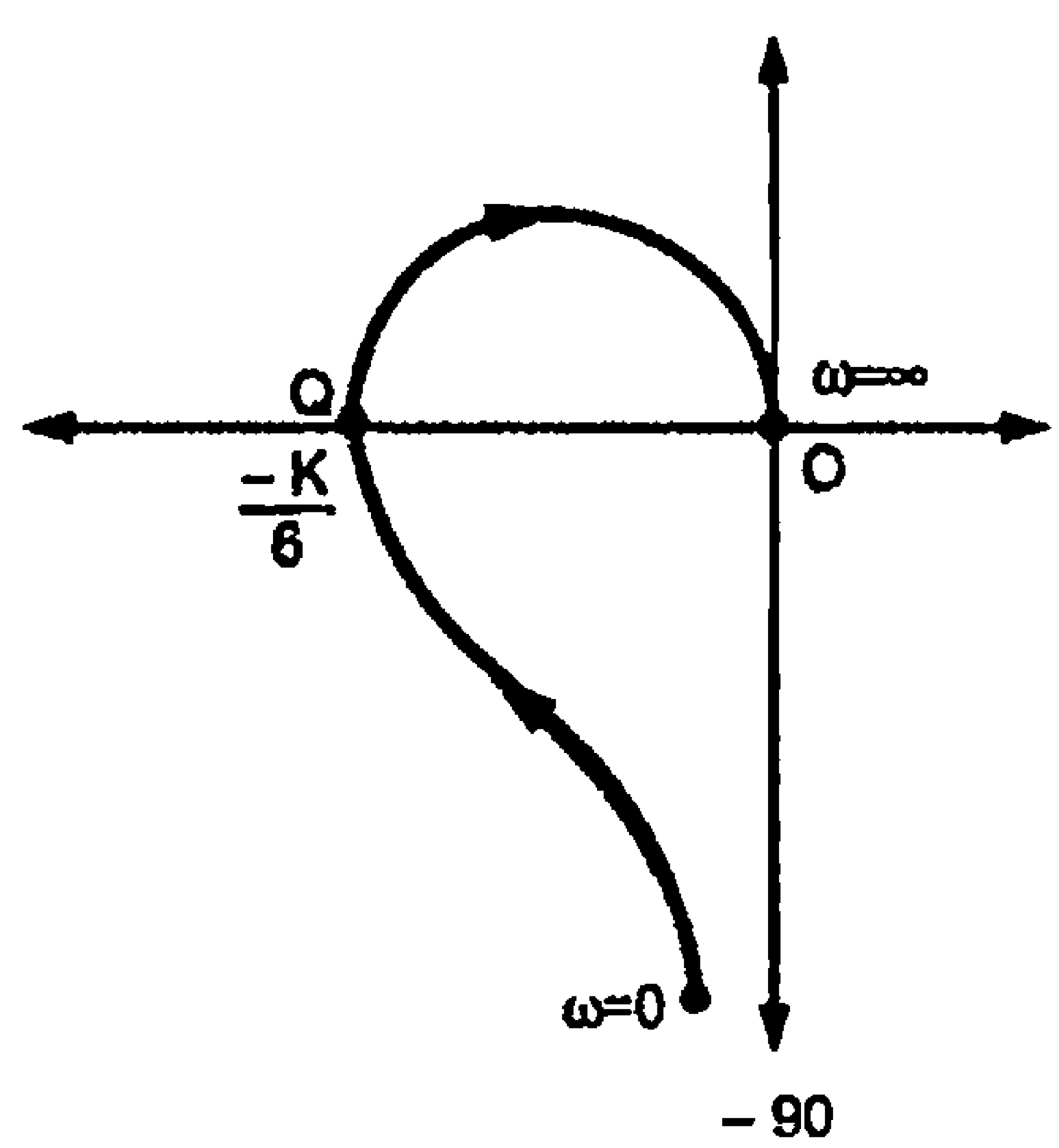


Fig. 12.64

For stability polar plot should not encircle $-1 + j0$

$\therefore \quad |OQ| < 1$

$\therefore \quad \frac{K}{6} < 1$

$\therefore \quad$ **$K < 6$ for stability**

Example 12.27 : *The open loop transfer function of a feedback system is* $G(s)H(s) = \frac{K(1+s)}{(1-s)}$.

Comment on stability using Nyquist criterion, with variation in K.

(M.U. : Dec.-2003, Dec.-2005)

Solution :

Step 1 : P = 1 as one open loop pole in right half of s-plane.

Step 2 : N = – P = –1 for stability.

Step 3 : The Nyquist path is as shown.

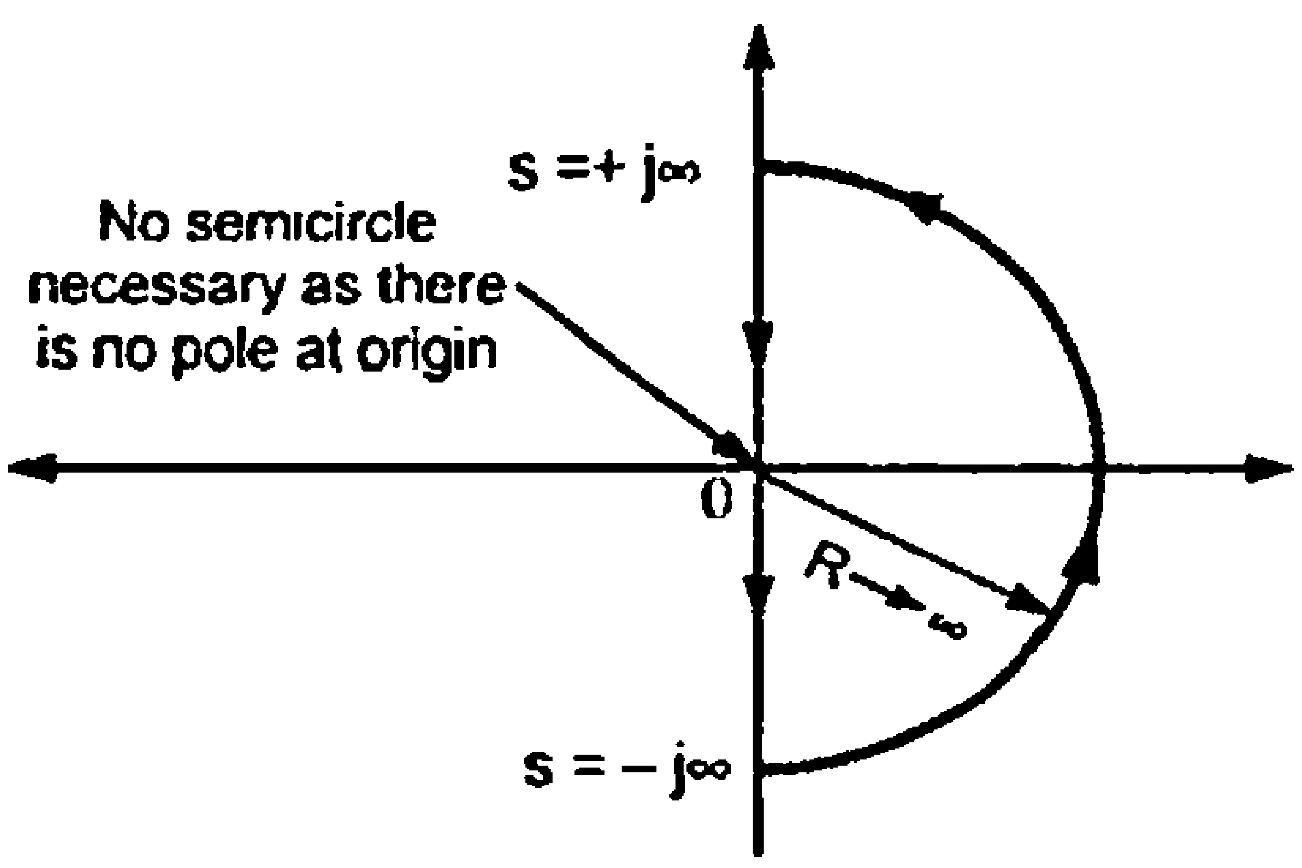


Fig. 12.65 (a)

Step 4 : Analysis of sections

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)}{(1-j\omega)}$$

Note that $\angle 1-j\omega = -90^\circ$ as $\omega \to +\infty$ and $\angle 1-j\omega = 0^\circ$ for $\omega = 0$

Section I : $s = +j\infty$ to $s = 0$

Start $\omega \to +\infty$	$K\angle\frac{90^\circ}{-90^\circ} = K\angle +180^\circ$	$0^\circ-(+180^\circ) = -180^\circ$ Clockwise
End $\omega = 0$	$K\angle\frac{0^\circ}{0^\circ} - K\angle 0^\circ$	

Section II : This is from $s = 0$ to $s = -j\infty$ hence its mapping is mirror image of ction I about real axis.

Section III : Not required.

Step 5 : Rationalize $G(j\omega)H(j\omega)$ for intersection with negative real axis.

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)(1+j\omega)}{(1-j\omega)(1+j\omega)} = \frac{K[1+2j\omega-\omega^2]}{(1+\omega^2)}$$

$$= \frac{K(1-\omega^2)}{(1+\omega^2)} + \frac{2jK\omega}{1+\omega^2}$$

For $\omega = 0$, imaginary part is zero. So apart form $\omega = 0$ there is no ω_{pc} hence there is no finite intersection with negative real axis, apart from K $\angle$+ 180° occuring at $\omega = 0$.

Step 6 : Nyquist plot is as shown.

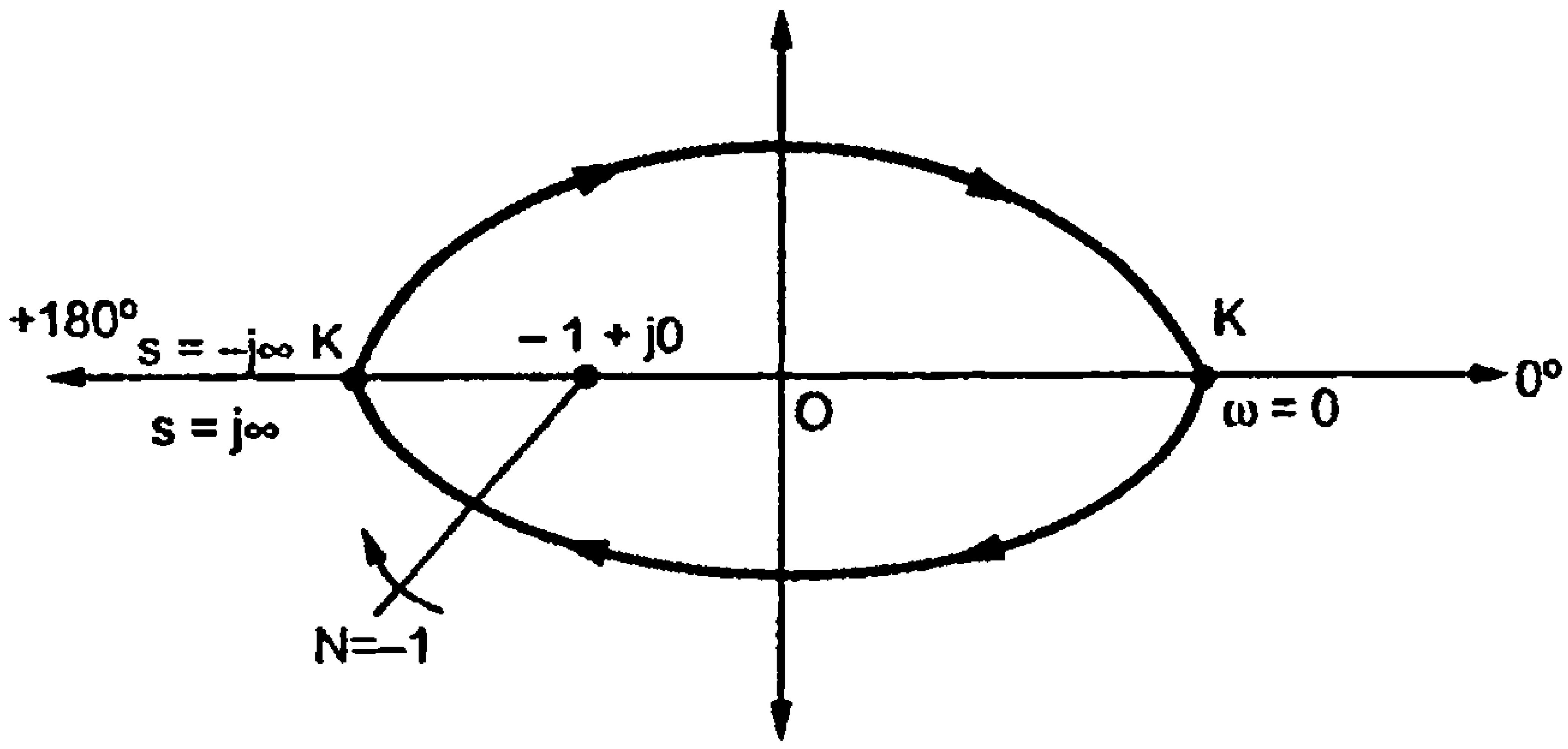


Fig. 12.65 (b)

Step 7 : For stability N = – 1. This is possible if the critical point lies to the right of K.

$\therefore$ l (OK) > 1

$\therefore$ K > 1 for stability. Hence for $1 < K < \infty$ the system is stable in nature.

Example 12.28 : *Sketch the polar plot of* $G(s)H(s) = \dfrac{45\,(s+2)}{s^2\,(s+4)(s+6)}$ *hence find stability, K marginal and range of K (if 45 is replaced by K).* **(M.U. : May-2004)**

Solution : For polar plot, obtain $G(j\omega)H(j\omega)$.

$$\therefore \quad G(j\omega)H(j\omega) = \frac{K\,(j\omega+2)}{(j\omega)^2(j\omega+4)(j\omega+6)}$$

$$M = \frac{K \times \sqrt{\omega^2+4}}{\omega^2 \times \sqrt{\omega^2+16} \times \sqrt{\omega^2+36}}, \quad \phi = +\tan^{-1}\frac{\omega}{2} - 180° - \tan^{-1}\frac{\omega}{4} - \tan^{-1}\frac{\omega}{6}$$

$\omega = 0$	$M = \infty$	$\phi = -180°$	$-270° - (-180°) = -90°$ Clockwise
$\omega = \infty$	$M = 0$	$\phi = -270°$	

To find intersection with negative real axis, rationalize $G(j\omega)H(j\omega)$.

$$G(j\omega)H(j\omega) = \frac{K(2+j\omega)(4-j\omega)(6-j\omega)}{-\omega^2 \times (4+j\omega)(4-j\omega)(6+j\omega)(6-j\omega)}$$

$$= \frac{K(2+j\omega)[24-10j\omega-\omega^2]}{-\omega^2 \times (16+\omega^2)(36+\omega^2)} = \frac{K\,[48-20j\omega-2\omega^2+24j\omega+10\omega^2-j\omega^3]}{-\omega^2(16+\omega^2)(36+\omega^2)}$$

$$= \frac{K(48+8\omega^2)}{-\omega^2(16+\omega^2)(36+\omega^2)} + \frac{Kj\omega(4-\omega^2)}{-\omega^2(16+\omega^2)(36+\omega^2)}$$

Hence at $4-\omega^2 = 0$ i.e. $\omega = 2$ rad/sec, the polar plot intersects negative real axis at point Q whose co-ordinates are given by real part at $\omega = 2$.

$$\therefore \quad Q = \frac{K(48+8\times 4)}{-(4)(16+4)(36+4)} = -\ 0.025\ K$$

The polar plot is shown in the Fig. 12.66.

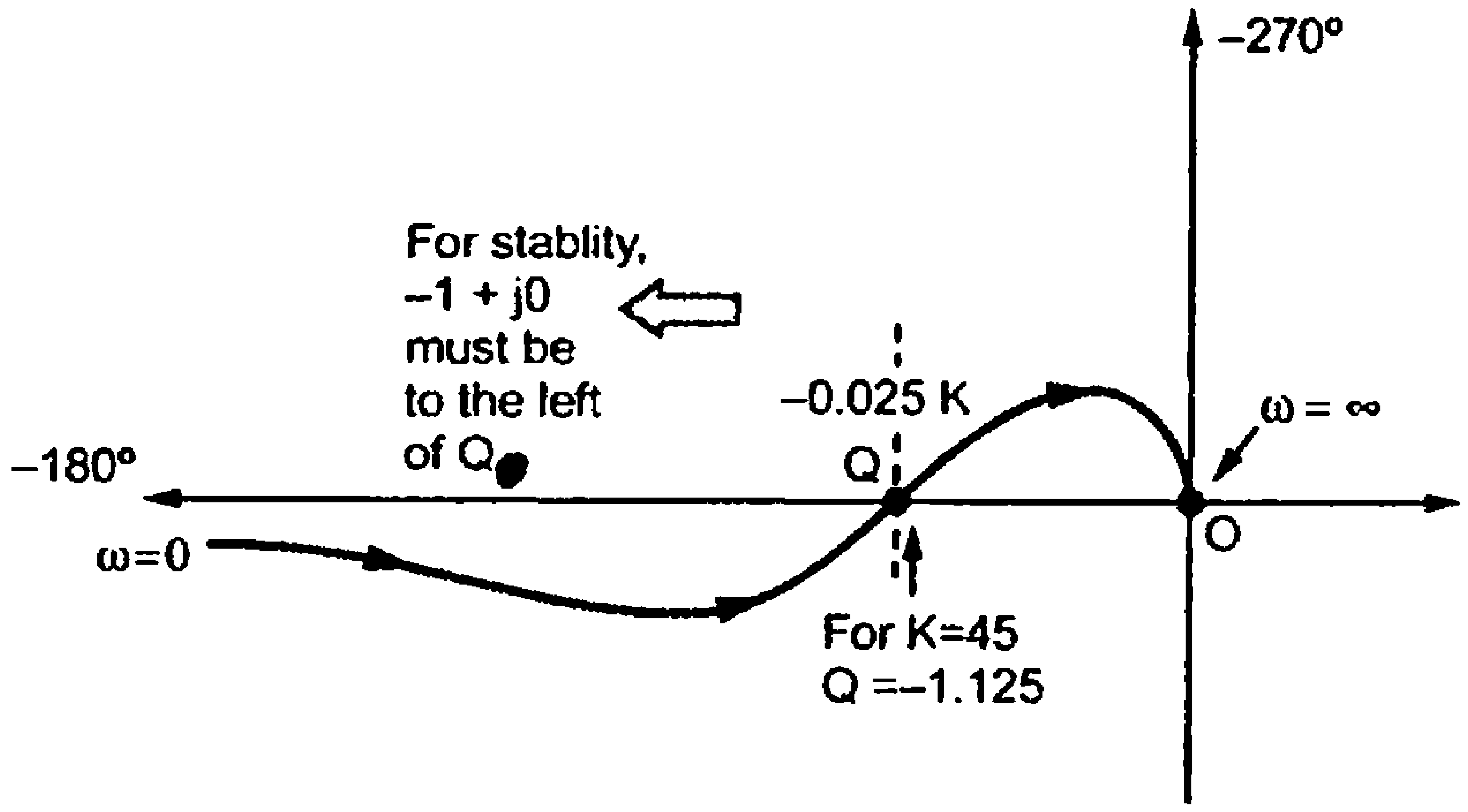


Fig. 12.66

For K = 45, critical point −1 + j0 gets enclosed by the polar plot hence system is **Unstable.** K must be such that the critical point − 1+j0, should not get enclosed by the polar plot.

$\therefore\ l\,(OQ) < 1$

$\therefore\ |0.025\ K| < 1$ i.e. $K < 40$ $\therefore$ $0 < K < 40$ is the range of K for stability.

Example 12.29 : *Sketch the Nyquist plot and hence comment on closed loop stability. Given* $G(s)H(s) = \dfrac{10(s+2)}{s(s-1)}$ **(M.U. : May-2004)**

Solution :

Step 1 : One open loop pole in right hand side of s - plane, P = 1.

Step 2 : N = − P = − 1 for stability.

Step 3 : Nyquist path is as shown.

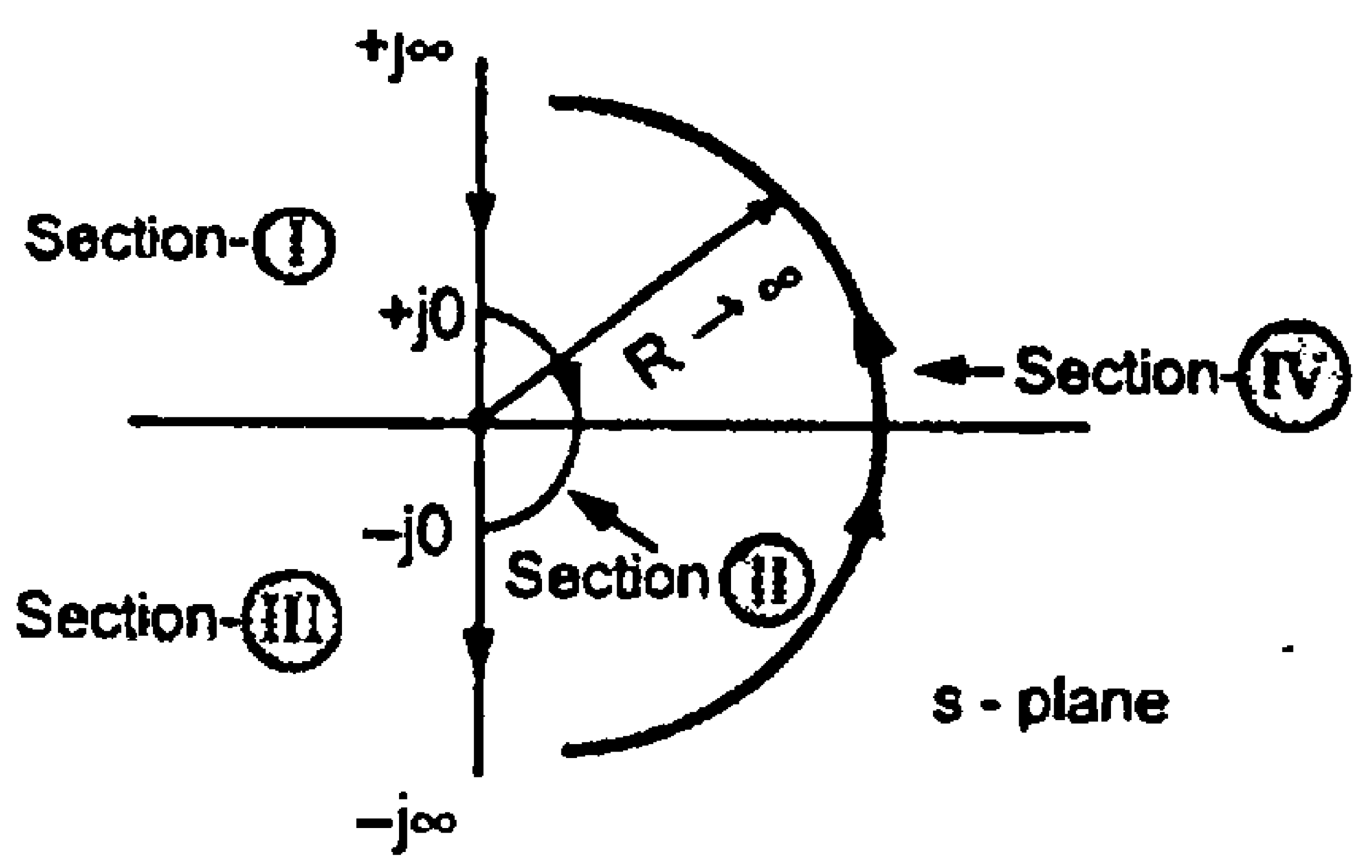


Fig. 12.67

Step 4 : Analysis of sections

$$G(j\omega)H(j\omega) = \frac{10(2+j\omega)}{j\omega(-1+j\omega)}$$

Note that $\angle -1+j\omega = +180°$ for $\omega \to \pm 0$
while $\angle -1+j\omega = +90°$ for $\omega \to \infty$

Step 6 : Nyquist plot is as shown.

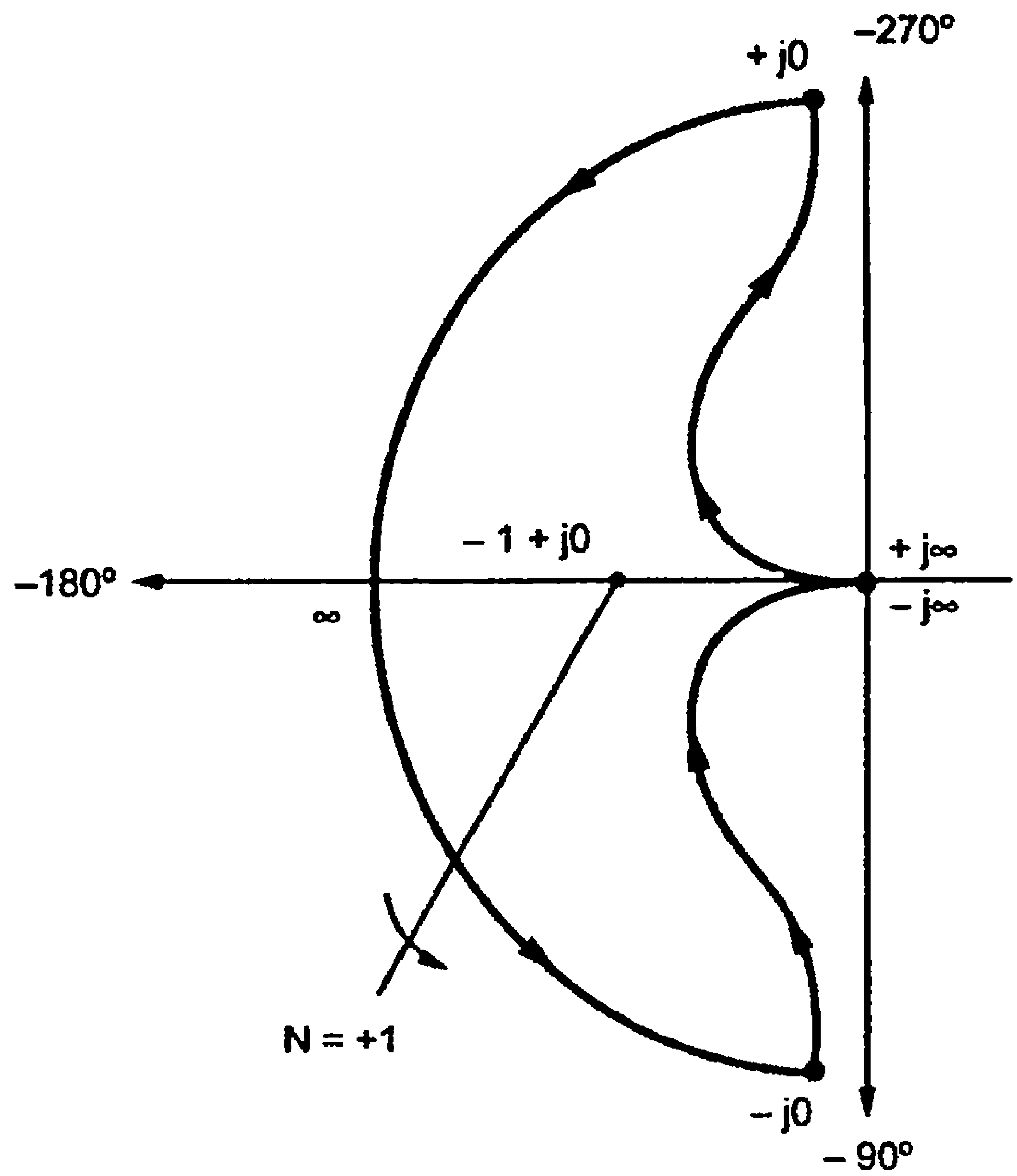


Fig. 12.70

Step 7 : N = + 1

But N = Z – P

∴ 1 = Z – 1

∴ Z = 2

So there are two roots in right half of s - plane making system **unstable.**

ii) $G(s)H(s) = \dfrac{4(s-1)}{s(s-2)}$

The steps 1 to 3 are same as above.

Step 4 : $G(j\omega)H(j\omega) = \dfrac{4(-1+j\omega)}{j\omega(-2+j\omega)}$

Note that $\angle -1+j\omega$ and $\angle -2+j\omega$ is + 180° for $\omega \to \pm 0$. While it is + 90 ° for $\omega \to +\infty$

Section I : s = + j∞ to s = + j0

Start $\omega \to +\infty$	$0\angle\dfrac{0°\ 90°}{90°\ 90°} = 0\angle -90°$	$-90° - (-90°) = 0°$
End $\omega \to +0$	$\infty\angle\dfrac{0°\ 180°}{90°\ 180°} = \infty\angle -90°$	

As rotation of plot is zero, calculate the actual angle for ω between + ∞ to + 0 and check the quadrant in which plot lies.

$$\angle G(j\omega)H(j\omega) = \frac{\angle -1 + j\omega}{90^\circ \angle -2 + j\omega}$$

For $\omega = 10$, $\angle G(j\omega)H(j\omega) = \frac{\angle -1 + j10}{90^\circ \angle -2 + j10} = \frac{95.71^\circ}{90^\circ \angle 101.31^\circ} = -95.6^\circ$ ... Use R → P

For all ω, $\angle -2 + j\omega$ is more than $\angle -1 + j10$ hence Nyquist plot will lie in 3rd quadrant **for section I.**

Section II : $s = +j0$ to $s = -j0$

Start ω = + 0	$\infty \angle -90^\circ$	$90^\circ - (-90^\circ) = +180^\circ$ Anticlockwise
End ω = − 0	$\infty \angle \frac{0^\circ\ 180^\circ}{-90^\circ\ 180^\circ} = \infty \angle +90^\circ$	

Section III : Mirror image of section I about real axis.

Section IV : Not required.

Step 5 : No finite intersection with negative real axis.

Step 6 : Nyquist plot is as shown.

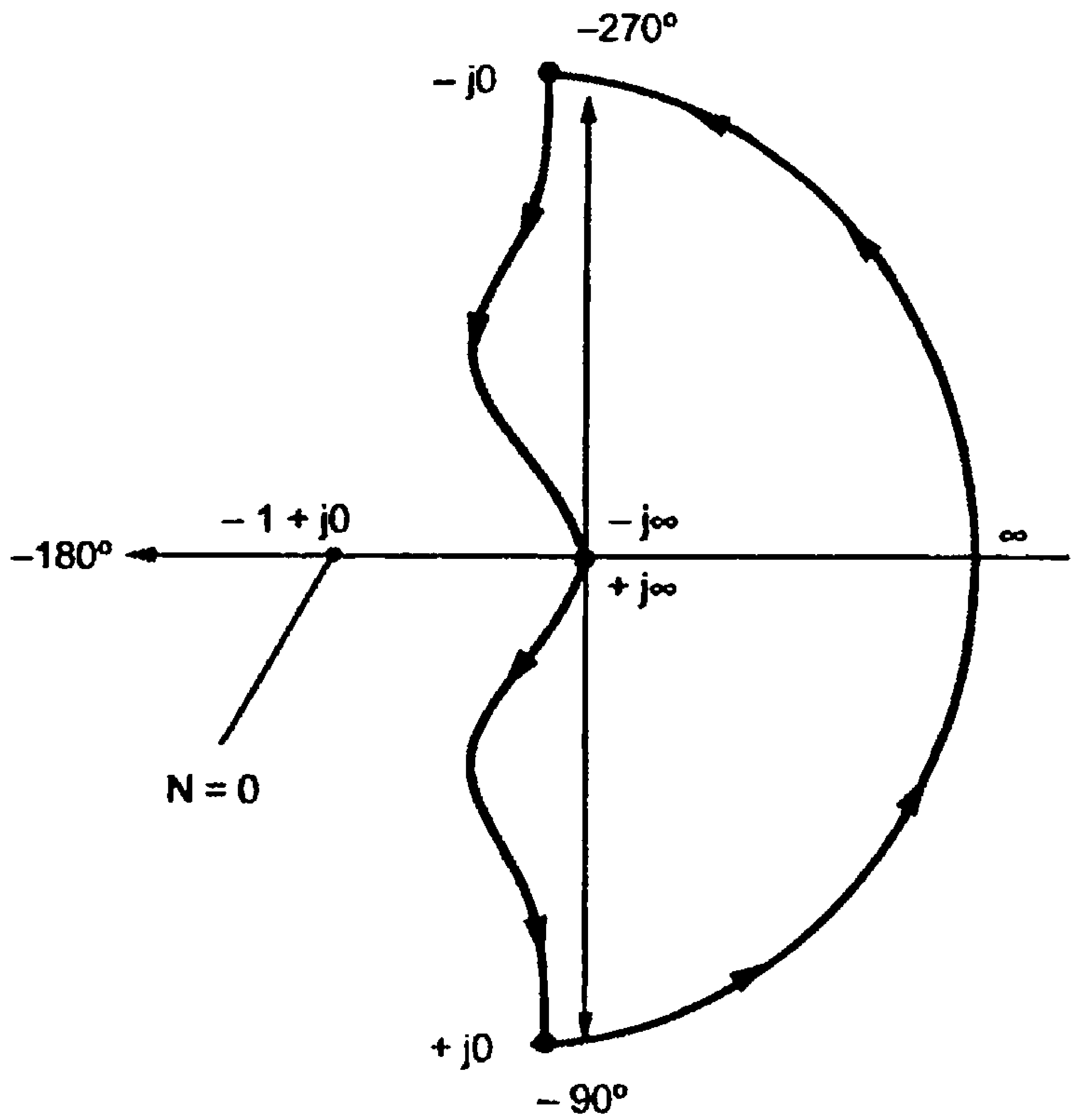


Fig. 12.71

Step 7 : N = 0

But N = Z − P

∴ 0 = Z − 1

∴ Z = 1

So there is one root in right half of s - plane making system **unstable.**

$$= \frac{2(0.25+j\omega)\,[0.5-1.5j\omega-\omega^2]}{(-\omega^2)\,(1+\omega^2)\,(0.25+\omega^2)}$$

$$= \frac{2\,(0.125+1.25\omega^2)}{(-\omega^2)\,(1+\omega^2)\,(0.25+\omega^2)} + \frac{2j\omega\,(0.125-\omega^2)}{(-\omega^2)\,(1+\omega^2)\,(0.25+\omega^2)}$$

Equating imaginary part to zero, $0.125 - \omega^2 = 0$

$\therefore \quad \omega^2 = 0.125 \quad$ i.e. $\omega_{pc} = \mathbf{0.3535}$

Using in real part,

$$Q = \frac{2\,[0.125+1.25\times 0.125]}{(-0.125)\,(1+0.125)\,(0.25+0.125)} = -\,\mathbf{10.667}$$

This is intersection of Nyquist plot with negative real axis.

Step 6 : Nyquist plot is as shown.

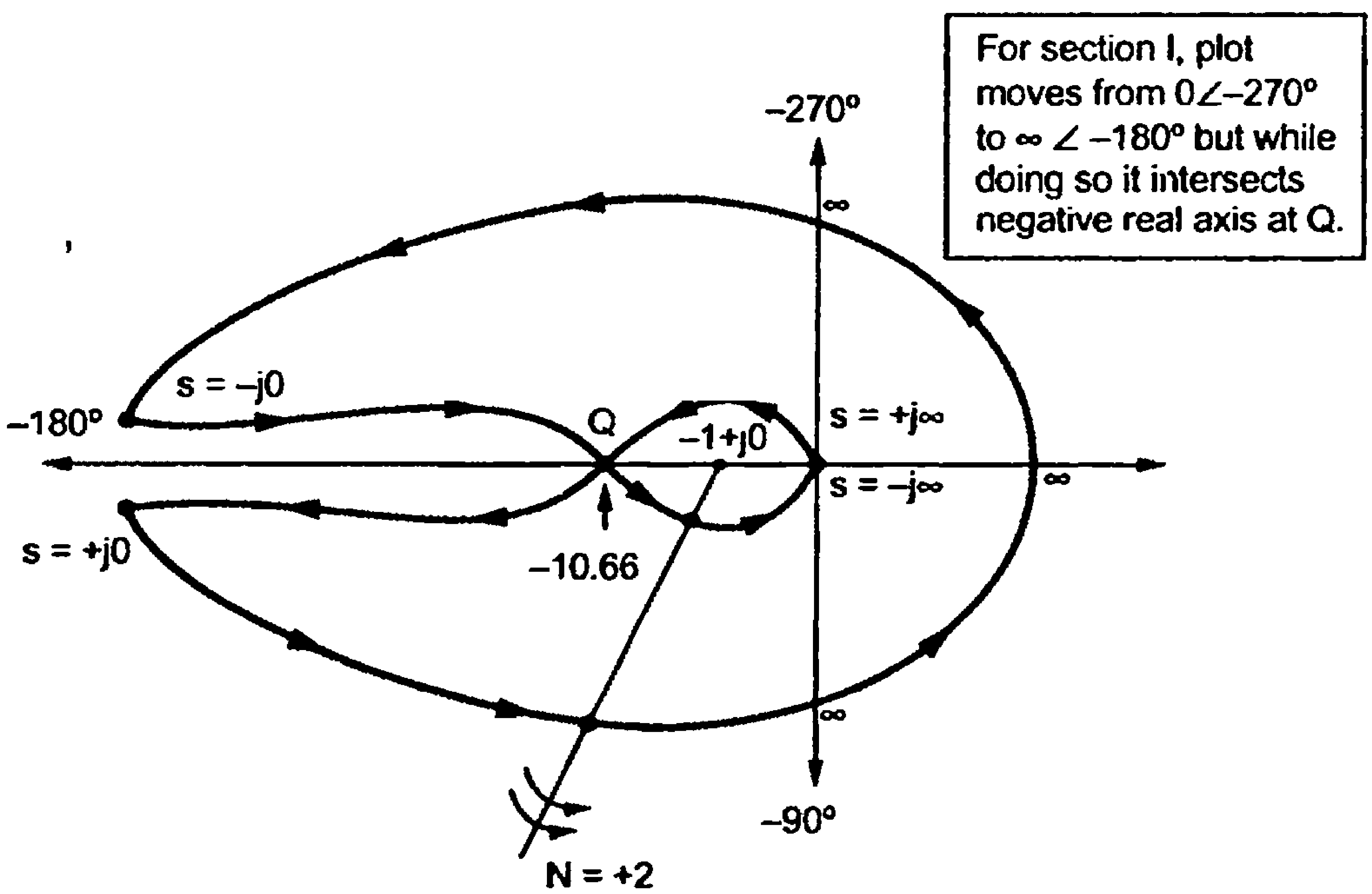


Fig. 12.73

Step 7 : Actually N = + 2 while N must be zero for stability. Hence the system is unstable with 2 roots located in the right half of s - plane.

Example 12.32 : *Obtain polar plot. Given :* $G(s)H(s) = \frac{5}{s(s+1)(s+2)}$. *Find* ω_{pc} *and G.M. If '5' is replaced by K then using polar plot, find range of K for stability and K marginal.* **(M.U. : May-2005)**

Solution : Convert G(s)H(s) to the frequency domain,

$$G(j\omega)H(j\omega) = \frac{K}{j\omega\,(1+j\omega)\,(2+j\omega)}$$

$$M = \frac{K}{\omega\sqrt{1+\omega^2}\times\sqrt{4+\omega^2}},\ \phi = -90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}$$

Key Point *: Angle contribution by positive K is zero.*

$\omega = 0$	$M = \infty$	$\phi = -90^\circ$	$-270^\circ - (-90^\circ) = -180^\circ$
$\omega = \infty$	$M = 0$	$\phi = -270^\circ$	Clockwise

To find intersection with negative real axis, rationalize $G(j\omega)H(j\omega)$.

$$\therefore \quad G(j\omega)H(j\omega) = \frac{K(-j\omega)\,(1-j\omega)\,(2-j\omega)}{(j\omega)\,(-j\omega)\,(1+j\omega)\,(1-j\omega)\,(2+j\omega)\,(2-j\omega)}$$

$$= \frac{-jK\omega\ [2-3j\omega-\omega^2]}{\omega^2(1+\omega^2)\ (4+\omega^2)} = \frac{-3K}{(1+\omega^2)\ (4+\omega^2)} - \frac{jK\omega(2-\omega^2)}{\omega^2(1+\omega^2)\ (4+\omega^2)}$$

To have zero imaginary part, $2-\omega^2 = 0$

$$\therefore \quad \omega_{pc} = \sqrt{2} = 1.4142 \text{ rad/sec}$$

Using in real part,

$$Q = \frac{-3K}{(1+2)\,(4+2)} = \frac{-K}{6}$$

The polar plot is as shown in the Fig. 12.74.

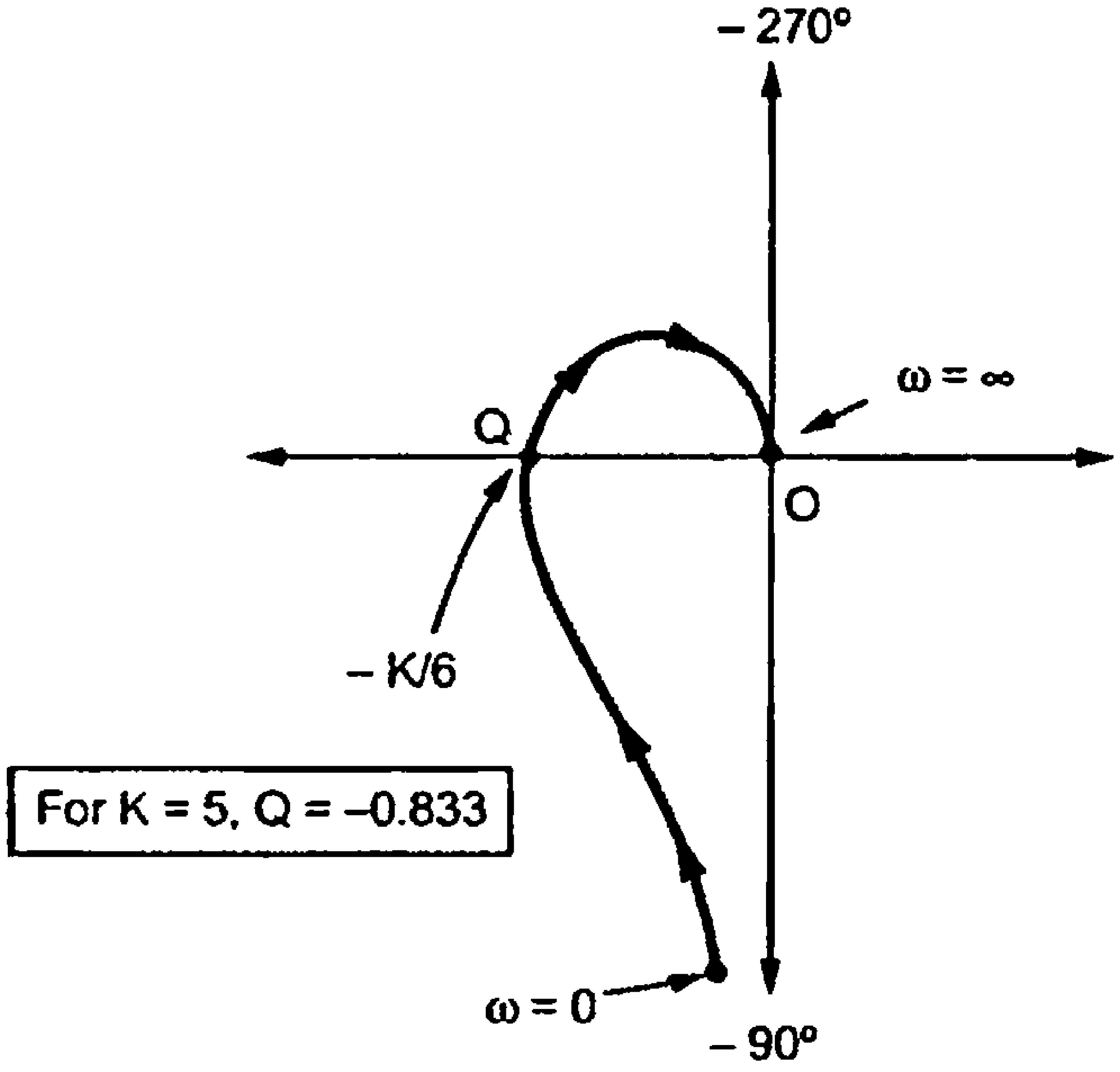


Fig. 12.74

For K = 5 , $Q = \frac{-5}{6} = -0.833$. So critical point $-1 + j0$ lies to the left of polar plot hence not enclosed. So **system is stable for K = 5.**

In general for stability, critical point must lie to the left of Q.

$\therefore\ l\,(OQ) < 1$

$\therefore\ \left|-\frac{K}{6}\right| < 1$

i.e. $K < 6$

So range of K for stability is given by, **0 < K < 6**

Hence the marginal value of **$K_{mar} = 6$.**

Example 12.33 : *Discuss the stability of the system using Nyquist plot for,*

$$G(s)H(s) = \frac{K(s-2)}{(s+1)^2}$$

(M.U. : May-2006)

Solution :

Step 1 : P = 0 as no open loop pole in right half of s-plane.

Step 2 : N = – P = 0 for stability .

Step 3 : The Nyquist path is as shown.

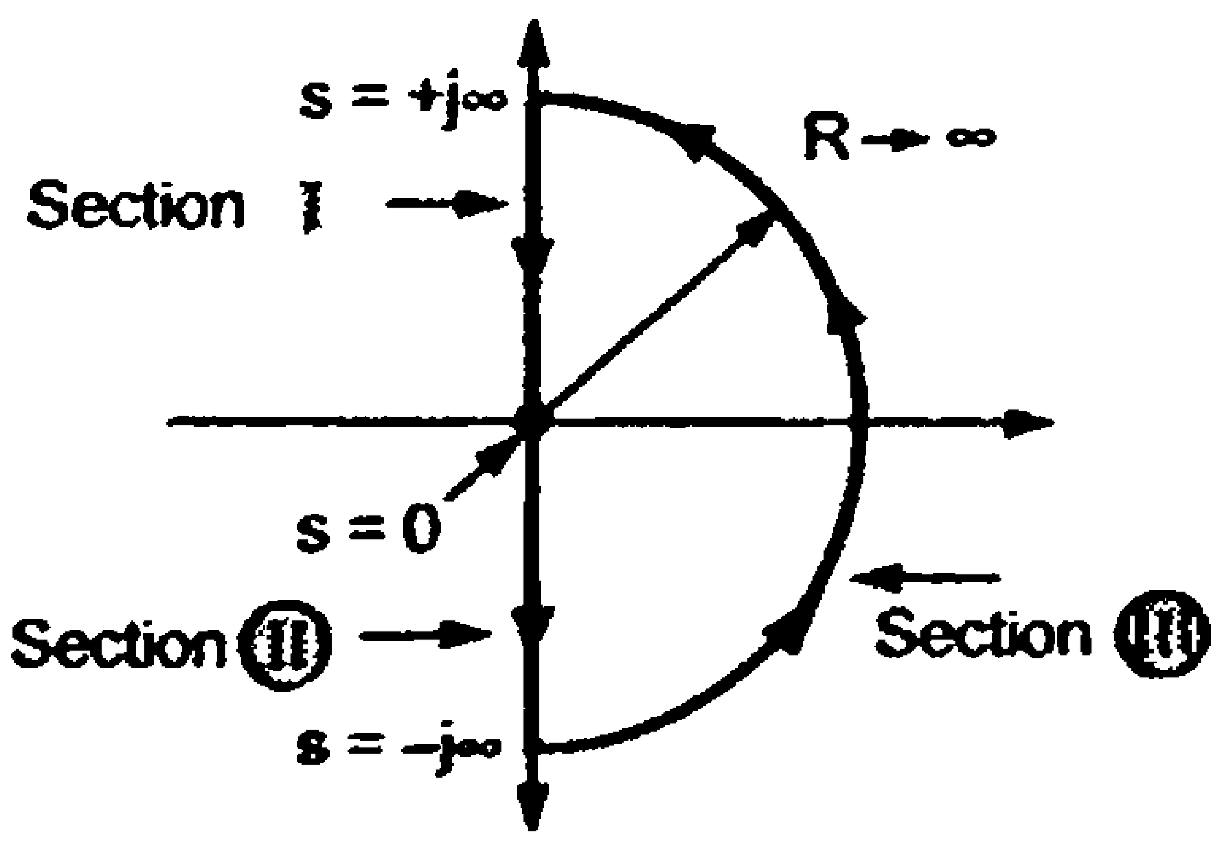


Fig. 12.75

Step 4 : $G(j\omega)\,H(j\omega) = \frac{K(-2+j\omega)}{(1+j\omega)\,(1+j\omega)}$

Note that $\angle -2+j\omega \rightarrow +90°$ when $\omega \rightarrow +\infty$ and $\angle -2+j\omega$ is $+180°$ when $\omega = 0$.

Section I : s = + j∞ to s = 0

Start $\omega \to +\infty$	$0\angle\frac{90°}{90°\,90°} = 0\angle -90°$	$180° - (-90°) = +270°$ Anticlockwise
End $\omega = 0$	$2K\angle\frac{180°}{0°\,0°} = 2K\angle 180°$	

Section II : Mirror image of section I about real axis.

Section III : Not required.

Step 4 : Rationalize G(jω)H(jω) to obtain intersection with negative real axis.

$$G(j\omega)H(j\omega) = \frac{K(-2+j\omega)(1-j\omega)(1-j\omega)}{(1+j\omega)(1-j\omega)(1+j\omega)(1-j\omega)} = \frac{K(-2+j\omega)(1-2j\omega-\omega^2)}{(1+\omega^2)^2}$$

$$= \frac{K[-2+4j\omega+2\omega^2+j\omega+2\omega^2-j\omega^3]}{(1+\omega^2)^2}$$

$$= \frac{K(-2+4\omega^2)}{(1+\omega^2)^2} + \frac{jK\omega(5-\omega^2)}{(1+\omega^2)^2}$$

Equating imaginary part to zero $5-\omega^2 = 0$

$\therefore \quad \omega^2 = 5 \quad$ i.e. $\omega - \sqrt{5}$ rad/sec

Using in real part, $Q = \frac{K(-2+4\times 5)}{(1+5)^2} = +\frac{K}{2}$

Thus actually at $\omega = \sqrt{5}$, **the plot intersects positive real axis at $\frac{K}{2}$ and not the negative real axis.**

Step 5 : The Nyquist plot is shown in the Fig. 12.76.

Step 6 : For N = 0, – 1 + j0 point must lie to the left of P.

$\therefore \quad l\,(OP) < 1$

$\therefore \quad |-2K| < 1$

$\therefore \quad \boxed{K < \frac{1}{2}}$

For stability, **K < 0.5**.

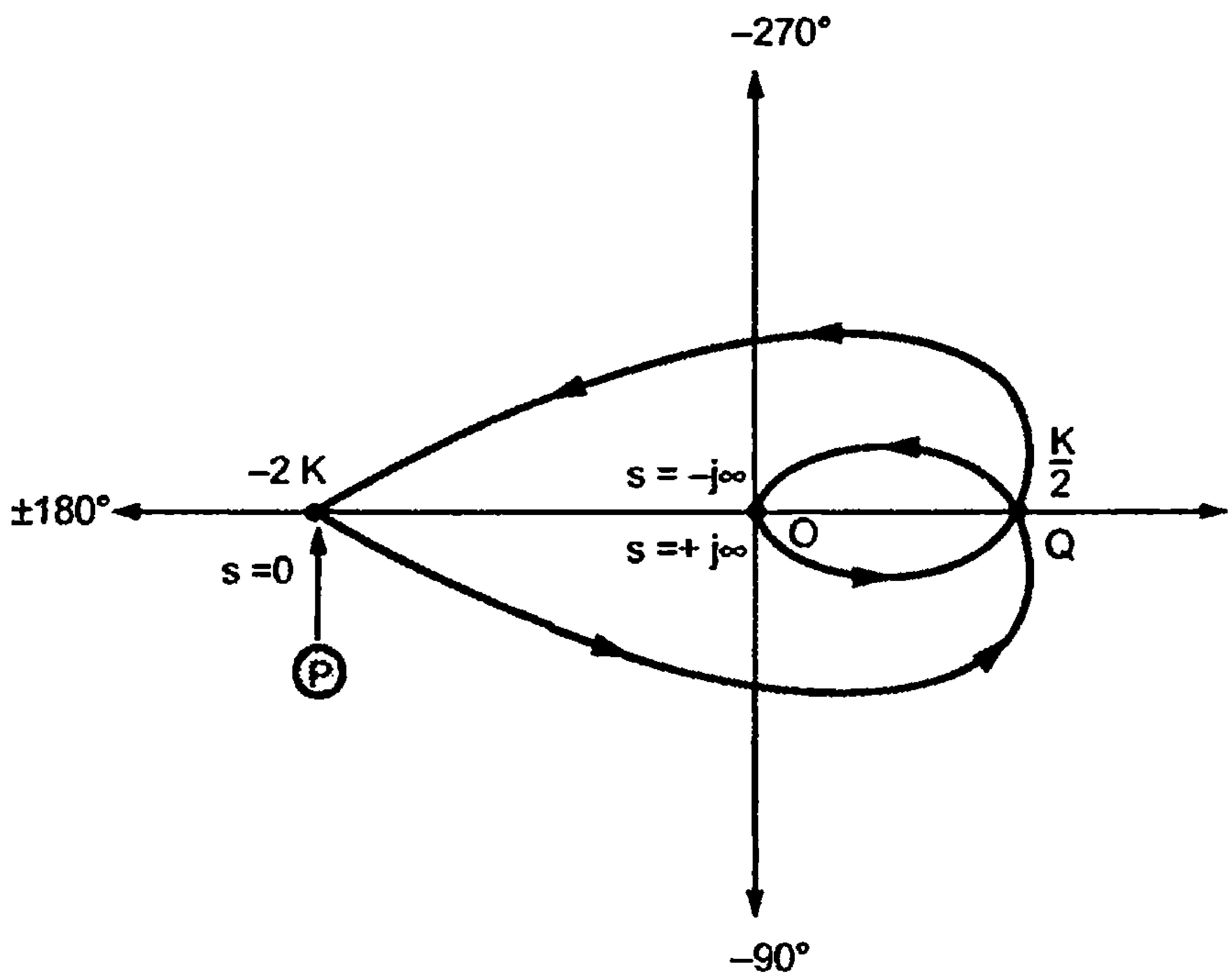


Fig. 12.76

Example 12.34 : *If* $G(s)H(s)=\frac{K(s+1)}{s^2(s+2)(s+4)}$*, using polar plot determine the range of K for stability. Verify your result by Routh's criterion.* **(M.U. : Dec. - 2006)**

Solution : The frequency domain transfer function is,

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)}{(j\omega)(j\omega)(2+j\omega)(4+j\omega)}$$

$$M = \frac{K\sqrt{1+\omega^2}}{\omega^2\sqrt{4+\omega^2}\sqrt{16+\omega^2}}, \phi = +\tan^{-1}\omega - 180° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

Start $\omega = 0$	$M = \infty$	$\phi = -180°$	$-270° - (-180°) = -90°$
End $\omega = \infty$	$M = 0$	$\phi = -270°$	Clockwise

Rationalize $G(j\omega)H(j\omega)$ to obtain intersection with negative real axis.

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)(2-j\omega)(4-j\omega)}{(-\omega^2)(2+j\omega)(2-j\omega)(4+j\omega)(4-j\omega)}$$

$$= \frac{K(1+j\omega)[8-6j\omega-\omega^2]}{(-\omega^2)(4+\omega^2)(16+\omega^2)} = \frac{K[8-6j\omega-\omega^2+8j\omega+6\omega^2-j\omega^3]}{(-\omega^2)(4+\omega^2)(16+\omega^2)}$$

$$= \frac{K(8+5\omega^2)}{(-\omega^2)(4+\omega^2)(16+\omega^2)} + \frac{Kj\omega(2-\omega^2)}{(-\omega^2)(4+\omega^2)(16+\omega^2)}$$

Equating imaginary part to zero, $2-\omega^2 = 0$

$\therefore \quad \omega^2 = 2 \quad$ i.e. $\omega_{pc} = \sqrt{2}$

Using in real part, $\quad Q = \frac{K(8+5\times 2)}{(-2)(4+2)(16+2)} = -0.08333K$

The polar plot is shown in the Fig. 12.77.

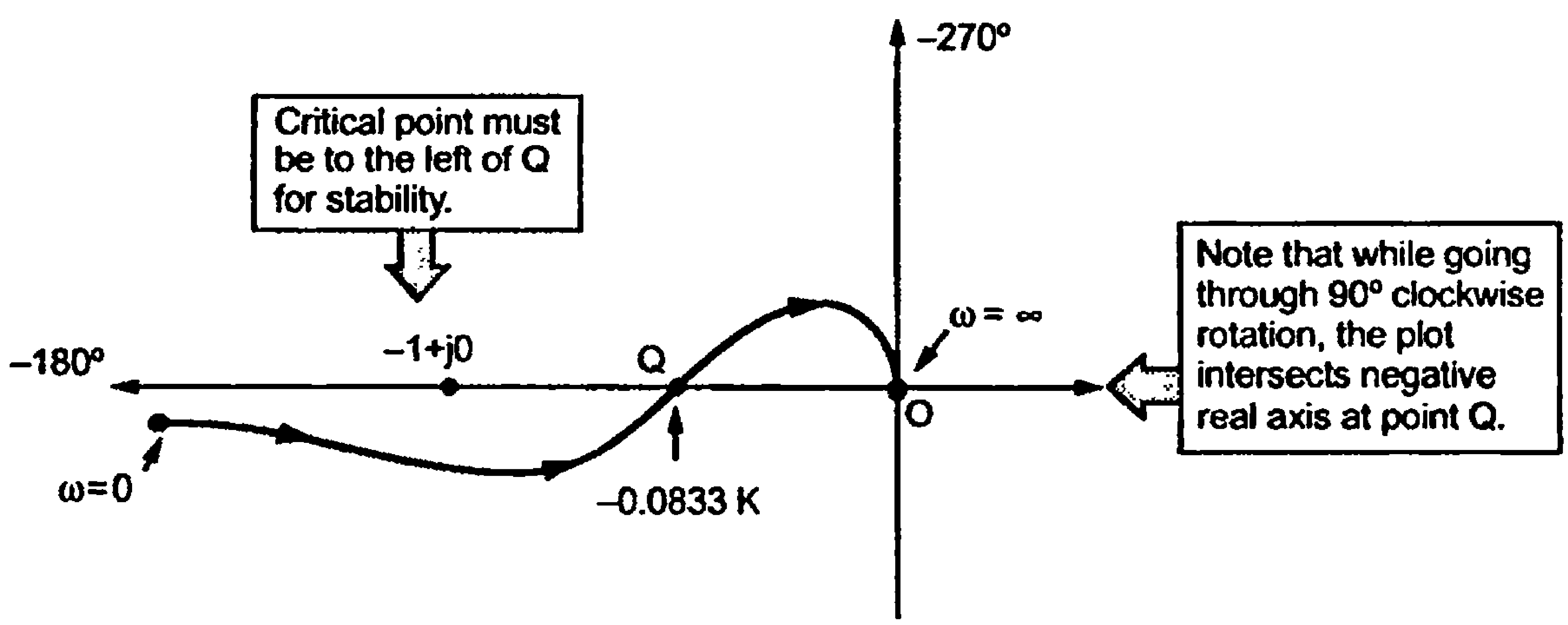


Fig. 12.77

$\therefore \quad l\,(OQ) < 1$ i.e. $|-0.08333\,K| < 1$ ... For stability

$\therefore$ $\boxed{K < 12}$

Routh's criterion

The characteristic equation is $1 + G(s)H(s) = 0$

$$1 + \frac{K(s+1)}{s^2(s+2)(s+4)} = 0 \quad \text{i.e. } s^4 + 6s^3 + 8s^2 + Ks + K = 0$$

s^4	1	8	K
s^3	6	K	0
s^2	$\frac{48-K}{6}$	K	0
s^1	$\frac{\left(\frac{48-K}{6}\right)(K)-6K}{\left(\frac{48-K}{6}\right)}$	0	
s^0	K		

For stability

$$48 - K > 0 \quad \text{... from } s^2$$

$$\therefore \quad K < 48 \quad \text{... (1)}$$

From row of s^1

$$\left(\frac{48-K}{6}\right)(K) - 6K > 0$$

i.e. $\quad 48 - K - 36 > 0$

i.e. $\quad 12 - K > 0$

i.e. $\quad K < 12 \quad$ ..(2)

Hence the ultimate condition for stability is **0 < K < 12** as obtained by the polar plot.

Example 12.35 : *If* $G(s)H(s) = \dfrac{K}{s\,(1+2s)\,(1+0.1s)}$, *using polar plot determine the range of K for stability. Verify your results by Routh's criterion.* **(M.U. : May-2007)**

Solution : Convert transfer function to the frequency domain.

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega)\,(1+2j\omega)\,(1+0.1j\omega)}$$

$$\therefore \quad M = \frac{K}{\omega\sqrt{1+4\omega^2}\,\sqrt{1+0.01\omega^2}}, \quad \phi = -90° - \tan^{-1} 2\omega - \tan^{-1} 0.1\omega$$

Start $\omega = 0$	$M = \infty$	$\phi = -90°$	$-270° - (-90°) = -180°$ clockwise
End $\omega = \infty$	$M = 0$	$\phi = -270°$	

Rationalize $G(j\omega)H(j\omega)$ to find intersection with negative real axis.

$$G(j\omega)H(j\omega) = \frac{K(-j\omega)(1-2j\omega)(1-0.1j\omega)}{(j\omega)(-j\omega)(1+2j\omega)(1-2j\omega)(1+0.1j\omega)(1-0.1j\omega)} = \frac{-jK\omega[1-2.1j\omega - 0.2\omega^2]}{\omega^2(1+4\omega^2)(1+0.01\omega^2)}$$

$$= \frac{-2.1K}{(1+4\omega^2)(1+0.01\omega^2)} - \frac{jK\omega(1-0.2\omega^2)}{\omega^2(1+4\omega^2)(1+0.01\omega^2)}$$

Equating imaginary part to zero, $1-0.2\omega^2 = 0$

$\therefore \quad \omega^2 = 5 \quad$ i.e. $\omega_{pc} = \sqrt{5}$ rad/sec

Using in real part,

$$Q = \frac{-2.1K}{[1+4\times 5][1+0.01\times 5]} = -0.09523\ K$$

The polar plot is shown in the Fig. 12.78.

For stability, the critical point must not be enclosed i.e. it must lie to the left of Q.

$\therefore\ l(OQ) < 1$

$\therefore\ 0.09523\ K < 1$

$\therefore\ K < 10.5$

$\therefore$ **0 < K < 10.5** ... Range for stability

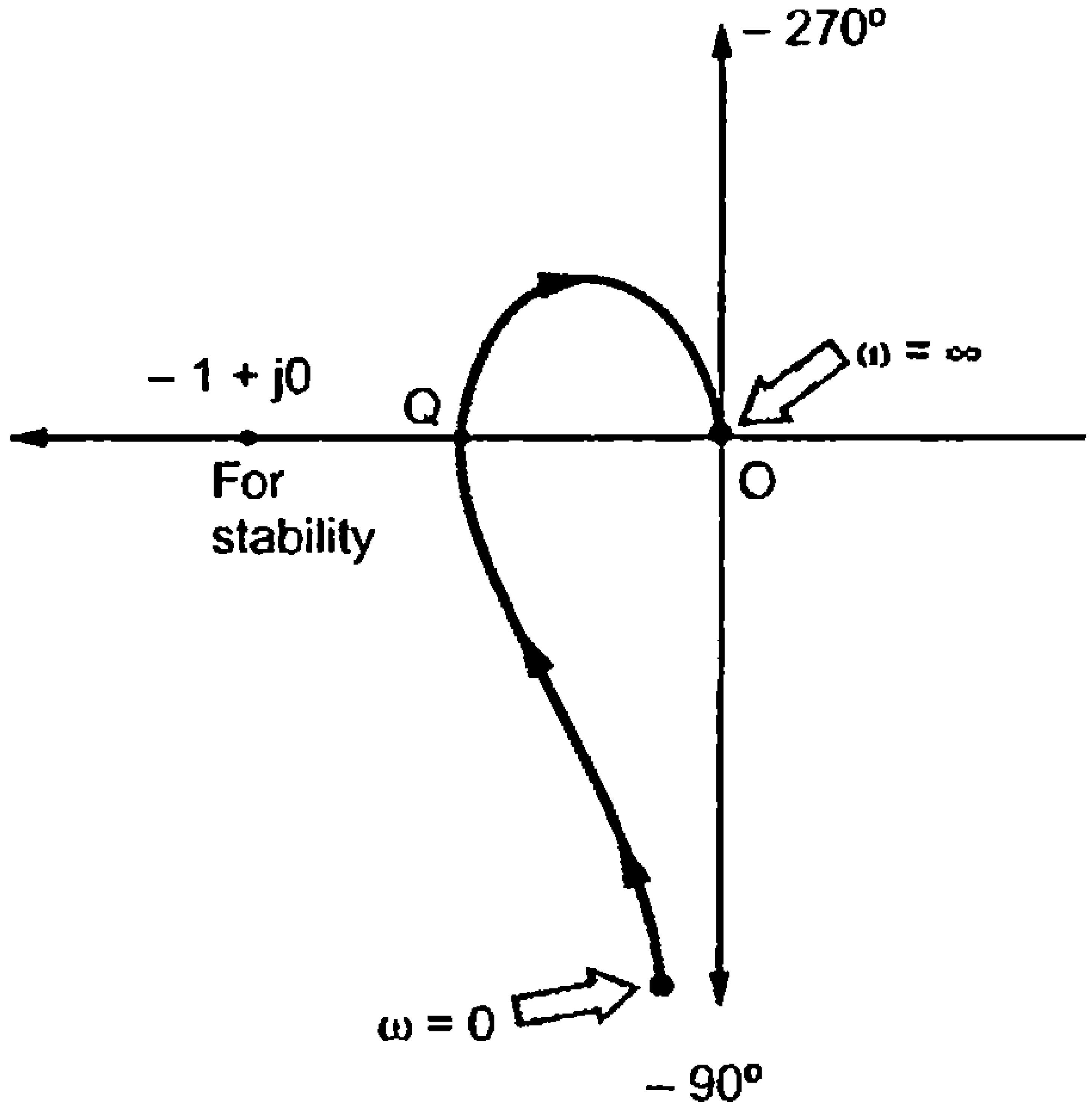


Fig. 12.78

Routh's method : The characteristic equation is,

$$1+G(s)H(s)=0 \text{ i.e. } 1+\frac{K}{s(1+2s)(1+0.1s)} = 0$$

$\therefore\ 0.2s^3 + 2.1s^2 + s + K = 0$

s^3	0.2	1
s^2	2.1	K
s^1	$\frac{2.1-0.2K}{2.1}$	
s^0	K	

From row of s^1, $2.1 - 0.2K > 0$ i.e. $2.1 > 0.2K$

$\therefore\ K < \frac{2.1}{0.2} < 10.5$

$\therefore$ **0 < K < 10.5** ... Range of stability

13 Closed Loop Frequency Response

13.1 Closed Loop Frequency Response

Uptill now we have seen that the closed loop stability and the frequency response specifications can be obtained by plotting open loop frequency response of the system. The open loop frequency respose is the graph of magnitude of $G(j\omega)\,H(j\omega)$ and phase angle of $G(j\omega)\,H(j\omega)$ against frequency ω. The methods discussed earlier to obtain open loop frequency response are Bode Plot, Polar Plot and Nyquist plot.

For a unity feedback closed loop stable system, the closed loop frequency response can be obtained from the open loop frequency response. The closed loop transfer function of a unity feedback (H(s) = 1) system is given by,

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\therefore \quad \boxed{M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}}$$

Key Point: *The closed loop frequency response is the graph of magnitude of $C(j\omega)$ / $R(j\omega)$ and phase angle of $C(j\omega)/R(j\omega)$ against frequency ω.*

The data required to obtain closed loop frequency response can be obtained from the polar plot plotted from open loop transfer function $G(j\omega)$. The Fig. 13.1 shows the polar plot of $G(j\omega)$ ploted by varying ω from 0 to ∞.

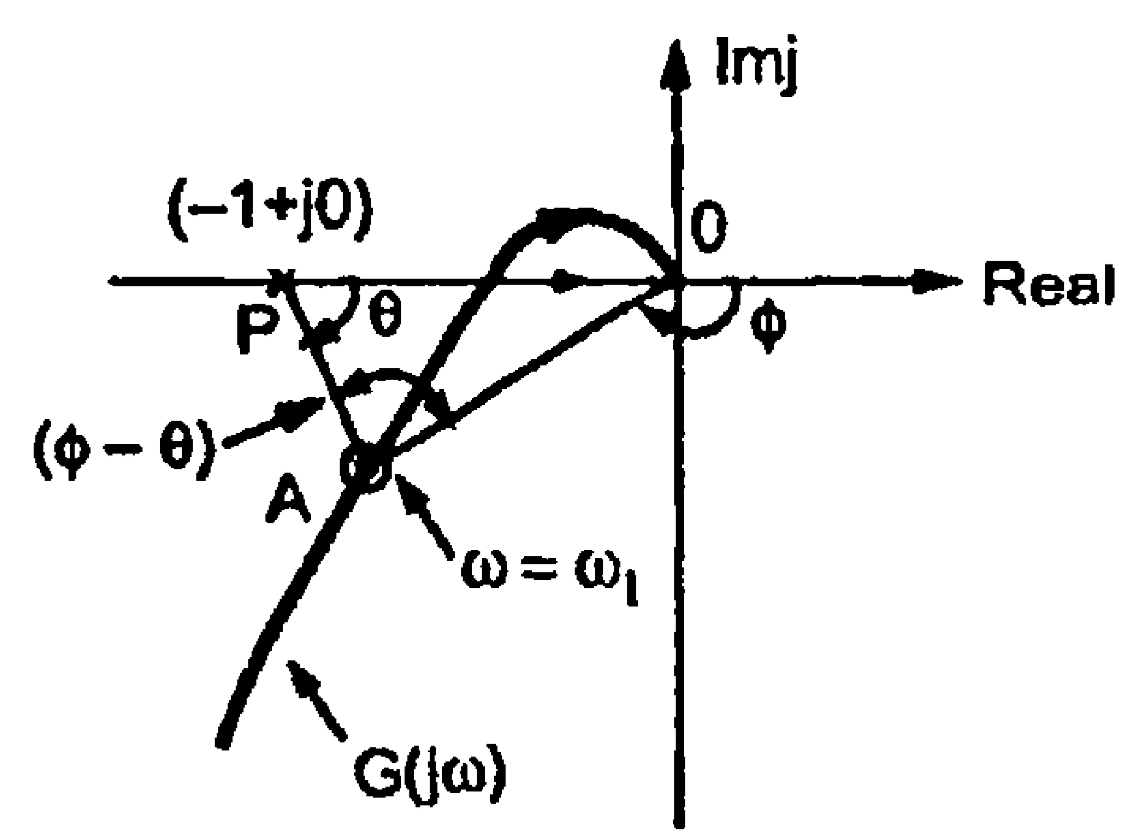


Fig. 13.1

The point A on the polar plot represent $|G(j\omega)|$ and $\angle G(j\omega)$ at $\omega = \omega_1$

$$\therefore \quad \vec{OA} = |G(j\omega)|$$

The point P is the critical point $-1 + j0$.

Now join point P to point A to get vector PA. The $\vec{PA}$ represents $1+G(j\omega)|_{\omega=\omega_1}$.

Key Point: *Hence the magnitude of C(jω) / R(jω) i.e. | M(jω) | at $\omega = \omega_1$ is the ratio of vectors $\vec{OA}$ and $\vec{PA}$.*

$$\therefore \quad \boxed{|M(j\omega)|_{\omega=\omega_1} = \left.\frac{C(j\omega)}{R(j\omega)}\right|_{\omega=\omega_1} = \frac{\vec{OA}}{\vec{PA}} = \left.\frac{G(j\omega)}{1+G(j\omega)}\right|_{\omega=\omega_1}}$$

Similar phase angle of $C(j\omega) / R(j\omega)$ is the angle formed by the vector $\vec{OA}$ and $\vec{PA}$.

$$\therefore \quad \angle M(j\omega)|_{\omega=\omega_1} = \angle \left.\frac{C(j\omega)}{R(j\omega)}\right|_{\omega=\omega_1} = \frac{\angle \vec{OA}}{\angle \vec{PA}} = \frac{\phi}{\theta}$$

$$\therefore \quad \boxed{\alpha = (\phi - \theta)}$$

This is shown in the Fig. 13.1.

The procedure can be repeated for various values of ω to obtain magnitudes and phase angles of closed loop transfer function $C(j\omega) / R(j\omega)$. From this data, the closed loop frequency response can be obtained.

The method discussed above is complicated to obtain closed loop frequency response. Let us study how to obtain constant magnitude loci and constant phase angle loci for the closed loop transfer function. Such loci are called M and N circles. These circles are very much convenient in determining the closed loop frequency response from the polar plot.

Key Point: *It must be remembered that these circles are useful only for unity feedback systems.*

13.2 M Circles [Constant Magnitude Loci]

The closed loop transfer function of a unity feedback system is,

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Now 's' is the complex variable hence $G(j\omega)$ is also a complex variable.

$$\therefore \quad G(j\omega) = X + jY$$

Now $$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} = \frac{X+jY}{1+X+jY}$$

Let M be the magnitude of $C(j\omega)/R(j\omega)$.

$$\therefore \quad M = \left|\frac{X+jY}{1+X+jY}\right| = \frac{\sqrt{X^2+Y^2}}{\sqrt{(1+X)^2+Y^2}}$$

Squaring both sides we get,

$$M^2 = \frac{X^2+Y^2}{(1+X)^2+Y^2}$$

$$\therefore \; M^2[(1+X)^2+Y^2] = X^2+Y^2$$

$$\therefore \; M^2[1+2X+X^2+Y^2]-X^2-Y^2 = 0$$

$$\therefore \; X^2(M^2-1)+2XM^2+Y^2(M^2-1)+M^2 = 0 \quad \text{... (1)}$$

Dividing by M^2-1,

$$X^2+\frac{2XM^2}{(M^2-1)}+Y^2+\frac{M^2}{(M^2-1)} = 0$$

$$\therefore \; X^2-\frac{2XM^2}{(1-M^2)}+Y^2 = \frac{M^2}{1-M^2}$$ absorbing negative sign in denominator

To complete the square on left hand side, add $\left[\frac{M^2}{(1-M^2)}\right]^2$ on both sides.

$$\therefore \quad X^2-\frac{2XM^2}{(1-M^2)}+\left[\frac{M^2}{(1-M^2)}\right]^2+Y^2 = \frac{M^2}{(1-M^2)}+\left[\frac{M^2}{(1-M^2)}\right]^2$$

$$\therefore \quad \left[X-\frac{M^2}{(1-M^2)}\right]^2+Y^2 = \frac{M^2}{(1-M^2)}\left[1+\frac{M^2}{1-M^2}\right]$$

$$\therefore \quad \boxed{\left[X-\frac{M^2}{1-M^2}\right]^2+Y^2 = \left[\frac{M}{(1-M^2)}\right]^2} \quad \text{... (2)}$$

This equation is the equation of a circle with the center at $X=\frac{M^2}{1-M^2}$ and Y = 0 while radius = $\left|\frac{M}{1-M^2}\right|$.

For a particular value of M, we are getting a circle on which the value of M, the magnitude of closed loop transfer function is constant. And for various values of M, we are getting family of such circles. Hence these are called constant magnitude loci or M circles. Let us calculate center and radius for various values of M The values are tabulated in the Table 13.1.

M	Center $X = \frac{M^2}{1-M^2}, Y = 0$	Radius $R = \left\lvert \frac{M}{1-M^2} \right\rvert$
0.2	(0.041, 0)	0.208
0.5	(0.33, 0)	0.667
1	∞	∞
2	(– 1.33, 0)	0.667
5	(– 1.041, 0)	0.208

Table 13.1

It can be observed from the Table 13.1 that :

1. As M becomes larger and larger compared to 1, the radius becomes smaller and smaller and finally circles converge to the critical point $-1 + j0$.
2. For $M > 1$, the centers are to the left of point $-1 + j0$.
3. As M becomes smaller and smaller compared to 1, the radius becomes smaller and smaller and finally circles converge to the origin.
4. For $M < 1$, the centers are to the right of origin.
5. For $M = 1$, the locus is of the points which are equidistant from origin as well as from $-1 + j0$ point. Substituting $M = 1$ in equation (1) we get a straight line of $X = \frac{-1}{2}$ parallel to Y axis, passing through point $\left(-\frac{1}{2}, 0\right)$. Thus at $M = 1$, the circle degenerates into a straight line. All the M circles are symmetrical with respect to this straight line corresponding to $M = 1$.

The family of constant M circles is shown in the Fig. 13.2.

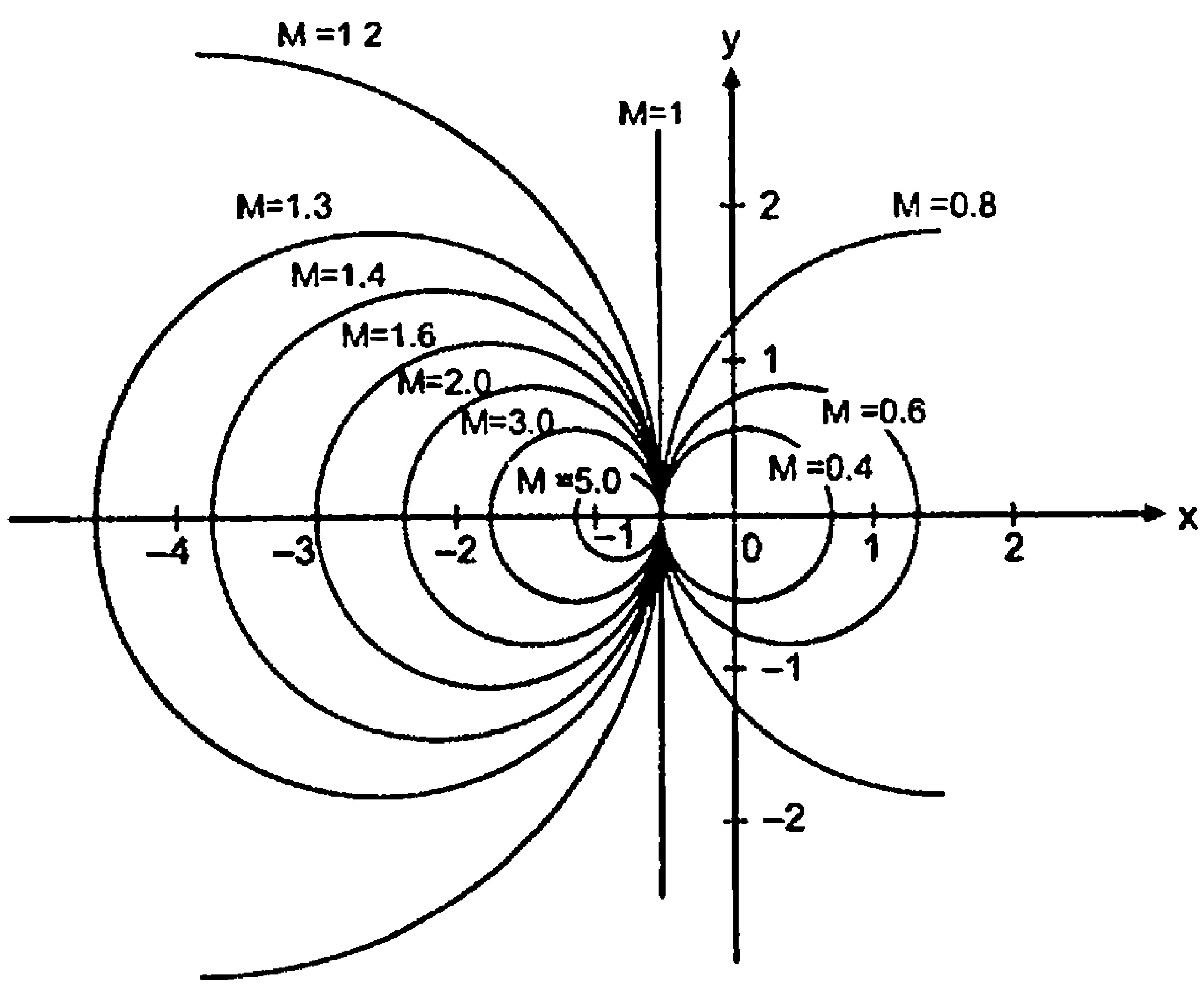


Fig. 13.2 A family of constant M circles

This is the equation of circle with centre at,

$$X = -\frac{1}{2},\ Y = \frac{1}{2N} \text{ with radius } \sqrt{\frac{1}{4}+\frac{1}{(2N)^2}}$$

For different values of N we get the family of circles. On a particular circle, the value of N i.e. phase angle α remains constant hence these circles are called constant phase loci or N circles.

Let us calculate center and radius for various values of N. The result is tabulated in the Table 13.2.

α	N = tan α	Centre $X = -\frac{1}{2}, Y = \frac{1}{2N}$	Radius $R = \sqrt{\frac{1}{4}+\frac{1}{(2N)^2}}$
-90°	∞	$\left(-\frac{1}{2}, 0\right)$	$\frac{1}{2} = 0.5$
-60°	– 1.732	$\left(-\frac{1}{2}, -0.288\right)$	0.577
-30°	– 0.5773	$\left(-\frac{1}{2}, -0.866\right)$	1
-0°	0	$\left(-\frac{1}{2}, \infty\right)$	∞
+30°	0.5773	$\left(-\frac{1}{2}, 0.866\right)$	1
+60°	1.732	$\left(-\frac{1}{2}, 0.288\right)$	0.577
+90°	∞	$\left(-\frac{1}{2}, 0\right)$	$\frac{1}{2} = 0.5$

Table 13.2

It can be observed that, the equation (1) gets satisfied for (X = 0, Y = 0) and (X = 1, Y = 0) irrespective of value of N.

Hence each circle represented by equation (1) passes through two points i.e. origin (0, 0) and point (– 1 + j0)

The family of constant N circles is shown in the Fig. 13.3.

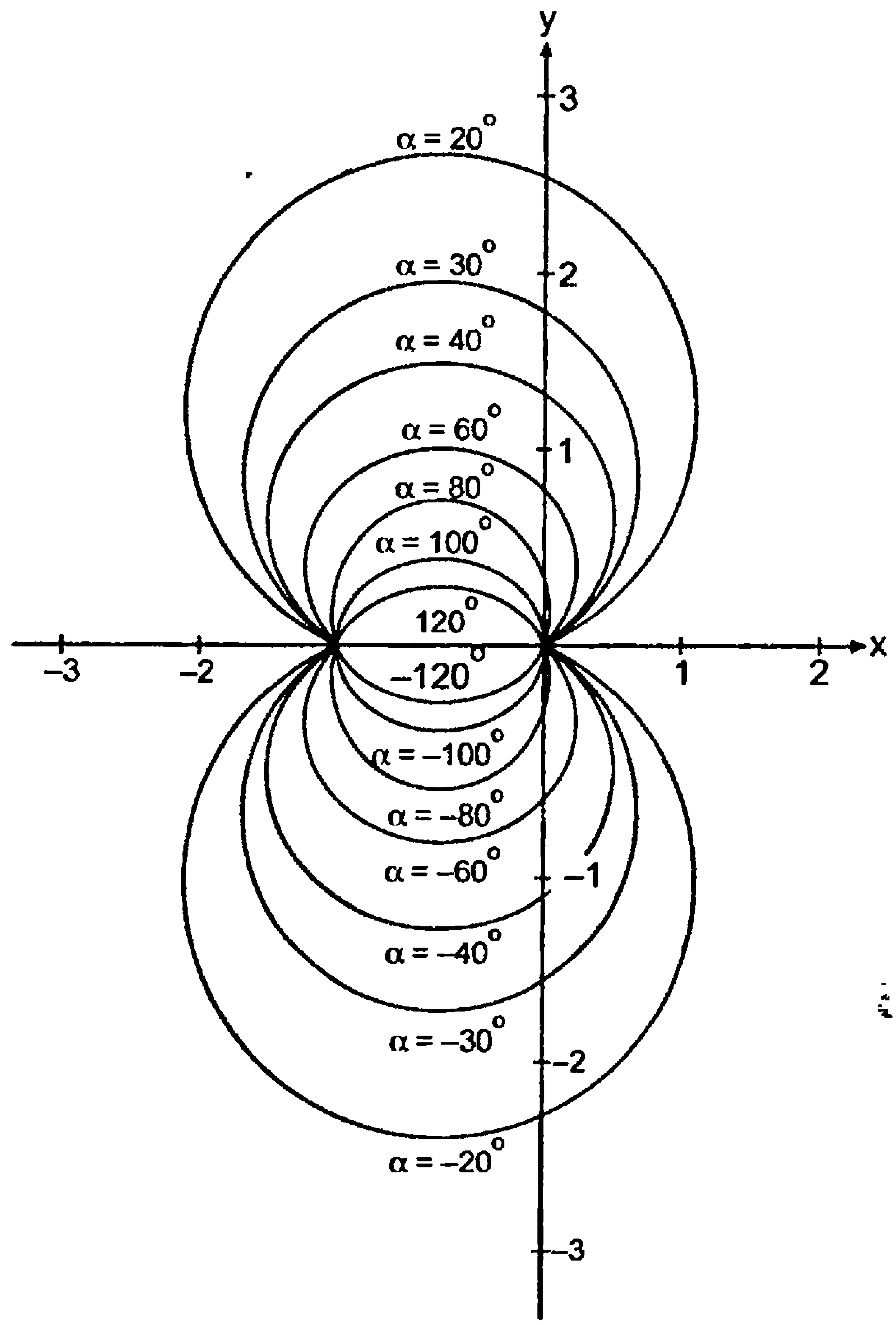


Fig. 13.3 A family of constant N circle

Note that the constant N locus for given value of α is not the entire circle but shown only an arc.

This is because the tangent of an angle remains same if ±180° or multiples of ± 180° is added to the angle.

Thus locus for $\alpha = 60^\circ$ and $\alpha = 60^\circ - 180^\circ = -120^\circ$ arcs are parts of same circle.

13.4 Use of M Circles

As stated earlier, using M and N circles, the closed loop frequency response from open loop frequency response can be obtained.

Consider that the polar plot of G(jω) i.e. open loop frequency response is super imposed on M-circles as shown in the Fig. 13.4.

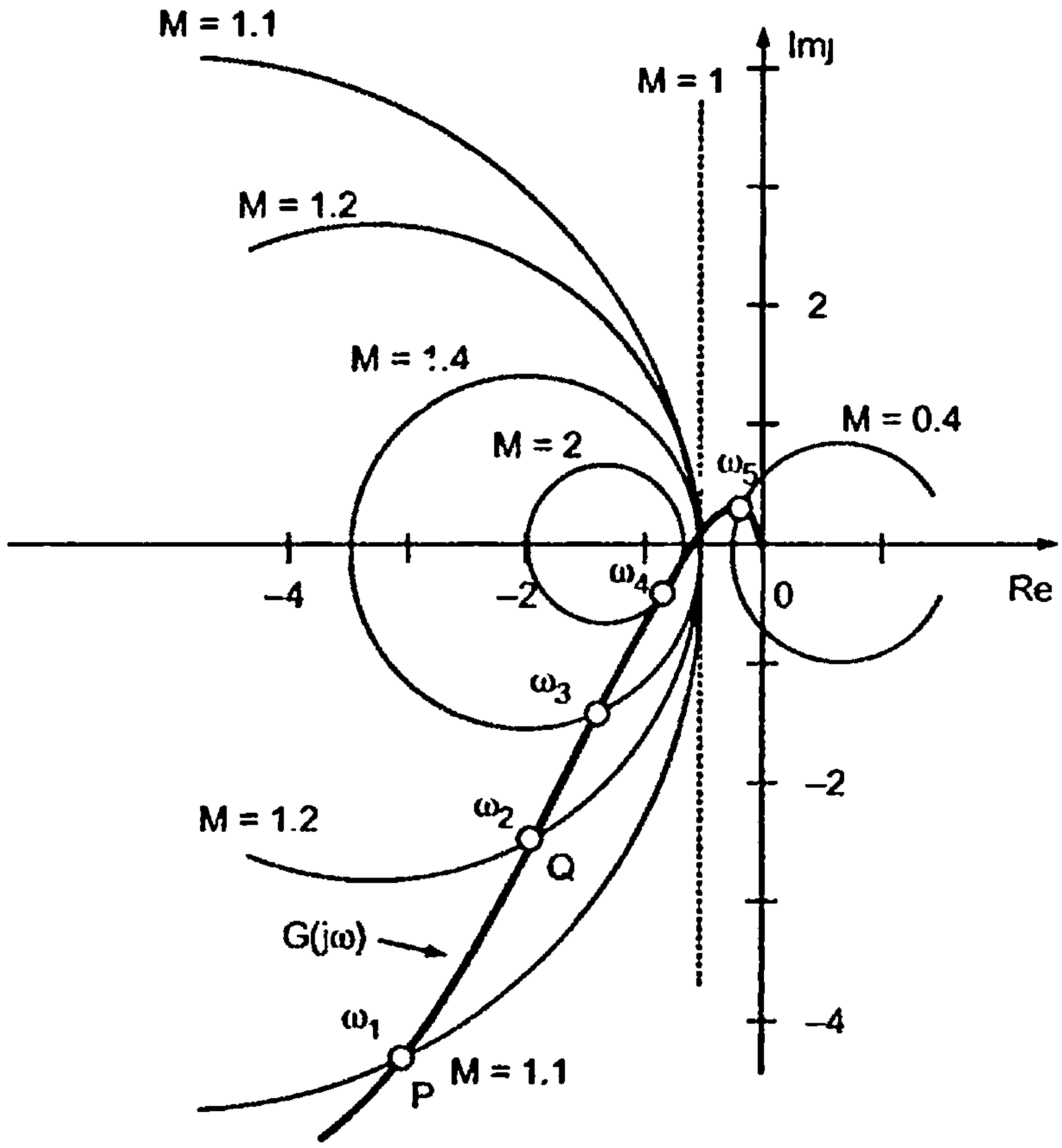


Fig. 13.4

Read the magnitude M of closed loop transfer function from the point of intersections of G(jω) polar plot with various M circles. The values of ω are known for polar plot of G(jω).

In the Fig. 13.4, at $\omega = \omega_1$ the point of intersection is P and corresponding magnitude is $M_1 = 1.1$, at $\omega = \omega_2$ the point of intersection is Q and corresponding magnitude is $M_2 = 1.2$. At $\omega = \omega_4$, the circle of M = 2 is just tangential to G(jω) plot. This represents maximum magnitude of closed loop transfer function at any frequency. This magnitude is resonant magnitude denoted as M_r and corresponding frequency is ω_r.

Thus for the G(jω) shown, $M_r = 2$ and $\omega_r = \omega_4$. Collecting this information, closed loop frequency response can be obtained as shown in the Fig. 13.5.

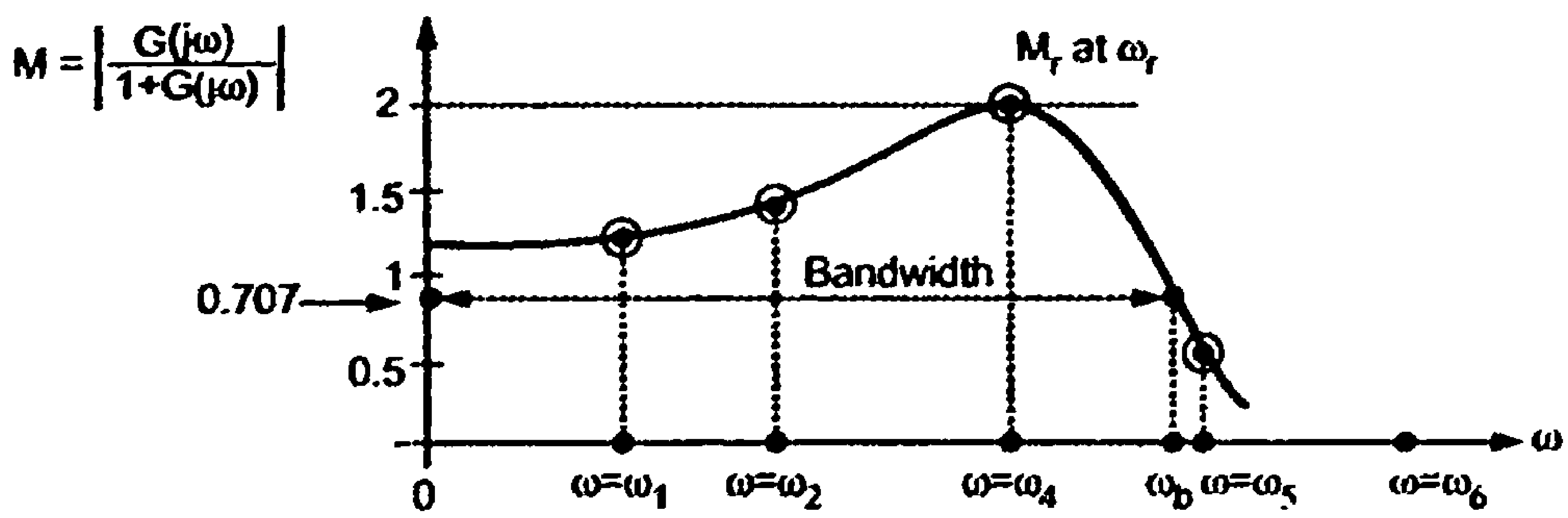


Fig.13.5 Closed loop frequency response (Magnitude plot)

13.7 Frequency Specifications from the Nichol's Chart

Consider a typical magnitude-phase plot obtained on the Nichol's chart as shown in Fig. 13.10.

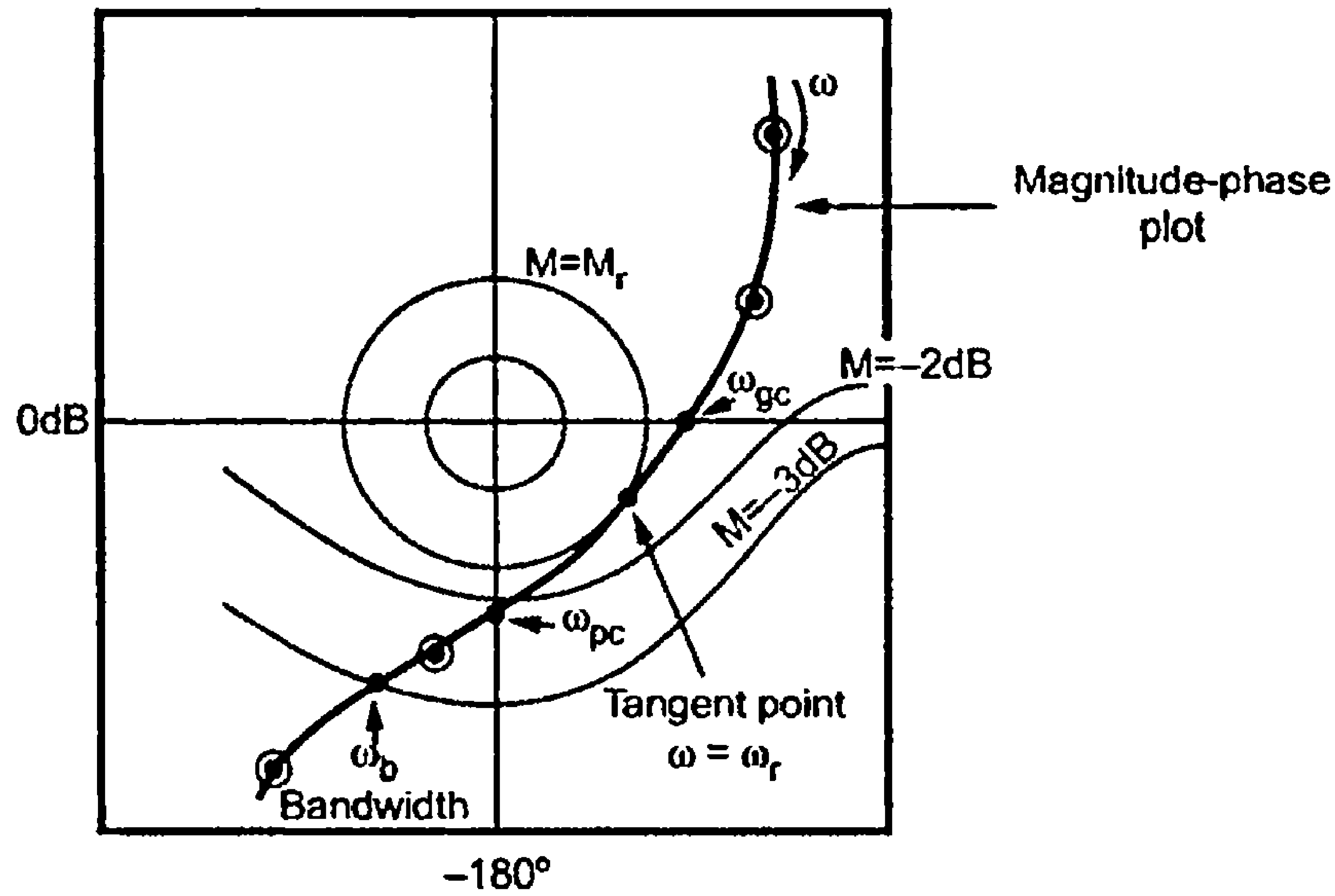


Fig. 13.10 Frequency response specifications from Nichol's chart

The M locus to which the magnitude-phase plot is tangential is resonant peak M_r for the system. The frequency at the point of tangency is resonant frequency ω_r.

The G.M. and P.M. can be obtained similar to the method used for obtaining them from magnitude-phase plot.

The frequency at which the magnitude-phase plot intersects the M = –3dB i.e. $|M| = 0.707$ locus is called bandwidth of the given system.

Thus all the frequency domain specifications M_r, ω_r, ω_b, ω_{gc}, ω_{pc}, G.M. and P.M. can be obtained by sketching open loop magnitude-phase plot on the Nichol's chart.

Example 13.1 : *The gain phase plot of the forward path transfer function G(jω)/K of a unity feedback control system is shown. Find the following performance characteristics of the system, (Refer Fig. 13.11 on next page)* ω_{gc}, ω_{pc}, *G.M., P.M.,* M_r *and* ω_r.

Solution : From the Fig. 13.11 (Refer Fig. 13.11 on next page)

a) Gain - crossover frequency (rad/sec) when K = 1

The gain crossover frequency is ω at which the magnitude of $\frac{G(j\omega)}{K}$ is 0 dB.

$$\therefore \qquad \omega_{gc} = \mathbf{8\ rad/sec}$$

b) Phase crossover frequency (rad/sec) when K = 1

The phase crossover frequency is ω at which the angle of $\frac{G(j\omega)}{K}$ is –180°.

$$\therefore \qquad \omega_{pc} = \mathbf{20\ rad/sec}$$

Example 13.3 : *Draw the plot for* $G(j\omega) = \frac{10}{s(1+0.1s)(1+0.01s)}$. *Use Nichol's chart and find the values of* ω_r, M_r *and BW of the closed loop system.*

Solution : We generate the data for plotting the M - ϕ plot i.e. gain in dB (G_{dB}) and phase in degrees against ω These values are given below.

ω	0.1	1.0	5	10	20	100	1000
G_{dB}	40	20	5	− 3	− 13.2	− 43	− 103
ϕ (deg.)	− 90.6	− 96.2	− 119.4	− 140.7	− 164.7	− 219.2	− 263.7

The $G_{dB} - \phi$ (normally called M - ϕ plot) has been drawn on the Nichol's chart.

(i) M_r is the M of that M- I ocus which is just tangential to the (M - ϕ) plot, and has highest M. In this case M_r is between + 1 dB and + 2 dB which we take as 1.5 dB. Thus M dB = 1.5 dB and M = 1.18 (since 1.5 = 20 log M giving M > 1.18)

(ii) ω_r is the value of M at the point of tangency. This may be roughly taken as approximately. Thus ω_r = 8.

(iii) Similarly BW is determined by finding crossing of the plot with − 3 dB locus. This occurs between ω = 10 and ω = 20. We may approximately take bandwidth = 15. Thus in this case in the closed loop M_r = 1.18, ω_r = 8 rad/sec and Bandwidth = 15 rad/sec. Incidentally from the plot GM = 24 dB and PM = 43°

Effect of increasing the gain :

If we increase the Bode's gain from K = 10 to K = 20 (i.e. by a factor 2) the value of M_r and ω_r should increase as we found on M circles.

The factor 2 in dB = 20 log 2 = 6 dB

The plot just shifts up by 6 dB and is shown in dotted.

From this

$$M_r = \mathbf{5\ dB\ (1.77)}$$

$$\omega_r = \mathbf{9\ rad/sec}$$

$$BW = \mathbf{22\ rad/sec}$$

$$GM = \mathbf{18\ dB} \quad \mathbf{PM = 20°}$$

(We have plotted an additional point at ω = 50, ϕ = 195.2 and G_{dB} = − 23 dB)

In fact by drawing the plot for various values of K we can select the one that we find most acceptable. (See Fig.13.13 on next page).

Finding K for a given M_r :

If we have to find the value of gain for a given M_m then the following procedure is useful.

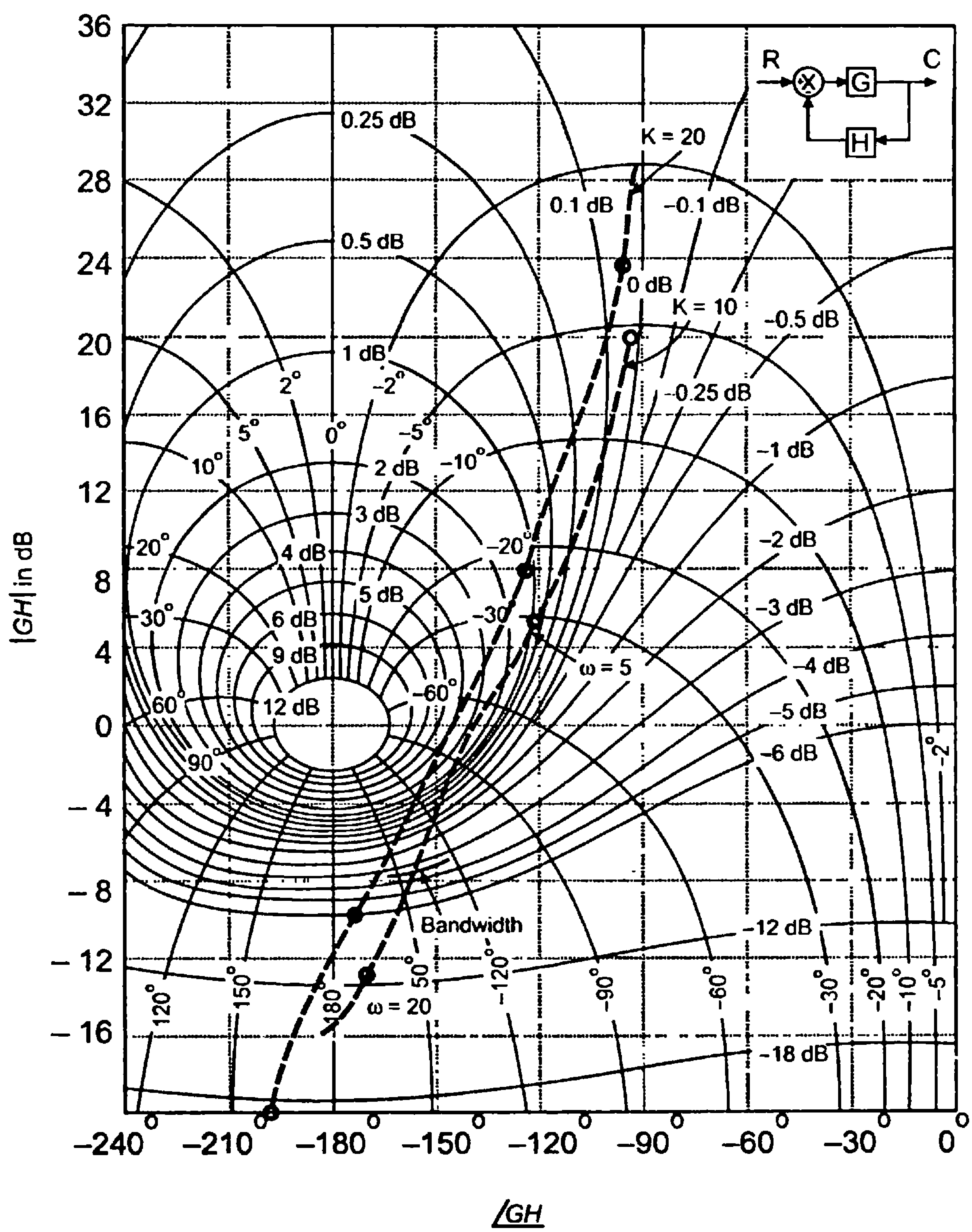


Fig. 13.13 Nichol's chart

1) We plot the M - ϕ plot for K = 1 on a chart containing Nichol's chart. (or any other convenient value of K).

2) This plot is shifted vertically up or down till the plot is just tangential to the locus of given M_r

If the shift is by x dB then it is obvious that K must be multiplied by x to achieve given M_r.

An example will clarify the idea.

Example 13.4 : *A unity feedback system has* $G(s) = \frac{K}{s(s+2)(s+8)}$. *Find the value of K to given* $M_r = 2$. *Under this condition find* ω_r, *BW, GM and PM.*

Solution : For K = 1

$$|G| = \frac{1}{\omega\sqrt{\omega^2+(2)^2} \times \sqrt{\omega^2+8^2}}$$

$$\phi = -90° - -\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{8}\right)$$

The values of G_{dB} and ϕ are calculated at ω = 0.1, 1, 2, 8, 10, 20, 50, 100, 1000 and M - ϕ is plotted on the Nichol's chart.

ω	0.1	1	2	8	10	20	50	100	1000
G (Apprx.)	0.625	0.055	0.02	0.00134	7.65×10^{-4}	1.16×10^{-4}	8×10^{-6}	1×10^{-6}	1×10^{-9}
				:					
G_{dB}	– 4.08	– 25.2	– 33.5	– 57.45	– 62.3	– 78.7	– 101.9	– 120	– 180
				:					
ϕ (deg.)	– 92.9	–123.7	– 149	2.11	– 220	– 242.4	– 258.6	– 264.2	– 269

We find that the Nichol's chart extends only from – 30 dB to + 30 dB and it will be more convenient to plot the above plot for K = 100 because the plot will be raised by 40 dB. This is shown in Fig. 13.14.

For K = 100

ω	0.1	1	2	8	10	20	50
G_{dB}	35.9	14.8	6.5	– 17.45	– 22.3	– 48.7	– 61.9
ϕ (deg.)	– 92.9	– 123.7	– 149	– 211	– 220	– 242.4	– 258.6

Now we want $M_r = 2$, [M_r dB = 20 log 2 = 6 dB]

Hence we slide the given plot down till it is just tangential to M_r = 6 dB locus. The new plot is obtained by sliding down the given plot by X_{dB} = – 5 dB (This is shown in dotted).

14 Compensation of Control Systems

14.1 Background

All the control systems are designed to achieve specific objectives. The certain requirements are defined for the control system. A good control system has less error, good accuracy, good speed of response, good relative stability, good damping which will not cause undue overshoots etc. For satisfactory performance of the system, gain is adjusted first. In practice, adjustment of gain alone cannot provide satisfactory results. This is because when gain is increased, steady state behaviour of the system improves but results into poor transient response, in some cases may even instability. In such cases it is necessary to redesign the entire system. Thus the design of control systems is a challenging job. Practically the design specifications are provided interms of precise numerical values according to which the system is designed. The set of such specifications include peak overshoot, peak time, damping ratio, natural frequency of oscillations, error coefficients, gain margin, phase margin etc.

In practice, if a system is to be redesigned so as to meet the required specifications, it is necessary to alter the system by adding an external device to it. Such a redesign or alteration of system using an additional suitable device is called **compensation of a control system.** While an external device which is used to alter the behaviour of the system so as to achieve given specifications is called **compensator**. The compensator provides whatever is missing in a system, so as to achieve required performance.

14.2 Types of Compensation

An external device, compensator can be introduced in a system anywhere as per the convenience and the requirement. Depending upon where the compensator is introduced in a system, the various types of compensation are,

1. Series compensation
2. Parallel compensation
3. Series-parallel compensation

14.2.1 Series Compensation

The compensator is a physical device whose transfer function is denoted as $G_c(s)$. If the compensator is placed in series with the forward path transfer function of the plant, the scheme is called series compensation. The arrangement is shown in the Fig. 14.1.

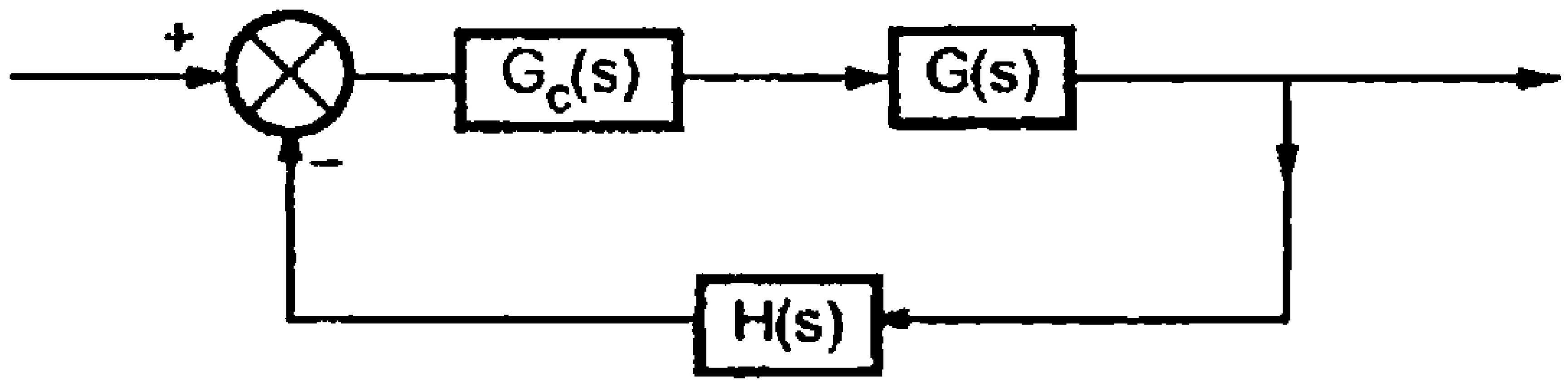


Fig. 14.1 Series compensation

This scheme is also called cascade compensation. The flow of signal in such a series scheme is from lower energy level towards higher energy level. This requires additional amplifiers to increase the gain and also to provide necessary isolation. The number of components required in series scheme is more than in parallel scheme.

14.2.2 Parallel Compensation

In some cases, the feedback is taken from some internal element and compensator is introduced in such a feedback path to provide an additional internal feedback loop. Such compensation is called feedback compensation or parallel compensation. The arrangement is shown in the Fig. 14.2.

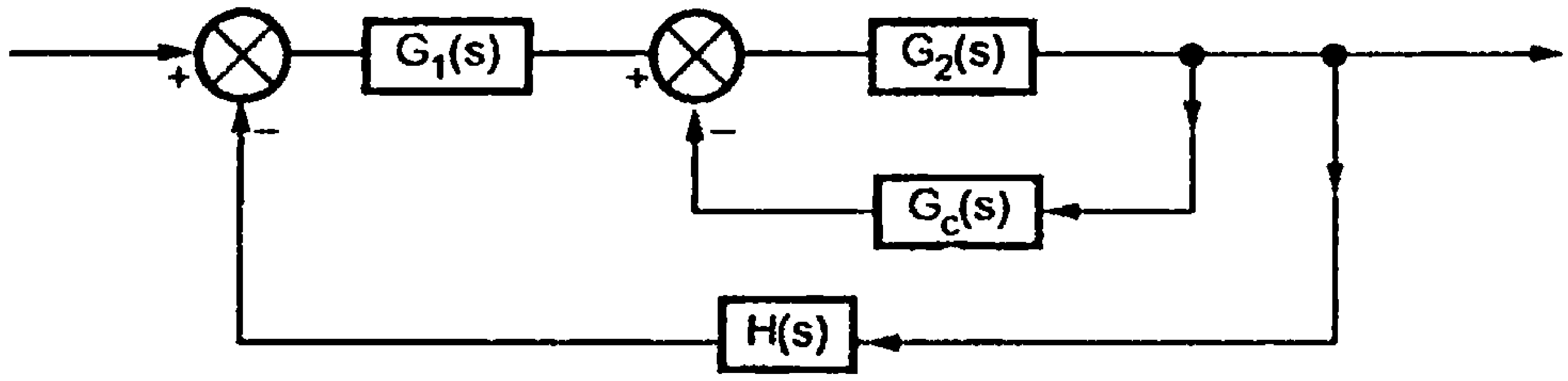


Fig. 14.2 Parallel compensation

The energy transfer in such parallel scheme is from higher energy level towards lower energy level point. Hence in such scheme the additional amplifiers are not required. Thus the number of components required are less than required in the series scheme.

14.2.3 Series-Parallel Compensation

In some cases, it is necessary to provide both types of compensations, series as well as feedback. Such a scheme is called series-parallel compensation. The arrangement is shown in the Fig. 14.3.

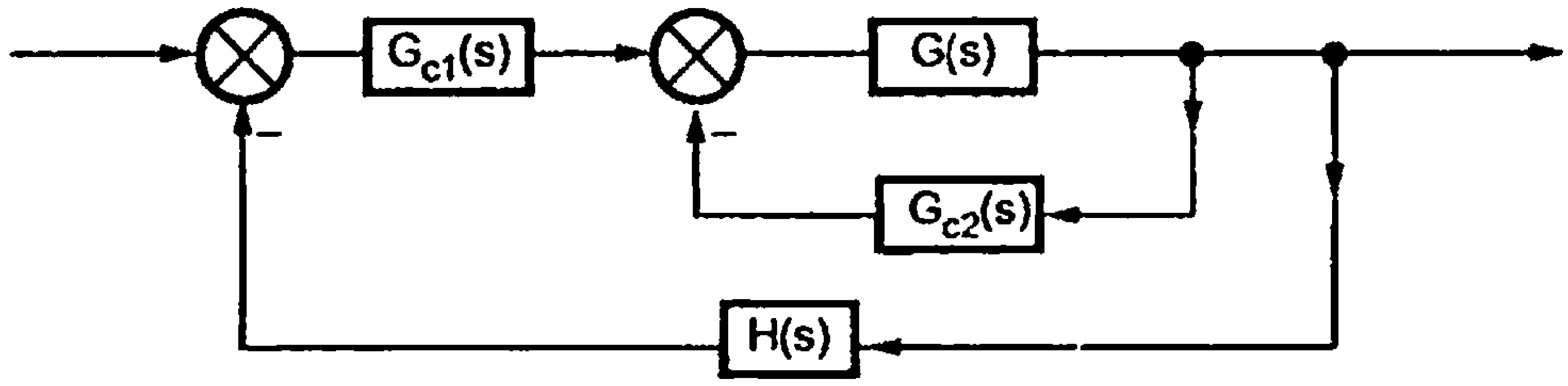


Fig. 14.3 Series-parallel compensation

The selection of the proper compensation scheme depends on the nature of the signals available in the system, the power levels at the various points, available components, the economic considerations and the designer's experience.

14.3 Compensating Networks

The compensator is a physical device. It may be an electrical network, mechanical unit, pneumatic, hydraulic or combinations of various types of devices. In this chapter we are going to study the electrical networks which are used for series compensation.

The commonly used electrical compensating networks are,

1. Lead network or Lead compensator
2. Lag network or Lag compensator
3. Lag-lead network or Lag-lead compensator

When a sinusoidal input is applied to a network and it produces a sinusoidal steady state output having a phase lead with respect to input then the network is called lead network. If the steady state output has phase lag then the network is called lag network. In the lag-lead network both phase lag and lead occur but in the different frequency regions. The phase lag occurs in the low frequency region while the phase lead occurs in the high frequency region.

Key Point: *The phase lag occurs in the low frequency region while the phase lead occurs in the high frequency region.*

Let us discuss in detail the characteristics of these three compensating networks.

14.4 Lead Compensator

Consider an electrical network which is a lead compensating network, as shown in the Fig. 14.4.

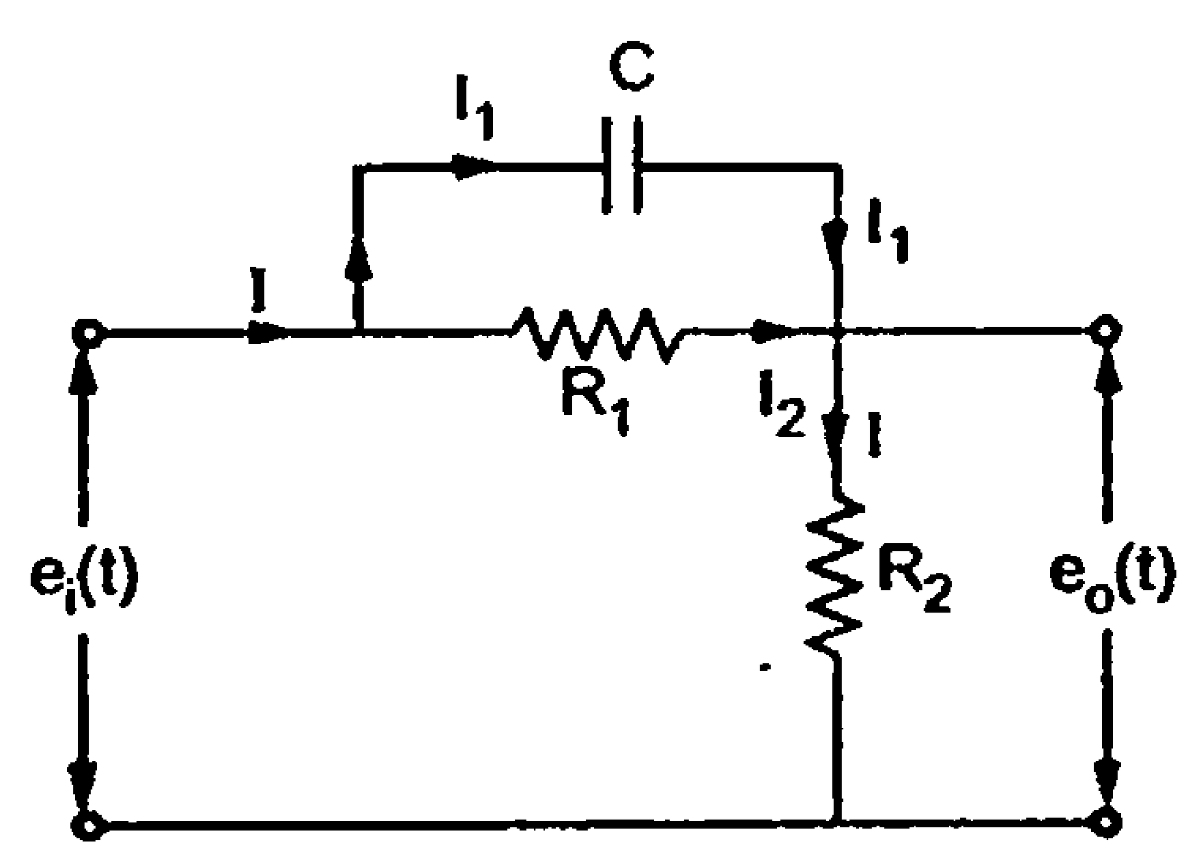


Fig. 14.4 Lead network

Let us obtain the transfer function of such an electrical lead network. Assuming unloaded circuit and applying KCL for the output node we can write,

$$I_1 + I_2 = I$$

$$C\frac{d(e_i - e_o)}{dt} + \frac{1}{R_1}(e_i - e_o) = \frac{1}{R_2}e_o$$

Taking Laplace transform of the equation,

$$sC\,E_i(s) - sC\,E_o(s) + \frac{1}{R_1}E_i(s) - \frac{1}{R_1}E_o(s) = \frac{1}{R_2}E_o(s)$$

$$\therefore \quad E_i(s)\left[sC+\frac{1}{R_1}\right] = E_o(s)\left[sC+\frac{1}{R_1}+\frac{1}{R_2}\right]$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{R_1R_2}{R_1+R_2+R_1R_2sC}\cdot\frac{1+sCR_1}{R_1}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{\left(s+\frac{1}{R_1C}\right)}{s+\frac{(R_1+R_2)}{R_1R_2C}} = \frac{\left(s+\frac{1}{R_1C}\right)}{\left[s+\frac{1}{\left(\frac{R_2}{R_1+R_2}\right)R_1C}\right]}$$

This is generally expressed as,

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}}$$

Where, $T = R_1C$ and $\alpha = \frac{R_2}{R_1+R_2} < 1$

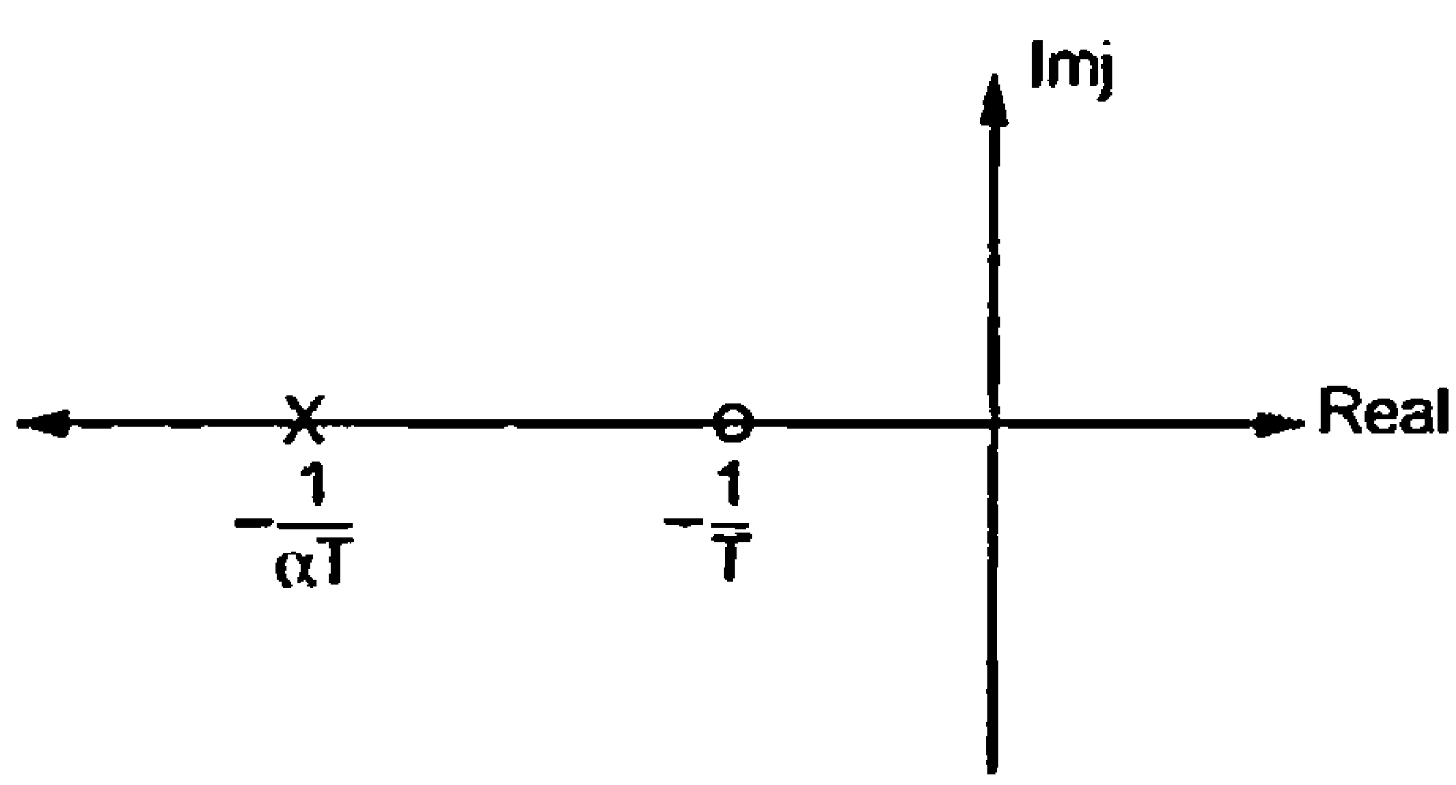


Fig. 14.5

The lead compensator has zero at $s = -\frac{1}{T}$ and a pole at $s = -\frac{1}{\alpha T}$.

As $0 < \alpha < 1$, the zero is always located to the right of the pole. The pole zero plot is shown in the Fig. 14.5. The minimum value of α is generally taken as 0.05.

14.4.1 Maximum Lead Angle ϕ_m and α

Let us see what is the maximum lead angle ϕ_m which lead compensator can provide and at what frequency it provides this angle.

$$\frac{E_o(s)}{E_i(s)} = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = \frac{\alpha(1+Ts)}{(1+\alpha Ts)}$$

Replace s by $j\omega$

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{\alpha(1+j\omega T)}{(1+j\omega\alpha T)}$$

$$\therefore \quad \tan\phi = \frac{\omega T - \alpha\,\omega T}{1+\omega T.\,\alpha\,\omega T} = \frac{\omega T(1-\alpha)}{1+\omega^2 T^2\,\alpha}$$

At $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$,

$$\tan\phi_m = \frac{(1-\alpha)}{\sqrt{\alpha}\,(1+1)} = \frac{1-\alpha}{2\sqrt{\alpha}} \quad \text{... (4)}$$

$$\therefore \quad \boxed{\sin\phi_m = \frac{1-\alpha}{1+\alpha}} \quad \text{...(5)}$$

The equation (5) is also used to get the relation between α and the maximum lead angle ϕ_m.

14.4.2 Polar Plot of Lead Compensator

From equation (1) and (2) it is easy to obtain polar plot of lead compensator.

When $\omega = 0$, $M = \alpha$ and $\phi = 0°$

When $\omega = \infty$, $M = 1$ and $\phi = 0°$

So both the points, starting as well as terminating points are on positive real axis. For any value of ω between 0 to ∞, magnitude is always positive while for $\alpha < 1$, $\tan^{-1}\omega T > \tan^{-1}\alpha\,\omega T$ hence ϕ will be always positive. Hence all the points of polar plot are always in the first quadrant of complex plane.

Thus the polar plot is as shown in the Fig. 14.6.

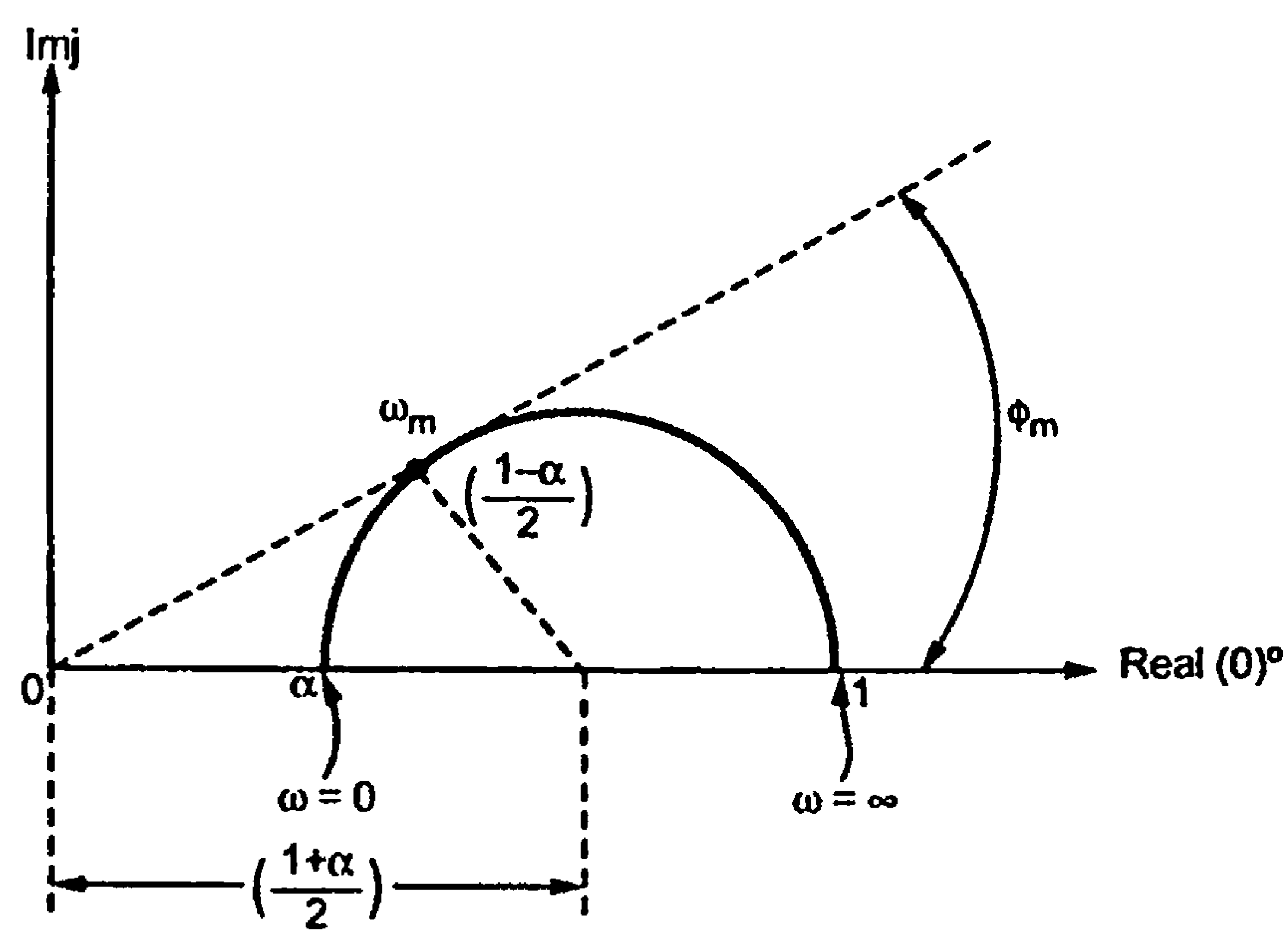


Fig. 14.6 Polar plot of lead compensator

$$\therefore \quad 20 = 10\text{ K}$$

$$\therefore \quad \text{K} = 2$$

$$\therefore \quad G_1(s) = \frac{20}{s(s+1)}$$

Step 2 : Sketch the Bode plot of $G_1(s)$ which is shown in the Fig. 14.10.

Factors : 20 Log 20 = 26 dB

1 pole at origin

1 simple pole with corner frequency $\omega_c = 1$.

Thus line of slope – 20 dB/dec till $\omega_c = 1$ and line of slope –40 dB/dec from 1 onwards.

Phase angle table : $G_1(j\omega) = \dfrac{20}{j\omega\,(1+j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	ϕ_R
0.1	– 90°	– 5.71°	– 95.71°
1	– 90°	– 45°	– 135°
2	– 90°	– 63.4°	– 153.4°
10	– 90°	– 84.2°	– 174.2°
∞	– 90°	– 90°	– 180°

From the Fig. 14.10,

$$\phi_1 = \text{P.M.} = 15°, \quad \omega_{gc} = 4\text{ rad/sec}, \quad \text{G.M.} = +\infty\text{ dB}$$

Step 3 : $\phi_s = 50°$

$$\therefore \quad \phi_m = \phi_s - \phi_1 + \varepsilon, \qquad \text{let } \varepsilon = 5°$$

$$= 50° - 15° + 5°$$

$$= 40°$$

Step 4 : $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$

$$\therefore \quad \sin 40° = \frac{1-\alpha}{1+\alpha} = 0.6427$$

$$\therefore \quad 1-\alpha = 0.6427\,(1+\alpha)$$

$$\therefore \quad \alpha = 0.2174$$

Choose $\alpha = 0.21$

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$\therefore \quad \frac{1}{T} = 2.7495$$

Step 6 : Two corner frequencies of the lead compensator are,

$$\omega_{C1} = \frac{1}{T} = 2.7495 \quad \text{and} \quad \omega_{C2} = \frac{1}{\alpha T} = 13.09$$

Step 7 :

$$K = K_c \alpha$$

$$\therefore \quad K_c = \frac{K}{\alpha} = \frac{2}{0.21} = 9.523$$

Step 8 :

$$G_c(s) = 9.523 \times 0.21 \frac{(1+0.3637\,s)}{(1+0.0763\,s)}$$

$$= \frac{2(1+0.3637\,s)}{(1+0.0763\,s)}$$

This is the designed lead compensator.

$$\therefore \quad \boxed{G_c(s)\,G(s) = \frac{20\,(1+0.3637\,s)}{s(1+s)\,(1+0.0763\,s)}}$$

Draw the Bode plot for this transfer function and obtain the values of G.M. and P.M. The plot is drawn on the same semilog paper shown in the Fig. 14.10.

Phase angle table for the compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	$+\tan^{-1} 0.3637\,\omega$	$-\tan^{-1} 0.0763\,\omega$	ϕ_R
0.1	– 90°	– 5.71°	+ 2.08°	– 0.43°	– 94.06°
1	– 90°	– 45°	+ 20°	– 4.36°	– 119.36°
2	– 90°	– 63.4°	+ 36°	– 8.67°	– 126.07°
10	– 90°	– 84.2°	+ 74°	– 37.3°	– 137.5°
100	– 90°	– 89.4°	+ 88°	– 82.53°	– 173.9°

For magnitude plot, K = 20

$$\therefore \quad 20 \text{ Log } 20 = 26 \text{ dB}$$

One pole at origin, straight line of slope – 20 dB/dec.

ω_{C1} = 1, slope becomes – 40 dB/dec due to simple pole.

$\omega_{C2} = \frac{1}{T} = 2.75$, slope becomes – 20 dB/dec due to simple zero.

$\omega_{C3} = \frac{1}{\alpha T} = 13.09$, slope becomes – 40 dB/dec due to simple pole.

From the Fig. 14.10, for compensated system

ω_{gc}	=	7 rad/sec
P.M.	=	+ 50°
G.M.	=	+ ∞ dB

Thus the compensated system satisfies all the specifications.

14.5 Lag Compensator

Consider an electrical network which is a lag compensating network, as shown in the Fig. 14.11.

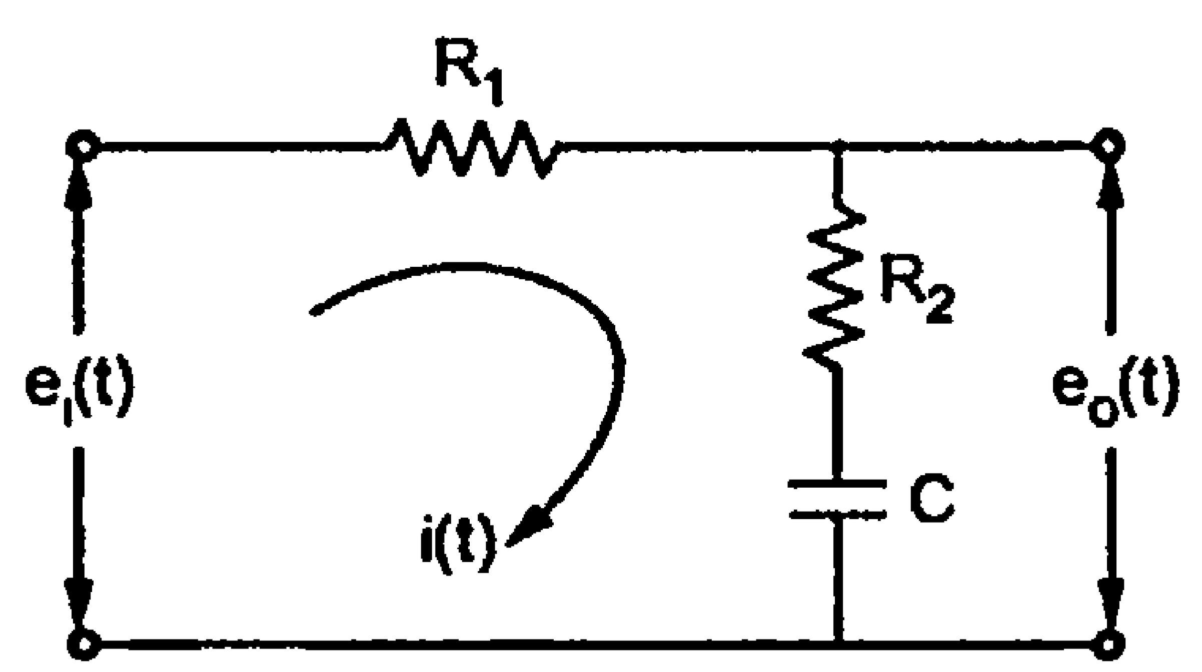


Fig. 14.11 Lag network

Let us obtain the transfer function of such an electrical lag network.

Assuming unloaded circuit and applying KVL to the loop we can write,

$$e_i(t) = i(t)\,R_1 + i(t)\,R_2 + \frac{1}{C}\int i(t)\,dt$$

Taking laplace transform of the equation,

$$E_i(s) = I(s)\left[R_1 + R_2 + \frac{1}{sC}\right] \quad ...(1)$$

Now the output equation is,

$$e_o(t) = i(t)\,R_2 + \frac{1}{C}\int i(t)\,dt$$

Taking laplace transform,

$$E_o(s) = I(s)\left[R_2 + \frac{1}{sC}\right] \quad ...(2)$$

Substituting I(s) from equation (2) in (1) we get,

$$E_i(s) = \frac{E_o(s)}{\left[R_2 + \frac{1}{sC}\right]}\left[R_1 + R_2 + \frac{1}{sC}\right]$$

While the phase angle is given by,

$$\therefore \quad \boxed{\phi = \tan^{-1}\omega T - \tan^{-1}\omega\beta T} \qquad \ldots(6)$$

The equation is exactly similar to the lead network, only $\beta > 1$. To find ϕ_m, let us find ω_m which maximises ϕ.

$$\therefore \quad \frac{d\phi}{d\omega} = 0$$

$$\therefore \quad \frac{d}{d\omega}[\tan^{-1}\omega T - \tan^{-1}\omega\beta T] = 0$$

Solving we get, $\boxed{\omega_m = \frac{1}{T\sqrt{\beta}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\beta T}}}$

This is the frequency at which the phase lag is at its maximum.

The two corner frequencies of lag compensator are,

$$\boxed{\omega_{c1} = \frac{1}{T} \text{ and } \omega_{c2} = \frac{1}{\beta T}}$$

Thus ω_m **is the geometric mean of the two corner frequencies.**

Key Point: *Note that the primary function of a lag compensator is to provide attenuation in the high frequency range to give a system sufficient phase margin.* ***The phase lag angle does not play a role in the lag compensation.***

14.5.2 Polar Plot of Lag Compensator

From the equations (5) and (6), polar plot of lag compensator can be achieved.

When $\omega = 0$, $M = 1$ and $\phi = 0°$

When $\omega = \infty$, $M = \frac{1}{\beta}$ and $\phi = 0°$

So both the points, starting and terminating are on positive real axis.

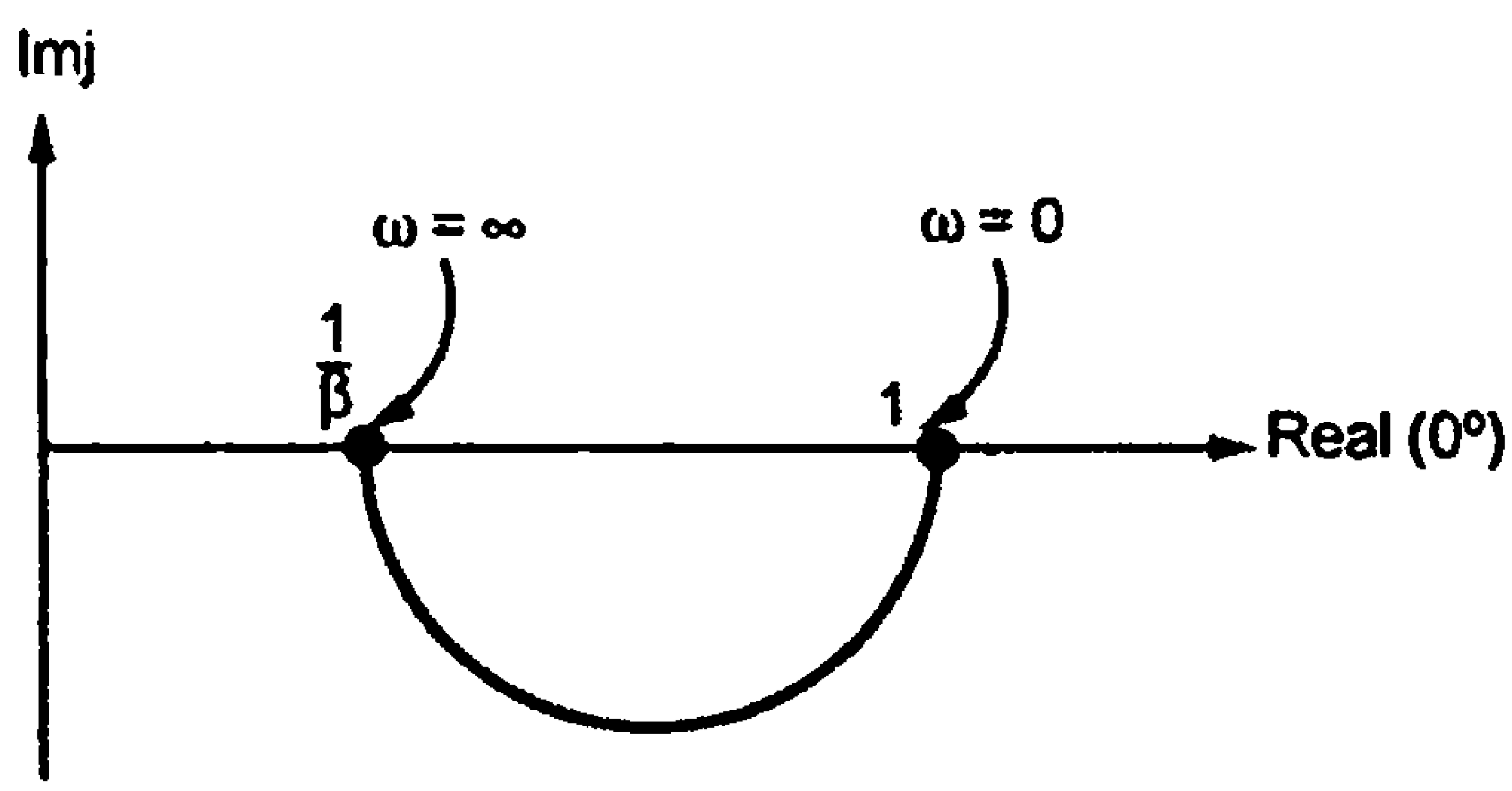


Fig. 14.13 Polar plot of lag compensator

For any value of ω between 0 to ∞, the magnitude is always positive while for $\beta > 1$, $\tan^{-1}\omega T < \tan^{-1}\omega\beta T$. Thus the resultant ϕ will be always negative for any value of ω between 0 to ∞. Hence all the points of polar plot are always in the third quadrant of the complex plane.

Thus the polar plot is as shown in the Fig. 14.13.

14.5.3 Bode Plot of Lag Compensator

The corner frequencies of the lag compensator are,

$$\omega_{C1} = \frac{1}{\beta T} \text{ for a pole at } s = -\frac{1}{\beta T}$$

$$\omega_{C2} = \frac{1}{T} \text{ for a zero at } s = -\frac{1}{T}$$

The pole is more dominating than zero. The transfer function of basic lag network is,

$$T(s) = \frac{1+sT}{1+s\beta T}$$

Thus as K = 1, the initial line is 0 dB line till $\omega_{C1} = \frac{1}{\beta T}$ where pole occurs. From $\omega_{C1} = \frac{1}{\beta T}$ to $\omega_{C2} = \frac{1}{T}$ there is a line of slope – 20 dB/dec. From $\omega_{C2} = \frac{1}{T}$ onwards again the net slope is zero due to existence of zero with $\omega_{C2} = \frac{1}{T}$.

The Bode plot is shown in the Fig. 14.14.

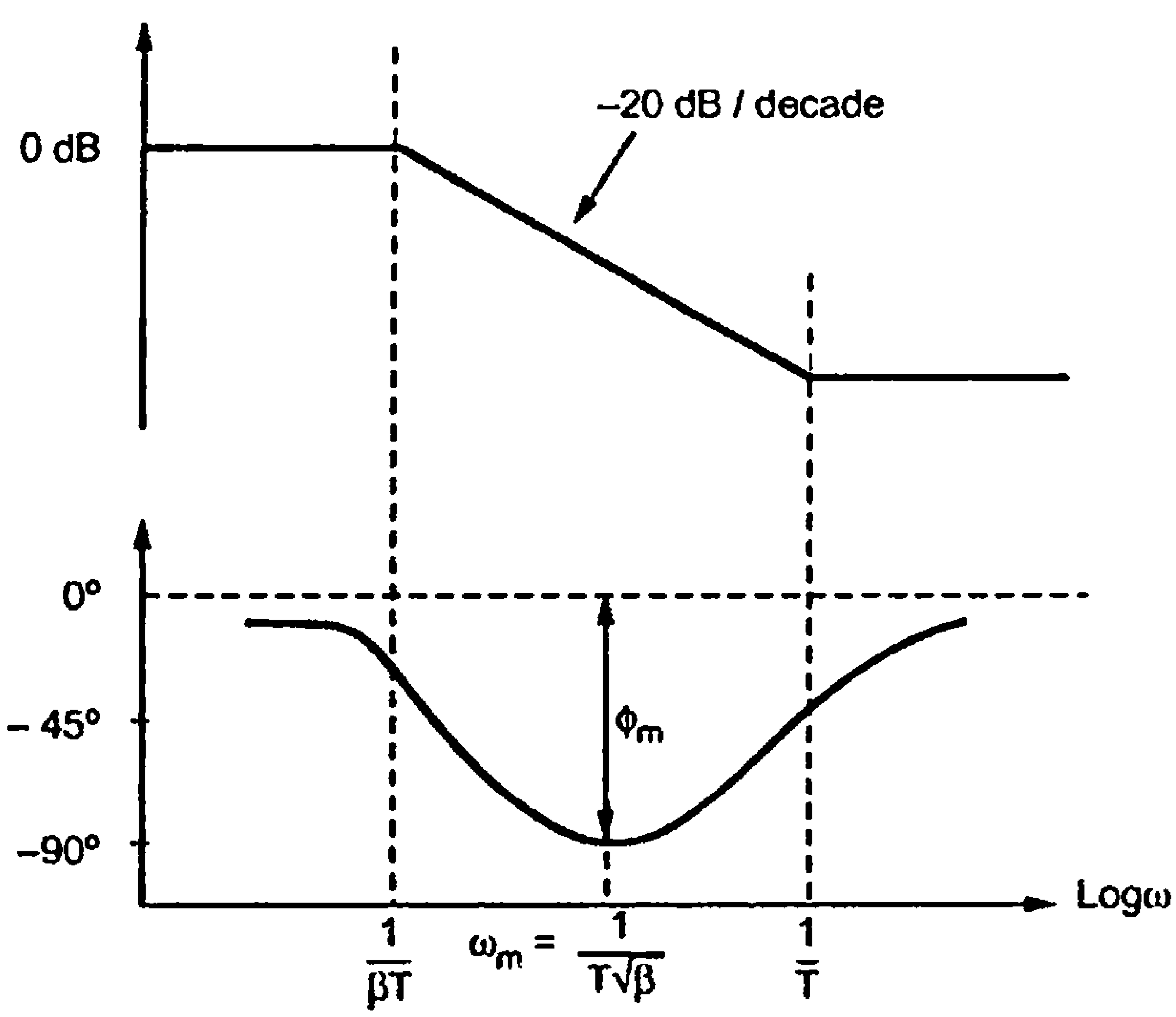


Fig. 14.14 Bode plot of lag compensator

The other corner frequency for the lag compensator is,

$$\omega_{C1} = \frac{1}{\beta T}$$

Step 7 : Thus once transfer function of the lag compensator is known, draw the Bode plot of compensated system and check the specifications. If specifications are not satisfied, repeat the design by modifying the pole-zero locations of the compensator till a satisfactory result is obtained.

14.5.5 Effects and Limitations of Lag Compensator

The various effects and limitations of lag compensator are,

1. Lag compensator allows high gain at low frequencies thus it is basically a lowpass filter. Hence it improves the steady state performance.
2. In lag compensation, the attenuation characteristics is used for the compensation. The phase lag characteristics is of no use in the compensation.
3. The attenuation due to lag compensator shifts the gain cross-over frequency to a lower frequency point. Thus the bandwidth of the system gets reduced.
4. Reduced bandwidth means slower response. Thus rise time and settling time are usually longer. The transient response lasts for longer time.
5. The system becomes more sensitive to the parameter variations.
6. Lag compensator approximately acts as proportional plus integral controller and thus tends to make system less stable.

Example 14.2 : *For a certain system,*

$$G(s) = \frac{0.025}{s(1+0.5s)(1+0.05s)}$$

Design a suitable lag compensator to give,

Velocity error constant = 20 sec^{-1}

Phase margin = 40°

Solution : Step 1 : Assume

$$G_1(s) = KG(s) = \frac{0.025K}{s(1+0.5s)(1+0.05s)}$$

$$\therefore \quad K_v = 20 = \lim_{s\to 0} s\,G_1(s)\,G_c(s)$$

$$\therefore \quad 20 = \lim_{s\to 0} s \cdot \frac{(1+Ts)}{(1+\beta Ts)} \cdot \frac{0.025K}{s(1+0.5s)(1+0.05s)}$$

$$\therefore \quad 20 = 0.025\,K$$

$$\therefore \quad K = 800$$

$\therefore \quad -20 \text{ Log } \beta = -23, \quad -\text{ve as down shift}$

$\therefore \quad \beta = 14.12$

Assuming $\beta = 14.$

Step 6 : Choose $\omega_{C2} = \frac{\omega_2}{10}$

$$= \frac{1.5}{10} = 0.15 \text{ rad/sec}$$

Now $\omega_{C2} = \frac{1}{T} = 0.15$

$\therefore \quad T = 6.66$

and $\omega_{C1} = \frac{1}{\beta T} = 0.0107$

Step 7 : Thus the lag compensator is,

$$G_c(s) = \frac{(1+6.66\, s)}{(1+93.46\, s)}$$

Thus the transfer function of the compensated system is,

$$\therefore \quad \boxed{G_c(s)G(s) = \frac{20(1+6.66\, s)}{s\,(1+0.5\, s)\,(1+0.05\, s)\,(1+93.46\, s)}}$$

To check for the specifications, draw the Bode plot of compensated system as shown in the Fig. 14.16. It is drawn on separate semilog paper as the starting frequency required for this system is 0.001.

Factors : 20 Log 20 = 26 dB

1 pole at origin

$$\omega_{C1} = \frac{1}{93.46} = 0.01, \text{ simple pole}$$

$$\omega_{C2} = \frac{1}{6.66} = 0.15, \text{ simple zero}$$

$$\omega_{C3} = \frac{1}{0.5} = 2, \text{ simple pole}$$

$$\omega_{C4} = \frac{1}{0.05} = 20, \text{ simple pole}$$

$$G_c(j\omega)\, G(j\omega) = \frac{20(1+6.66\, j\omega)}{j\omega\,(1+0.5 j\omega)\,(1+0.05\, j\omega)\,(1+93.46\, j\omega)}$$

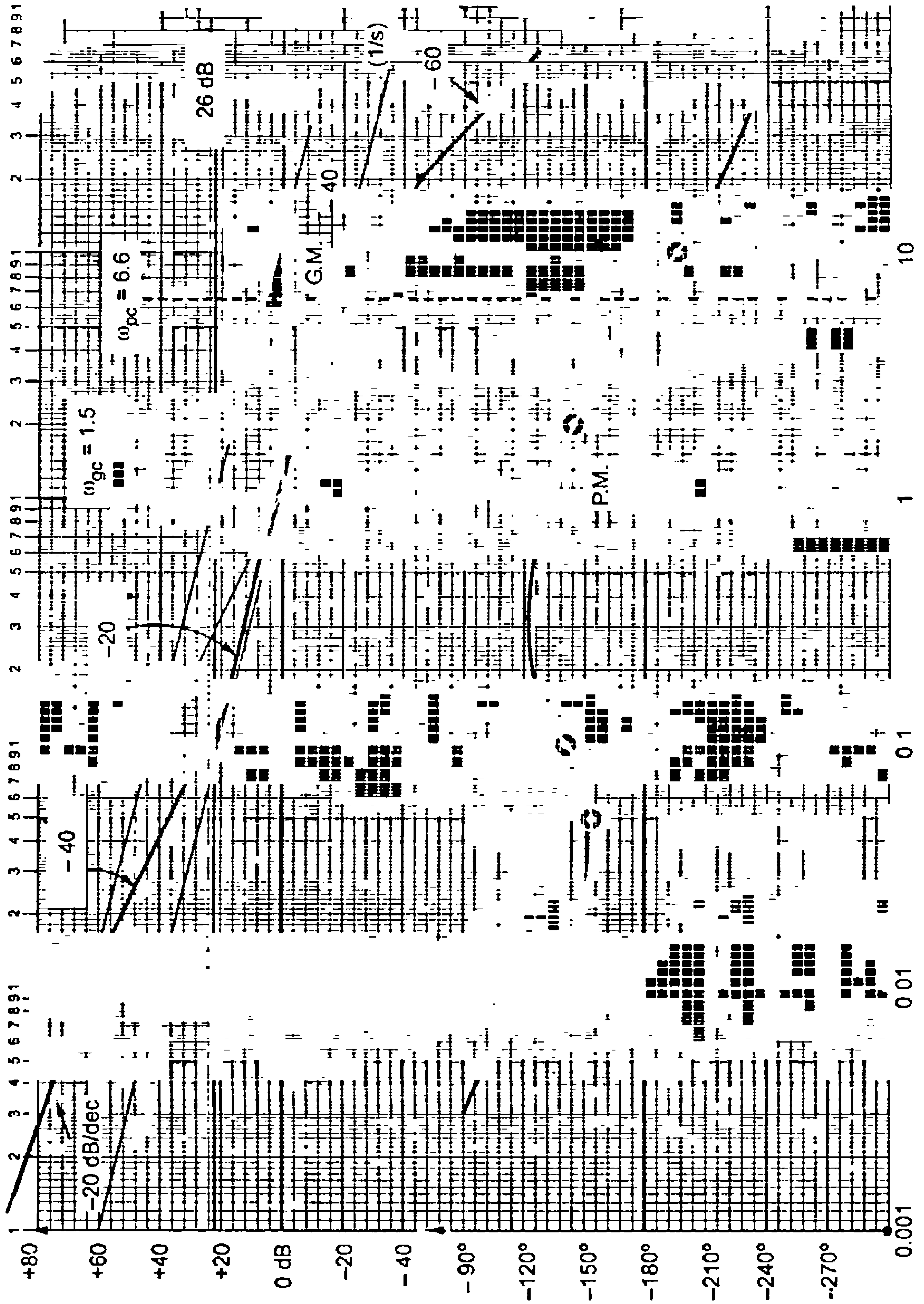


FIG. 14.16 Bode plot of compensated system in Ex.14.2

Phase angle table for compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1} 93.46\ \omega$	$+\tan^{-1} 6.66\ \omega$	$-\tan^{-1} 0.5\ \omega$	$-\tan^{-1} 0.05\ \omega$	ϕ_R
0.01	– 90°	– 43.06°	+ 3.81°	– 0.28°	– 0.028°	– 129.5°
0.05	– 90°	– 77.92°	+ 18.41°	– 1.43°	– 0.14°	– 151.08°
0.1	– 90°	– 84°	+ 33.66°	– 2.86°	– 0.28°	– 143.48°
1	– 90°	– 89.38°	+ 81.46°	– 26.56°	– 2.86°	– 127.34°
2	– 90°	– 89.7°	+ 85.7°	– 45°	– 5.71°	– 144.7°
10	– 90°	– 90°	+ 89°	– 78.6°	– 26.56°	– 195.5°

From the Fig. 14.16, the various specifications are,

ω_{gc} = 1.5 rad/sec

ω_{pc} = 6.6 rad/sec

G.M. = + 24 dB

P.M. = 48°

Thus the compensated system satisfies all the specifications.

14.6 Lag-Lead Compensator

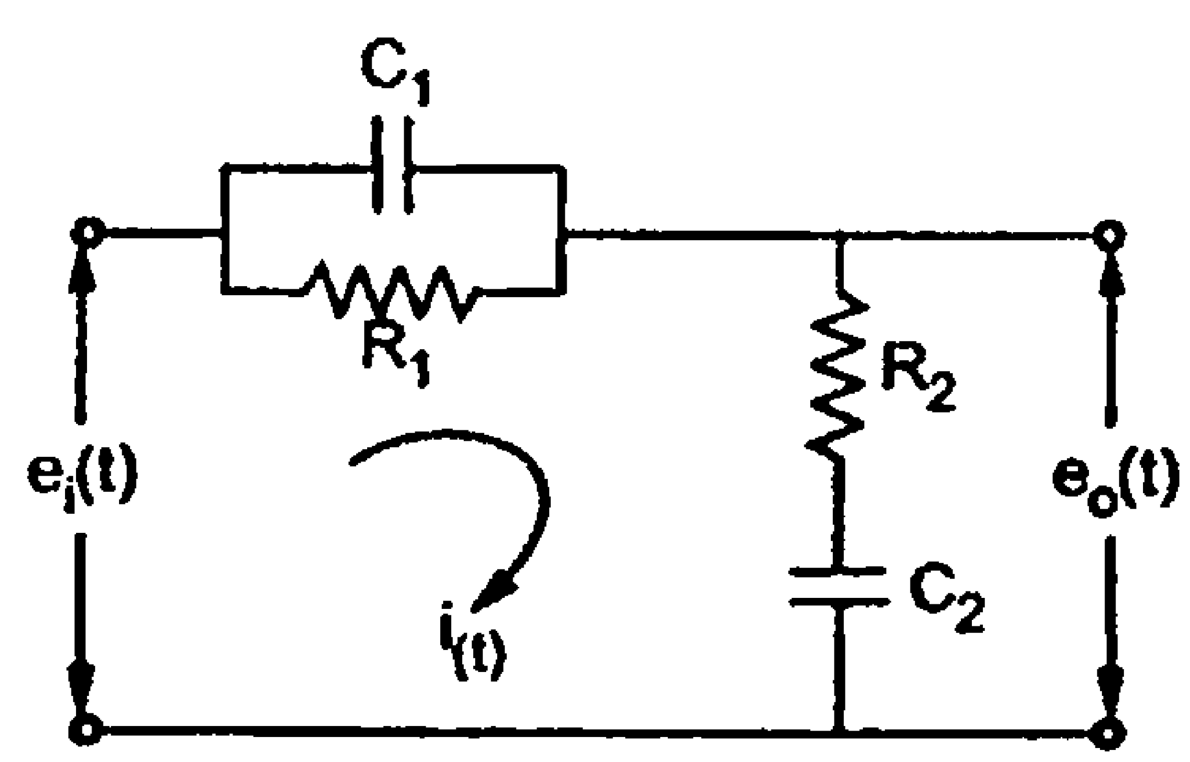


Fig. 14.17 Lag-lead network

A combination of a lag and lead compensators is nothing but a lag-lead compensator. Consider an electrical network which is lag-lead network, as shown in the Fig. 14.17.

Let us obtain the transfer function of the electrical lag-lead network.

Now sum of the current through R_1 and C_1 is nothing but current i(t).

$$\therefore \quad \frac{e_i - e_o}{R_1} + C_1 \frac{d(e_i - e_o)}{dt} = i(t)$$

Taking Laplace transform we get,

$$\frac{1}{R_1} E_i(s) - \frac{1}{R_1} E_o(s) + sC_1 E_i(s) - sC_1 E_o(s) = I(s) \quad \ldots(1)$$

It also can be expressed as,

$$\frac{E_o(s)}{E_i(s)} = \frac{(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)(1+T_2\beta s)} \quad \text{... (4)}$$

Where, $\beta > 1$

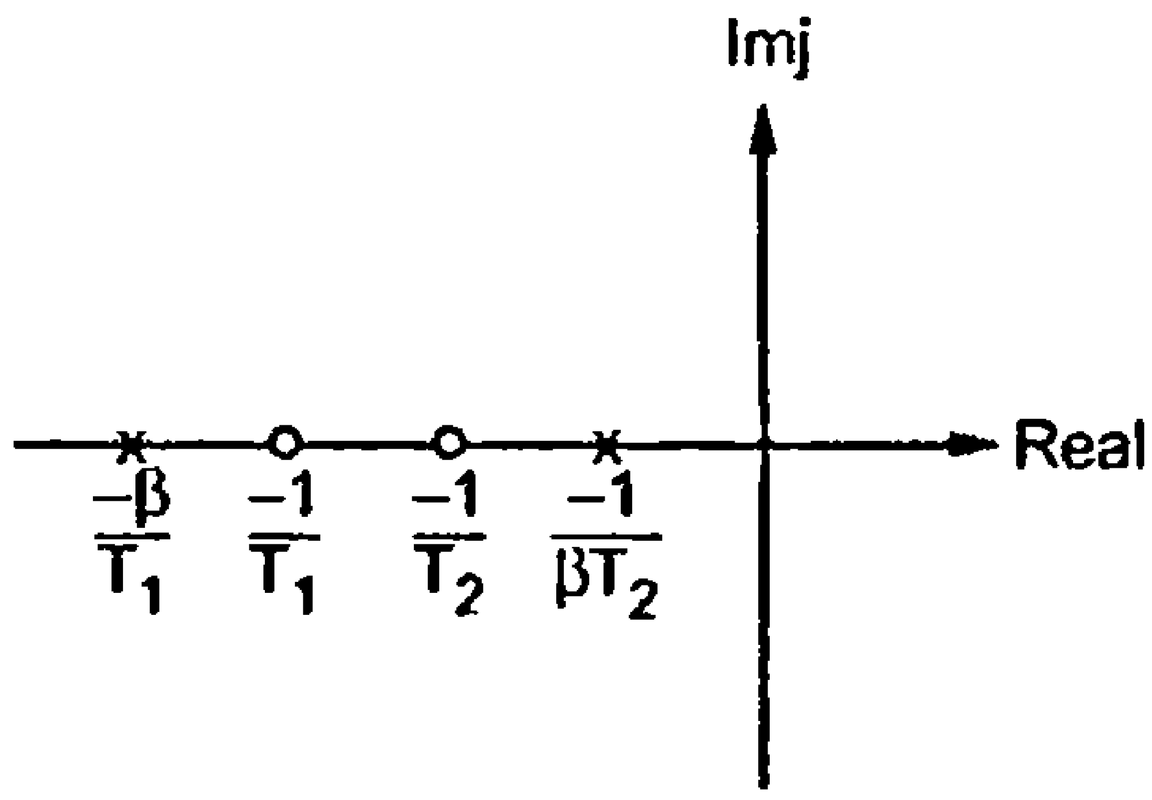


Fig. 14.18

The phase lead portion involving T_1, adds phase lead angle while the phase lag portion involving T_2 provide attenuation near and above the gain cross-over frequency.

The pole are $s = -\frac{\beta}{T_1}, -\frac{1}{\beta T_2}$ while the zeros are at $s = -\frac{1}{T_1}, -\frac{1}{T_2}$. The pole-zero plot for the lag-lead compensator is shown in the Fig. 14.18.

14.6.1 Polar Plot of Lag-Lead Compensator

The transfer function of Lag-lead compensator in the frequency domain can be obtained as,

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{(1+T_1 j\omega)(1+T_2 j\omega)}{\left(1+\frac{T_1}{\beta}j\omega\right)(1+jT_2\beta\omega)} \quad \text{... (5)}$$

The magnitude of the transfer functions,

$$M = \frac{\sqrt{1+\omega^2 T_1^2}\cdot\sqrt{1+\omega^2 T_2^2}}{\sqrt{1+\frac{T_1^2\omega^2}{\beta^2}}\cdot\sqrt{1+\beta^2 T_2^2\omega^2}} \quad \text{... (6)}$$

The phase angle is given by,

$$\phi = +\tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 - \tan^{-1}\frac{T_1\omega}{\beta} - \tan^{-1}\beta T_2\omega \quad \text{... (7)}$$

Now for $\omega = 0$, $M = 1$ and $\phi = 0^\circ$

While for $\omega = \infty$, $M = 1$ and $\phi = 0^\circ$

Thus the starting and terminating points of polar plot is same as $1 \angle 0^\circ$. But there is certain frequency ω_1 upto which it acts as lag network giving polar plot in third quadrant. While for $\omega > \omega_1$ till ∞ it acts as lead network giving polar plot in the first quadrant. Hence the entire polar plot, is circular in nature. The polar plot is as shown in the Fig. 14.19.

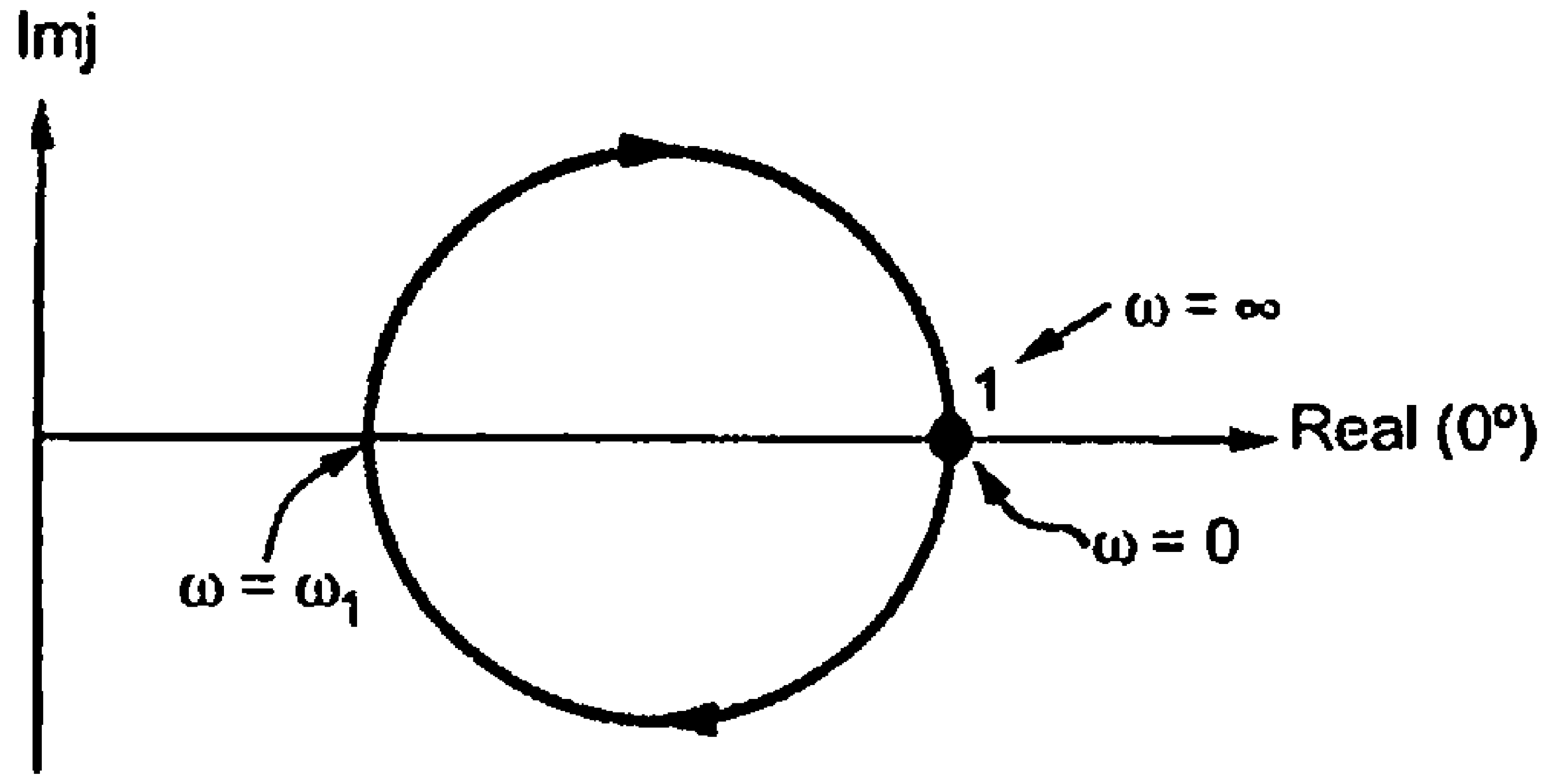


Fig. 14.19 Polar plot of lag-lead compensator

In the lag-lead network the frequency ω_1 is given by,

$$\omega_1 = \frac{1}{\sqrt{T_1 T_2}} \qquad \ldots(8)$$

14.6.2 Bode Plot of Lag-Lead Compensator

The various corner frequencies of the lag-lead compensator are,

$$\omega_{c1} = -\frac{\beta}{T_1}, \text{ for a simple pole}$$

$$\omega_{c2} = -\frac{1}{T_1}, \text{ for a simple zero}$$

$$\omega_{c3} = -\frac{1}{T_2}, \text{ for a simple zero}$$

$$\omega_{c4} = -\frac{1}{-\beta T_2}, \text{ for a simple pole}$$

So line of zero slope continues till ω_{c1}. At ω_{c1} the slope becomes – 20 dB/decade. At ω_{c2} the slope again becomes zero and continues till ω_{c3} . At ω_{c3} the slope becomes + 20 dB/decade and continues till ω_{c4}. And at ω_{c4} it again becomes zero. The Bode plot is shown in the Fig.14.20.

14.6.3 Effects of Lag-Lead Compensator

Lag-lead compensator is used when both fast response and good static accuracy are desired. Use of lag-lead compensator increases the low frequency gain which improves the steady state. While at the same time it increases bandwidth of the system, making the system response very fast.

In general, the phase lead portion of this compensator is used to achieve large bandwidth and hence shorter rise time and settling time. While the phase lag portion provides the major damping of the system.

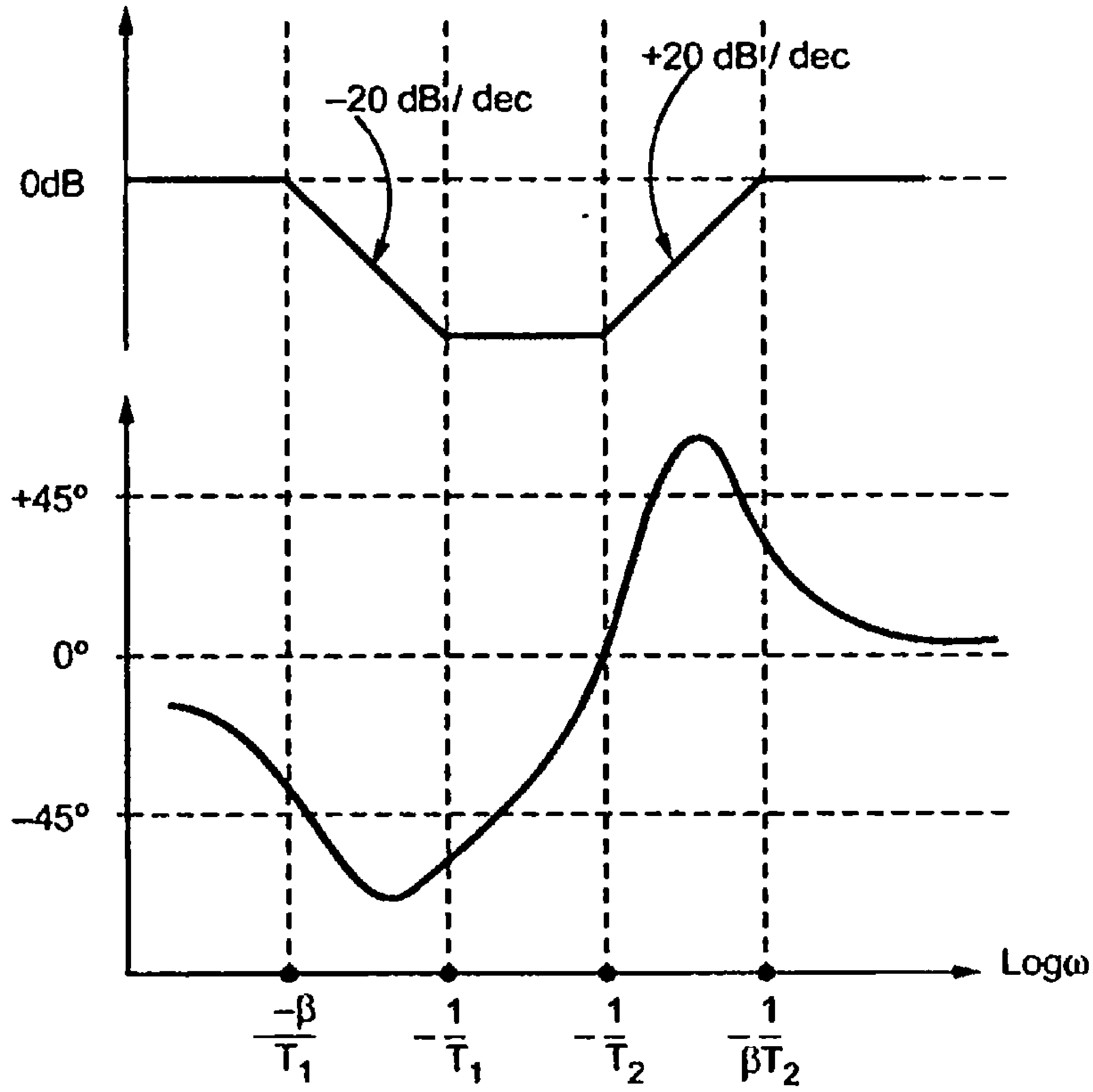


Fig. 14.20 Bode plot of lag-lead compensator

The design of lag-lead compensator is illustrated with an example.

Example 14.3 : *Consider the unity feedback system whose open loop transfer function is,*

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Design suitable lag-lead compensator so as to achieve,

Static velocity error constant = 10 sec^{-1}

Phase margin=50°

Gain margin ≥ 10 dB

Solution : Let the transfer function of the compensator be,

$$G_c(s) = \frac{(1+T_1 s)\,(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)\left(1+\beta T_2\,s\right)}$$

$$K_v = \lim_{s \to 0} s\, G_c(s)\, G(s)$$

$$\therefore \quad 10 = \lim_{s \to 0} \frac{s\,.\,(1+T_1 s)\,(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)\left(1+\beta T_2\,s\right)} \cdot \frac{K}{s(s+1)(s+2)}$$

as $T_1 = \frac{1}{\omega_{c1}} = \frac{1}{0.7} = 1.43$

and $\frac{T_1}{\beta} = \frac{1.43}{10} = 0.143$

Hence the transfer function of lag-lead compensator is,

$$G_c(s) = \frac{(1+1.43\,s)\,(1+6.67\,s)}{(1+0.143\,s)\,(1+66.67\,s)}$$

Thus the transfer function of the compensated system is,

$$G_c(s)G(s) = \frac{10(1+1.43\,s)\,(1+6.67\,s)}{s(1+0.143\,s)\,(1+66.67\,s)(1+s)\,(1+0.5\,s)}$$

Let us obtain the Bode plot of this compensated system and check the specifications.

Factors : K = 10, 20 Log 10 = 20 dB

1 Pole at origin pole, – 20 dB/dec

ω_{C1} = 0.015, simple pole, – 20 dB/dec

ω_{C2} = 0.15, simple zero, + 20 dB/dec

ω_{C3} = 0.7, simple zero, + 20 dB/dec

ω_{C4} = 1, simple pole, – 20 dB/dec

ω_{C5} = 2, simple pole, – 20 dB/dec

ω_{C6} = 7, simple pole, – 20 dB/dec

$$G_c(j\omega)\,G(j\omega) = \frac{10(1+1.43\,j\omega)\,(1+6.67\,j\omega)}{j\omega(1+0.143\,j\omega)(1+66.67\,j\omega)(1+j\omega)\,(1+0.5\,j\omega)}$$

Phase angle table for the compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1}66.67\omega$	$+\tan^{-1}6.67\omega$	$+\tan^{-1}1.43\omega$	$-\tan^{-1}\omega$	$-\tan^{-1}0.5\,\omega$	$-\tan^{-1}0.143\,\omega$	ϕ_R
0.02	– 90°	– 58.13°	+ 7.6°	+ 1.63°	– 1.1°	– 0.57°	– 0.16°	– 135.7°
0.1	– 90°	– 81.46°	+ 33.7°	+ 8.13°	– 5.7°	– 2.86°	– 0.81°	– 139°
0.5	– 90°	– 88.2°	+ 73.3°	+ 35.5°	– 26.56°	– 14.03°	– 4.08°	– 114.1°
1	– 90°	– 89.1°	+81.47°	+ 55°	– 45°	– 26.56°	– 8.13°	– 122.3°
2	– 90°	– 89.5°	+ 85.7°	+ 70.1°	– 63.4°	– 45°	– 15.96°	– 148°
4	– 90°	– 89.7°	+ 87.8°	+ 80°	– 75.9°	– 63.4°	– 29.8°	– 181°

The specifications are,

ω_{gc}	=	1.3 rad/sec
ω_{pc}	=	3.9 rad/sec
G.M.	=	+ 16 dB
P.M.	=	+ 50°

Thus compensated system satisfies the specifications.

14.7 Compensation using Root Locus

Uptill now we have seen the compensation using Bode plot where the specifications are given in the frequency domain. When the specifications are given in the time domain the root locus approach of design is very powerful.

The damping factor and undamped natural frequency are the two main specifications used in the root locus compensation. These two specifications decide the dominant closed loop poles near jω axis. Thus compensation using root locus means to reshape the root locus near jω axis and origin in order to place the dominant closed loop poles at the desired locations.

The root locus compensation also can be achieved using series compensation with lead, lag or lag-lead network.

Let us revise the relation between damping ratio ξ and angle θ.

The dominant complex conjugate poles are expressed as $-\xi\omega_n \pm j\omega_d$. Then in the complex plane we have $\xi = \sin\theta$ where θ is measured from j ω axis or $\xi = \cos\theta$ if θ is measured from negative real axis.

Thus lines of constant ξ are radial lines passing through the origin as shown in the Fig.14.23 (b).

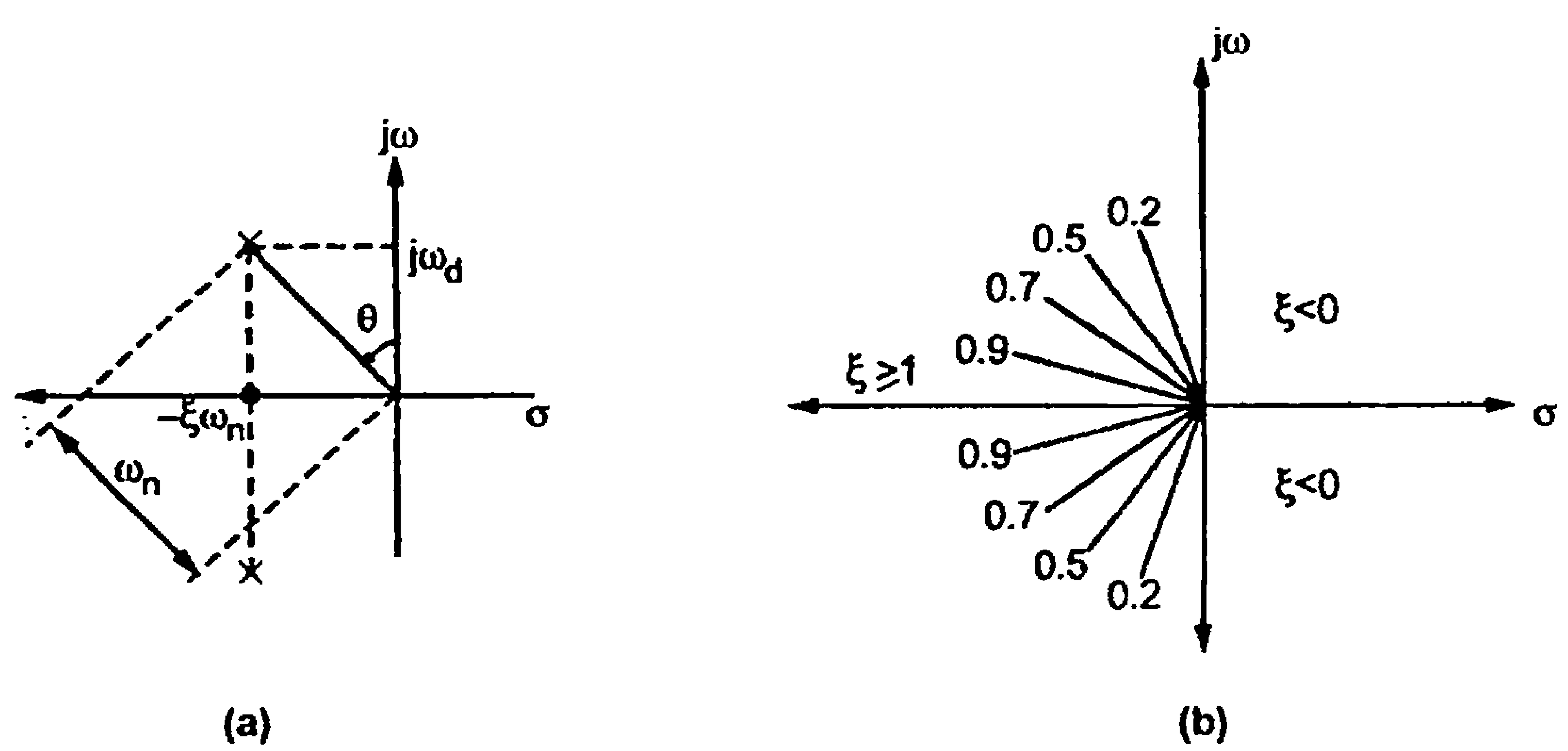


Fig. 14.23

For the sake of design purpose we will use,

$$\xi = \sin\theta, \theta \text{ measured from } j\omega \text{ axis}$$

With this background let us see the design procedures for lead, lag and lag-lead compensator.

14.8 Designing Lead Compensator using Root Locus

The procedure to design lead compensator is,

Step 1 : From the given specifications, find the desired locations of the dominant closed loop poles.

Step 2 : Assume the lead compensator as,

$$G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)}, \alpha < 1$$

K_c is determined from the requirement of open loop gain.

Step 3 : Find the sum of the angles at the desired location of one of the dominant closed loop poles with the open loop poles and zeros of the original system. This angle must be an odd multiple of 180°. If it is not, calculate the necessary angle ϕ to be added to get the sum as an odd multiple of 180°. This ϕ must be contributed by lead compensator. If ϕ is more than 60° then two or more lead networks may be needed. This ϕ, helps to determine values of α and T.

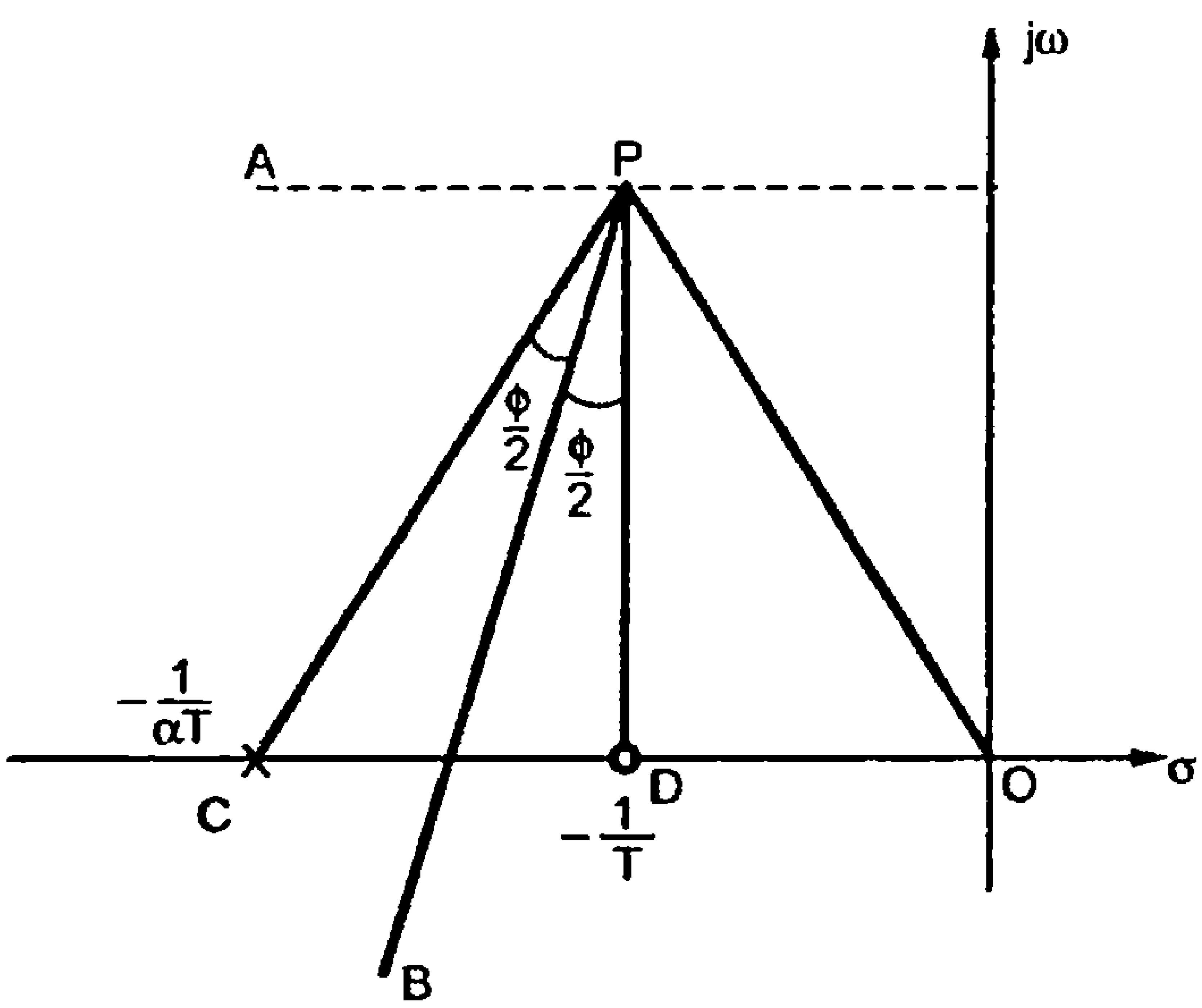


Fig. 14.24

Step 4 : To determine α and T for known ϕ, draw the horizontal line from one of the dominant closed loop pole say P. Join origin to P, as shown in the Fig. 14.24. Bisect the angle between the lines PA and PO. Draw the two line PC and PD that makes angle $\pm\frac{\phi}{2}$ with the bisector PB. The intersection of PC and PD with the negative real axis gives the necessary pole and zero of compensator.

Step 5 : The open loop gain can be determined by applying the magnitude condition at point P.

Step 6 : Check that the compensated system satisfies all the specifications. If not adjust the compensator pole and zero till all the specifications are satisfied.

Example 14.4 : *Design a suitable lead compensator for a system with unity feedback and having open loop transfer function :*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the specifications :

1. *Damping ratio* $\xi = 0.5$
2. *Undamped natural frequency* $\omega_n = 2$ *rad/sec*

Solution : Let us sketch the root locus of the uncompensated system.

$$P = 3, \; Z = 0, N = P = 3, P - Z = 3$$

Starting points : 0, –1, – 4

Terminating points : ∞, ∞, ∞

The angles of asymptotes : 60°, 180°, 300°

Centroid : $\frac{0-1-4}{3} = -1.67$

Breakaway point : $1 + G(s)H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s(s+1)(s+4)} = 0$$

$$\therefore \quad s^3 + 5s^2 + 4s + K = 0$$

$$\therefore \quad K = -s^3 - 5s^2 - 4s$$

$$\therefore \quad \frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

$$\therefore \quad s^2 + 3.33s + 1.33 = 0$$

$$\therefore \quad s = -0.464, -2.86$$

The point s = – 0.464 is valid breakaway point.

Intersection with imaginary axis : The Routh's array is,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

$$\frac{\angle K}{\angle -1+j\,1.73\,\angle -1+j\,1.73+1\,\angle -1+j\,1.73+4} = \frac{\angle K}{\angle -1+j\,1.73\;\;\angle j\,1.73\;\;\angle 3+j\,1.73}$$

$$= \frac{0^{\circ}}{120.029^{\circ}\;90^{\circ}\;29.97^{\circ}} = -240^{\circ}$$

$\therefore$ Angle to be contributed by lead compensator is,

$$-180^{\circ} - (-240^{\circ}) = 60^{\circ}$$

Step 4 : Find locations of pole and zero.

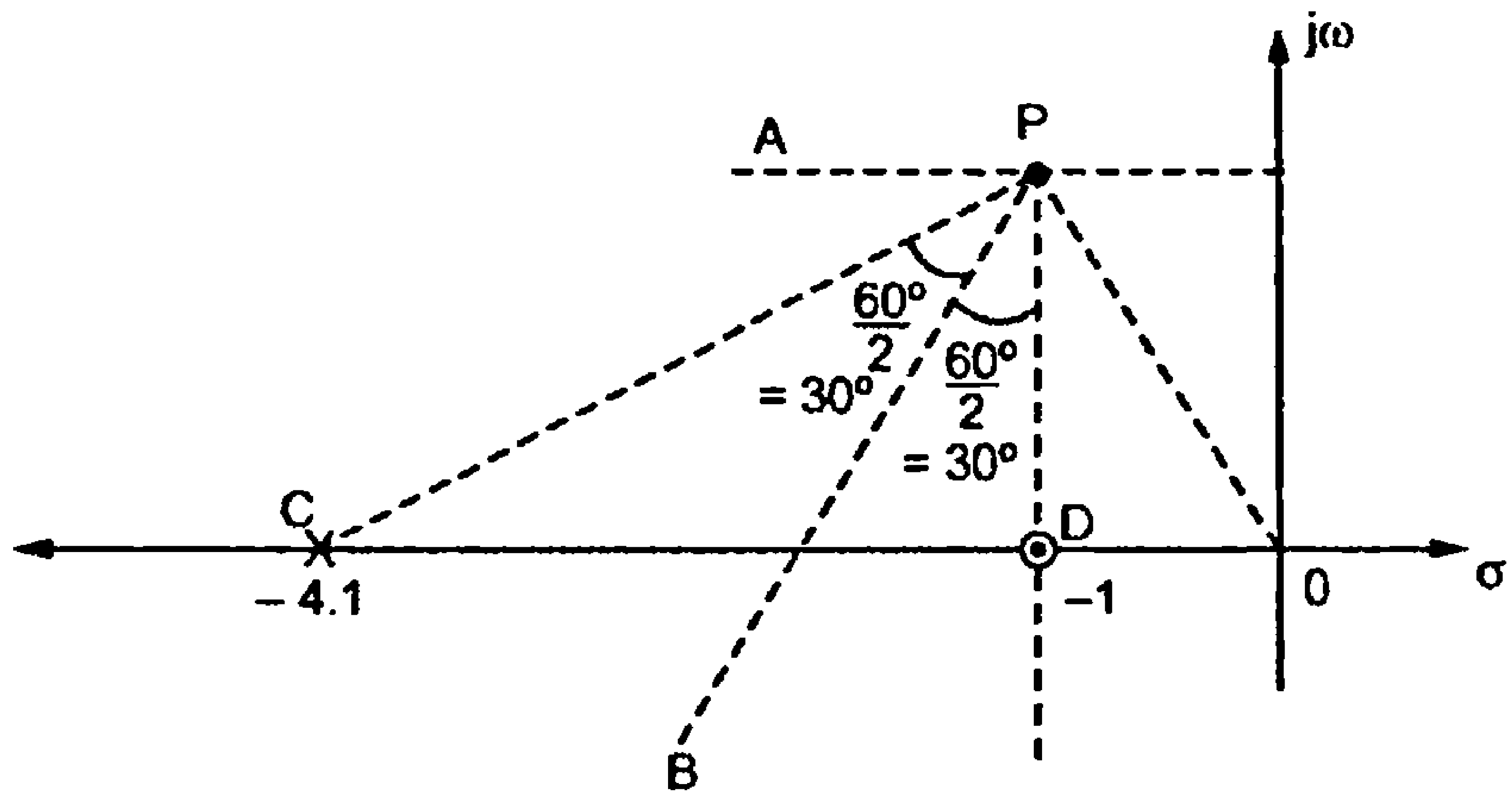


Fig. 14.26

$\therefore$ Zero at s= –1 will cancel the pole at s = –1 of the given system hence we will select the zero closer to s = –1 but slightly to left of s = –1.

$\therefore$ Zero of compensator at s = – 1.2

and pole of compensator at s = – 4.1 which is very close to pole s = –4 of original system hence we will select it at s = – 4.6.

$$\therefore \qquad \frac{1}{T} = 1.2 \qquad \therefore\ T = 0.833$$

$$\text{and} \qquad \frac{1}{\alpha T} = 4.6 \qquad \therefore\ \alpha T = 0.2173$$

$$\therefore \qquad \alpha = 0.26 < 1$$

The transfer function of the compensated system is,

$$G_c(s)\,G(s) = \frac{K\,(1+0.833\,s)}{s\,(1+s)\,(s+4)\,(1+0.2173\,s)}$$

Step 5 : Use the magnitude condition to obtain value of K at s=–1 + j 1.73.

$$|G_c(s)\,G(s)|_{s=-1+j\,1.73} = 1$$

$$\therefore \qquad \frac{K\,[1+0.833\,(-1+j\,1.73)]}{(-1+j\,1.73)\,(j\,1.73)\,(3+j\,1.73)\,[1+0.213\,(-1+j\,1.73)]} = 1$$

$$\therefore \qquad \frac{K|0.167 + j\,1.44|}{|-1+j\,1.73||0+j\,1.73||3+j\,1.73||0.7827+j\,0.3759|} = 1$$

$$\therefore \qquad \frac{K \times 1.449}{1.998 \times 1.73 \times 3.46 \times 0.8682} = 1$$

$$\therefore \qquad K = 7.166$$

$$\therefore \qquad G_c(s)\,G(s) = \frac{7.166\,(1+0.833\,s)}{s\,(1+s)\,(s+4)\,(1+0.2173\,s)}$$

$$\boxed{G_c(s)\,G(s) = \frac{27.47\,(s+1.2)}{s\,(s+1)\,(s+4)\,(s+4.6)}}$$

The root locus of the compensated system is shown in the Fig. 14.27. Students are expected to calculate breakaway points and intersection with the imaginary axis.

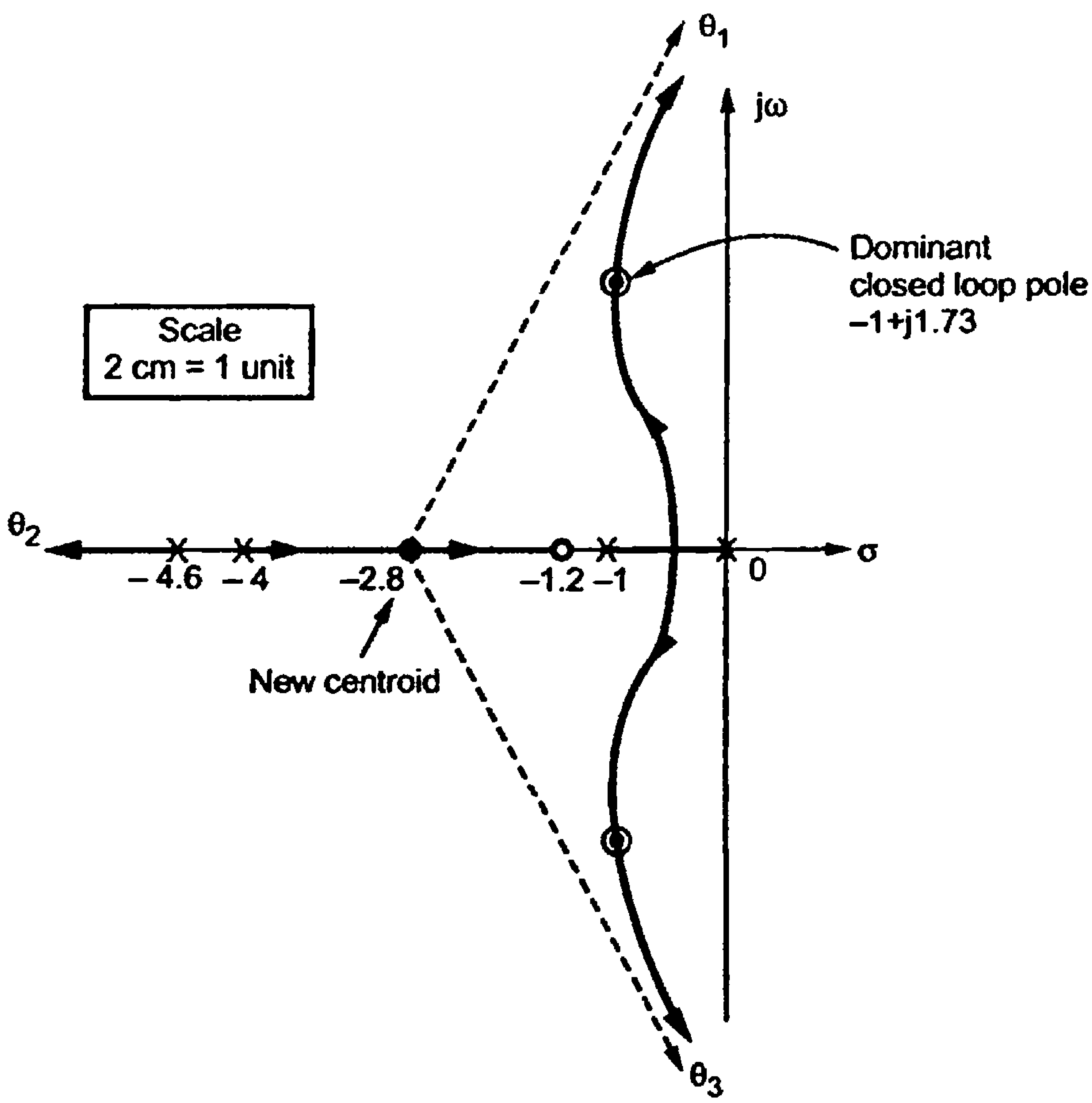


Fig. 14.27

14.9 Designing Lag Compensator using Root Locus

The lag compensator is generally used when the system is showing good transient response which is satisfactory but unsatisfactory steady state characteristics. This compensation needs not to change the root locus appreciably but to increase the open loop gain.

Thus lag compensator essentially does not change the root locus appreciably near the dominant closed loop poles but increases the open loop gain as per the requirement. The angle contribution by the lag compensator is very small of about 5° and to achieve this the pole and zero of the lag compensator are placed close together and near the origin of the s-plane.

Increase in gain means increase in static error constant. So when it is necessary to satisfy static error constant specification using root locus, lag compensator is used.

Steps to design the lag compensator are,

Step 1 : Draw the root locus of the uncompensated system and locate the dominant closed loop poles on the root locus.

Step 2 : Assume the lag compensator having transfer function,

$$G_c(s) = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}, \beta > 1$$

Step 3 : Calculate the static error constant specified in the problem.

Step 4 : Determine the amount of increase in static error constant necessary to satisfy the specification.

Step 5 : Determine pole and zero of the compensator, such that they do not produce appreciable change in the original root locus but produce necessary increase in static error constant.

Note that the ratio of value of gain required in the specifications and the gain found in the uncompensated system is the required ratio between the distance of the zero from origin and that of pole from the origin.

Step 6 : Draw the root locus of compensated system. Locate the dominant closed loop poles.

Step 7 : Adjust $\hat{K}_c$ from the magnitude condition so as to place the dominant closed loop poles at the desired location.

Example 14.5 : *Design a lag compensator for a system with open loop transfer function as,*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the following specifications :

1. Damping ratio=0.5 *2. Velocity error constant ≥ 5 sec^{-1}*

3. Settling time=10 sec

Solution : Let us draw the root locus of the uncompensated system. This is already obtained in Ex. 14.4 which is shown again in the Fig. 14.28.

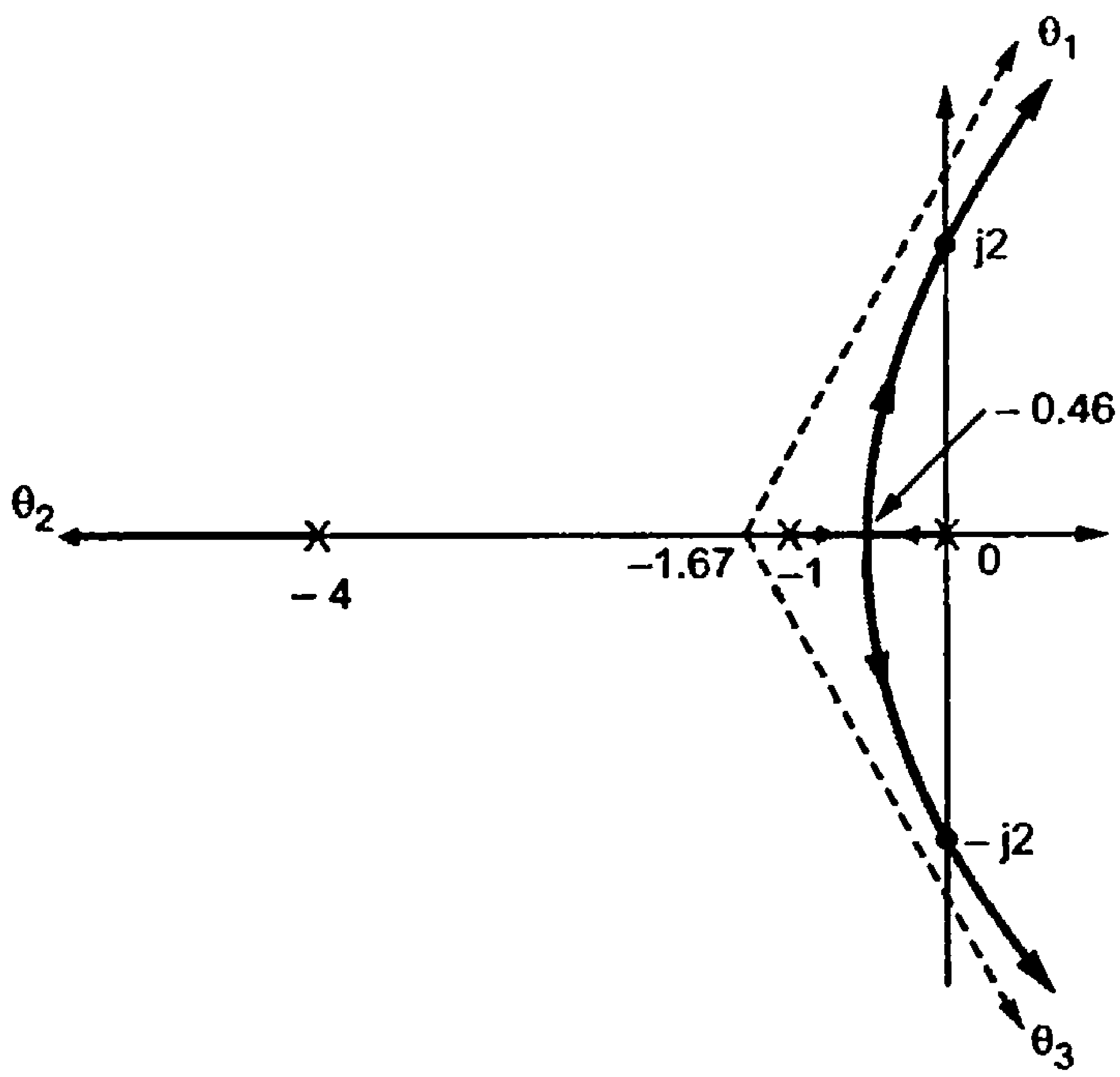


Fig. 14.28

From the given specifications, $\xi = 0.5$

and $$T_s = \frac{4}{\xi\omega_n} = 10$$

$$\therefore \quad \omega_n = 0.8 \text{ rad/sec}$$

Hence the dominant closed loop poles are,

$$= -\xi\omega_n \pm j\,\omega_n\sqrt{1-\xi^2}$$

$$= -0.4 \pm j\,0.693$$

The gain K at dominant pole $-0.4 + j\,0.693$ can be obtained by magnitude condition.

$$|G(s)H(s)|_{s=-0.4+j0.693} = 1$$

$$\therefore \quad \frac{|K|}{|-0.4+j0.693||0.6+j0.693||3.6+j0.693|} = 1$$

$$\therefore \quad \frac{K}{0.8 \times 0.9167 \times 3.667} = 1$$

$$\therefore \quad K = 2.688$$

The transfer function of uncompensated system is,

$$G(s) = \frac{2.688}{s(s+1)(s+4)}$$

$$K_v = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} \frac{s.2.688}{s(s+1)(s+4)}$$

$$= 0.672$$

It is desired to have K_v of 5 sec^{-1}. The factor by which static error constant is to be increased is,

$$\text{Factor} = \frac{K_v \text{ desired}}{K_v \text{ of uncompensated system}} = \frac{5}{0.672}$$

$$= 7.44$$

Let us choose this factor be 10.

$$\therefore \quad \beta = 10$$

Place the zero and pole of the lag compensator very close to the origin.

Let zero of compensator at s = – 0.1

∴ Pole of compensator at s = – 0.01

So transfer function of the lag compensator is,

$$G_c(s) = \hat{K}_c \frac{s+0.1}{s+0.01}$$

Hence the transfer function of the compensated system becomes,

$$G_c(s)G(s) = \hat{K}_c \frac{s+0.1}{s+0.01} \cdot \frac{2.688}{s(s+1)(s+4)}$$

$$\therefore \quad \boxed{G_c(s)G(s) = \frac{K(s+0.1)}{s(s+1)(s+4)(s+0.01)}}$$

Where, $K = 2..6\,88\,\hat{K}_c$

The new dominant pole is – 0.43 + j 0.67.

Apply magnitude condition at this new location of dominant closed loop pole.

$$|G_c(s)H(s)|_{s=-0.43+j0.67} = 1$$

$$\therefore \quad \frac{|K|\,|-0.33+j\,0.67|}{|-0.43+j\,0.67|\,|0.57+j\,0.67|\,|3.57+j\,0.67|\,|-0.42+j\,0.67|} = 1$$

$$\therefore \quad \frac{K \times 0.7468}{0.7961 \times 0.8786 \times 3.6323 \times 0.7907} = 1$$

$\therefore \quad K = 2.69$

But $\quad K = 2.688$

$\therefore \quad \hat{K}_c = 1.0019$

Hence the compensated system has the transfer function,

$$G_c(s)G(s) = \frac{2.69(s+0.1)}{s(s+0.01)(s+1)(s+4)}$$

This gives static error constant of 6.725 which is greater than 5.

14.10 Designing Lag-Lead Compensator using Root Locus

It is known that the lead compensator increases the speed of the response and improves stability. The lag compensator improves the steady state response. If improvement in both transient as well as the steady state response is desired then the lag-lead compensator is used. Lag-lead compensator combines the advantages of both lead and lag compensators.

Assume the transfer function of the lag-lead compensator as,

$$G_c(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{\gamma}{T_1}\right)} \cdot \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)}, \quad \beta > 1, \gamma > 1$$

Let us consider two cases to design such compensator.

Case 1 : $\gamma \neq \beta$

In this design, determine the locations of dominant closed loop poles from the given specifications.

From uncompensated G(s), calculate the angle deficiency ϕ which must be contributed by lead portion of the compensator.

Choose T_2 sufficiently large so that magnitude of lag portion is unity.

$$\left| \frac{\left(s_1 + \frac{1}{T_2}\right)}{\left(s_1 + \frac{1}{\beta T_2}\right)} \right| = 1$$

Where $s = s_1$ is one of the dominant closed loop pole. From the angle deficiency ϕ determine values of T_1 and γ.

$$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\gamma}{T_1}} = \phi$$

The determine value of K_c from the magnitude condition.

$$\left| K_c \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\gamma}{T_1}} G(s_1) \right| = 1$$

If the static velocity error constant K_v is given then,

$$K_v = \lim_{s \to 0} s\, G_c(s)\, G(s)$$

$$= \lim_{s \to 0} sK_c \frac{\left(s + \frac{1}{T_1}\right)}{\left(s + \frac{\gamma}{T_1}\right)} \cdot \frac{\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{1}{\beta T_2}\right)} \cdot G(s)$$

$$= \lim_{s \to 0} s\, K_c \cdot \frac{\beta}{\gamma} G(s)$$

As K_c and γ are known, β can be obtained.

Using this value of β, choose T_2 such that the magnitude of the lag portion is unity.

Case 2 : $\gamma = \beta$

Determine the locations of dominant closed loop poles according to given specifications.

As $\gamma = \beta$, the transfer function of compensator becomes,

$$G_c(s) = \frac{\left(s + \frac{1}{T_1}\right)}{\left(s + \frac{\beta}{T_1}\right)} \cdot \frac{\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{1}{\beta T_2}\right)}, \quad \beta > 1$$

From the requirement of K_v, K_c can be determined.

To have the dominant closed loop poles at desired location, calculate the angle deficiency ϕ.

Choose T_2 very large at the end so that

$$\left| \frac{\left(s_1 + \frac{1}{T_2}\right)}{\left(s_1 + \frac{1}{\beta T_2}\right)} \right| = 1$$

Where, $s = s_1$ is one of the dominant closed loop pole.

Now $$\left| K_c \frac{\left(s_1 + \frac{1}{T_1}\right)}{\left(s_1 + \frac{\beta}{T_1}\right)} G(s_1) \right| = 1$$

and $$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} = \phi$$

These two equations give the values of T_1 and β.

Once β is known, choose large T_2 so that the magnitude of the lag portion is approximately unity and

$$-5^\circ < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

βT_2 should be high but should be physically realizable.

Review Questions

1. *What is compensation ? What is compensator ? Which are the various compensation schemes used in practice ?*
2. *Which are the important electrical networks used practically for the compensation of the control systems ?*
3. *Derive the transfer functions of,*

   a. Lead network *b. Lag network* *c. Lag-lead network*
4. *Draw and explain the polar plot of,*

   a. Lead network *b. Lag network* *c. Lag-lead network*

5. *Draw and explain the Bode plot of,*

   a. Lead network b. Lag network c. Lag-lead network

6. *Derive the relation between ϕ_m and α for the lead compensator.*
7. *Compare the characteristics of three types of compensators.*
8. *Explain the design in frequency domain of,*

   a. Lead compensator b. Lag compensator c. Lag-lead compensator

9. *Explain the design of three compensators using Root locus method.*
10. *Explain the effects and limitations of the three types of electrical compensators.*
11. *Design a suitable lead compensator for system with,*

    $G(s) = \frac{4}{s(s+2)}$ *to meet the specifications as,*

    a. $K = 20\ sec^{-1-1}$ *b.* $P.M. = +\ 50^\circ$ *c.* $G.M. \geq +\ 10\ dB$

12. *Design a suitable lag compensator for a system with,*

    $G(s) = \frac{1}{s(s+1)(1+0.5s)}$ *to meet the following specifications :*

    a. $K \geq 5\ sec^{-1}$ *b.* $P.M. \geq +\ 40^\circ$ *c.* $G.M. \geq +\ 10\ dB$

13. *Design a lead compensator using root locus for the system with,*

    $G(s) = \frac{4}{s(s+2)}$ *to meet the following specifications :*

    a. Damping ratio = 0.5 b. Setting time = 2 sec

14. *Design a suitable lag compensator root locus for the system with,*

    $G(s) = \frac{K}{s(s+1)(s+2)}$ *to meet the following specifications :*

    a. Damping ratio = 0.5 b. $K \geq 5\ sec^{-1}$ *c. Undamped natural frequency = 0.7 rad/sec*

□□□

15 State Variable Analysis

15.1 Background

The conventional approach used to study the behaviour of linear time invariant control systems, uses the time domain or frequency domain methods. In all these methods, the systems are modelled using transfer function approach, which is the ratio of Laplace transform of output to input, neglecting all the initial conditions Thus this conventional analysis faces all the limitations associated with the transfer function approach.

Some of its limitations can be stated as :

1) Naturally, significant initial conditions in obtaining precise solution of any system, loose their importance in conventional approach.
2) The method is insufficient and troublesome to give complete time domain solution of higher order systems.
3) It is not very much convenient for the analysis of Multiple Input Multiple Output systems.
4) It gives analysis of system for some specific types of inputs like Step, Ramp etc.
5) It is only applicable to Linear Time Invariant Systems.
6) The classical methods like Root locus, Bode plot etc. are basically trial and error procedures which fail to give the optimal solution required.

Hence it is absolutely necessary to use a method of analysing systems which overcomes most of the above said difficulties. The modern method discussed in this chapter uses the concept of total internal state of the system considering all initial conditions. This technique which uses the concept of state is called **State Variable Analysis** or **State Space Analysis**. It is essentially a time domain approach but it has number of advantages compared to conventional methods of analysis.

15.1.1 Advantages of State Variable Analysis

The various advantages of state variable analysis are,

1) The method takes into account the effect of all initial conditions.
2) It can be applied to nonlinear as well as time varying systems.
3) It can be conveniently applied to Multiple Input Multiple Output systems.
4) The system can be designed for the optimal conditions precisely by using this modern method.
5) Any type of the input can be considered for designing the system.
6) As the method involves matrix algebra, can be conveniently adopted for the digital computers.
7) The state variables selected need not necessarily be the physical quantities of the system.
8) The vector matrix notation greatly simplifies the mathematical representation of the system.

15.2 Concept of State

Consider a football match. The score in the football match must be updated at every instant from the knowledge and information of the total score before that instant. This procedure of updating the score continues till the end of the match when we get exact and precise score of the entire match. This updating procedure has main importance in the understanding of the concept of state.

Consider the network as shown in the Fig. 15.1.

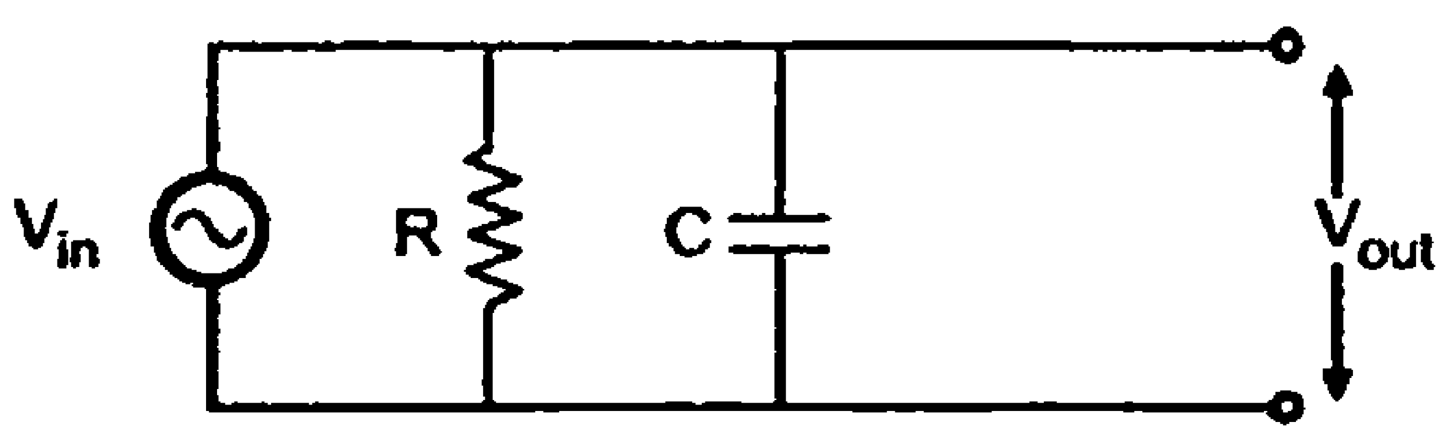


Fig. 15.1

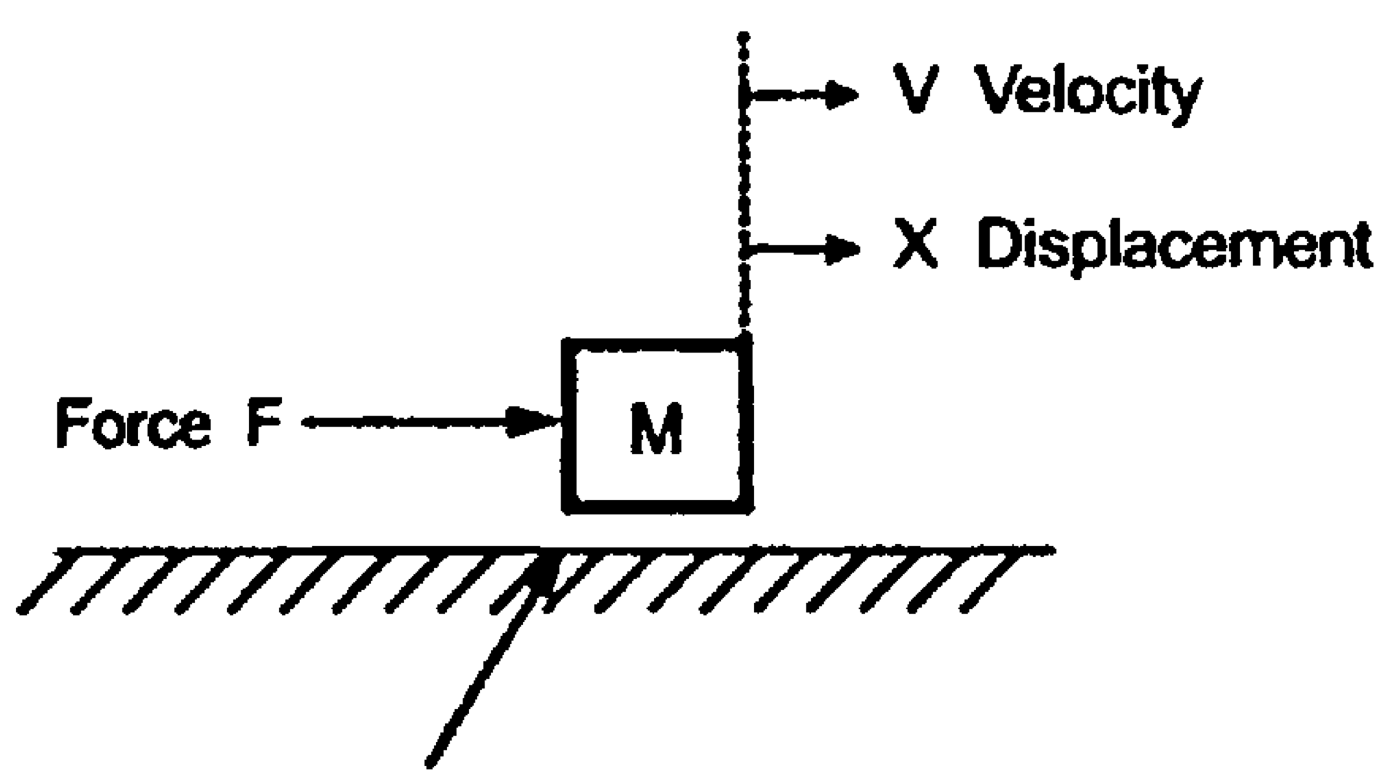


Fig. 15.2

To find V_{out}, the knowledge of the initial capacitor voltage must be known. Only information about V_{in} will not be sufficient to obtain precisely the V_{out} at any time $t \geq 0$. Such systems in which the output is not only dependent on the input but also on the initial conditions are called the *systems with memory* or *dynamic systems.*

While if in the above network capacitor 'C' is replaced by another resistance 'R_1' then output will be dependent only on the input applied V_{in}.

Such systems in which the output of the system depends only on the input applied at t = 0, are called *systems with zero memory or static systems.*

Consider another example of simple mechanical system as shown in the Fig. 15.2

Now according to Newton's law of motion,

$$F = Ma$$

$$a = \text{acceleration of mass M}$$

$$\therefore \quad F = M\frac{d}{dt}(v(t))$$

i.e.

$$v(t) = \frac{1}{M}\int f(t)\,dt$$

Now velocity at any time 't' is the result of the force F applied to the particle in the entire past,

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} f(t)\,dt = \frac{1}{M}\int_{-\infty}^{t_0} f(t)\,dt + \frac{1}{M}\int_{t_0}^{t} f(t)\,dt$$

where t_0 = initial time.

Now $\frac{1}{M}\int_{-\infty}^{t_0} f(t)\,dt = v(t_0)$

$$\therefore \quad v(t) = v(t_0) + \int_{t_0}^{t} f(t)\,dt$$

From the above equation it is clear that for the same input f(t), we get the different values of the velocities v(t) depending upon our choice of parameter t_0 and the value of $v(t_0)$.

If $v(t_0)$ is known and the input vector from 't_0' to 't' is known then we can obtain a unique value of the output v(t) at any time $t > t_0$.

Key Point: *Thus initial conditions i.e. memory affects the system characterisation and subsequent behaviour.*

Thus initial conditions describe the status or state of the system at $t = t_0$. **The state can be regarded as a compact and concise representation of the past history of the system.** So the state of the system in brief separates the future from the past so that the state contains all the information concerning the past history of the system necessarily required to determine the response of the system for any given type of input.

The state of the system at any time 't' is actually the combined effect of the values of all the different elements of the system which are associated with the initial conditions of the system. **Thus the complete state of the system can be considered to be a vector having components which are the variables of system which are closely associated with initial conditions. So state can be defined as vector X(t) called state vector.** This X(t) i.e. state at any time 't' is 'n' dimensional vector i.e. column matrix $n \times 1$ as indicated below.

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}$$

Now these variables $X_1(t)$, $X_2(t)$, ... $X_n(t)$ which constitute the state vector X(t) are called the *state variables of the system.*

If state at $t = t_1$ is to be decided then we must know $X(t_0)$ and knowledge of the input applied between $t_0 \rightarrow t_1$. This new state will be $X(t_1)$ which will act as initial state to find out the state at any time $t > t_1$. This is called the updating of the state. The output of the system at $t = t_1$ will be the function of $X(t_1)$ and the instantaneous value of the input at $t = t_1$, if any. The number of the state variables for a system is generally equal to the order of the system. The number of independent state variables is normally equal to the number of energy storing elements (e.g.: capacitor voltage, current in inductor) contained in the system.

15.2.1 Important Definitions

1) **State** : The state of a dynamic system is defined as a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

2) **State Variables** : The variables involved in determining the state of a dynamic system X(t), are called the state variables. $X_1(t)$, $X_2(t)$ ......$X_n(t)$ are nothing but the state variables. These are normally the energy storing elements contained in the system.

3) **State Vector** : The 'n' state variables necessary to describe the complete behaviour of the system can be considered as 'n' components of a vector X(t) called the state vector at time 't'. The state vector X(t) is the vector sum of all the state variables.

4) **State Space** : The space whose co-ordinate axes are nothing but the 'n' state variables with time as the implicit variable is called the state space.

5) **State Trajectory** : It is the locus of the tips of the state vectors, with time as the implicit variable.

Definitions can be explained by considering second order system :

Order is 2 so number of state variables required is 2 say $X_1(t)$ and $X_2(t)$.

State vector will be the matrix of order 2×1.

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

i.e. $$\vec{X(t)} = \vec{X_1(t)} + \vec{X_2(t)} \quad \text{(vector addition)}$$

The state space will be a plane in this case as the number of variables are two. Thus state space can be shown as in the Fig. 15.3.

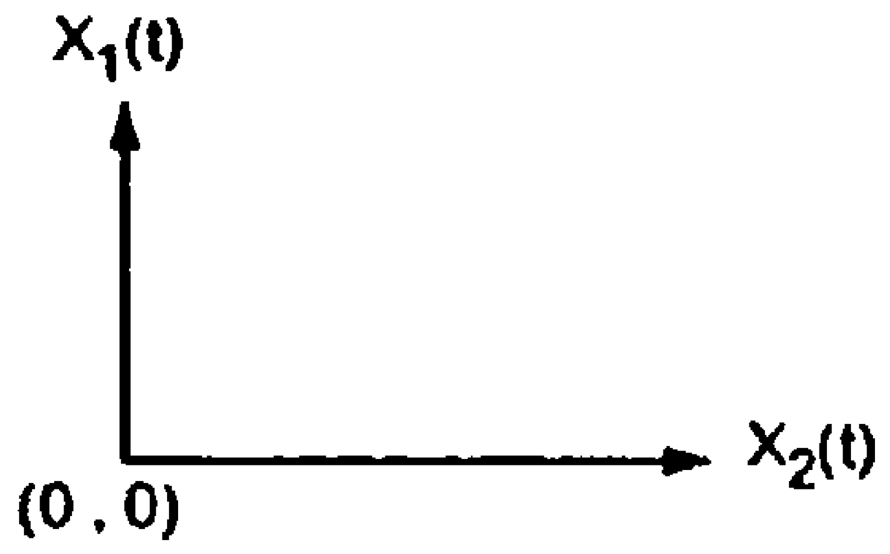


Fig. 15.3

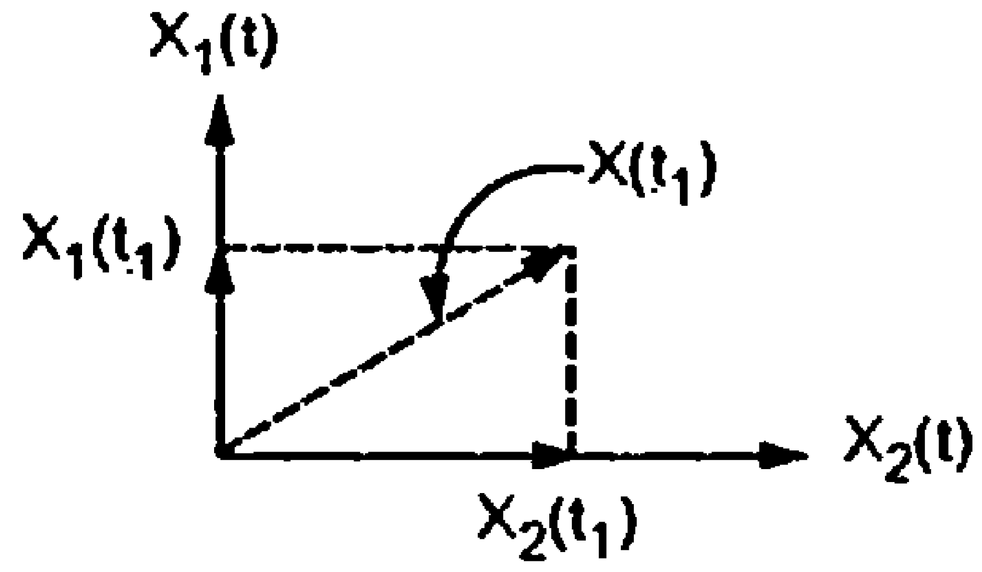


Fig. 15.4

Now consider $t = t_1$

$\therefore$ $$\vec{X(t_1)} = \vec{X_1(t_1)} + \vec{X_2(t_1)}$$

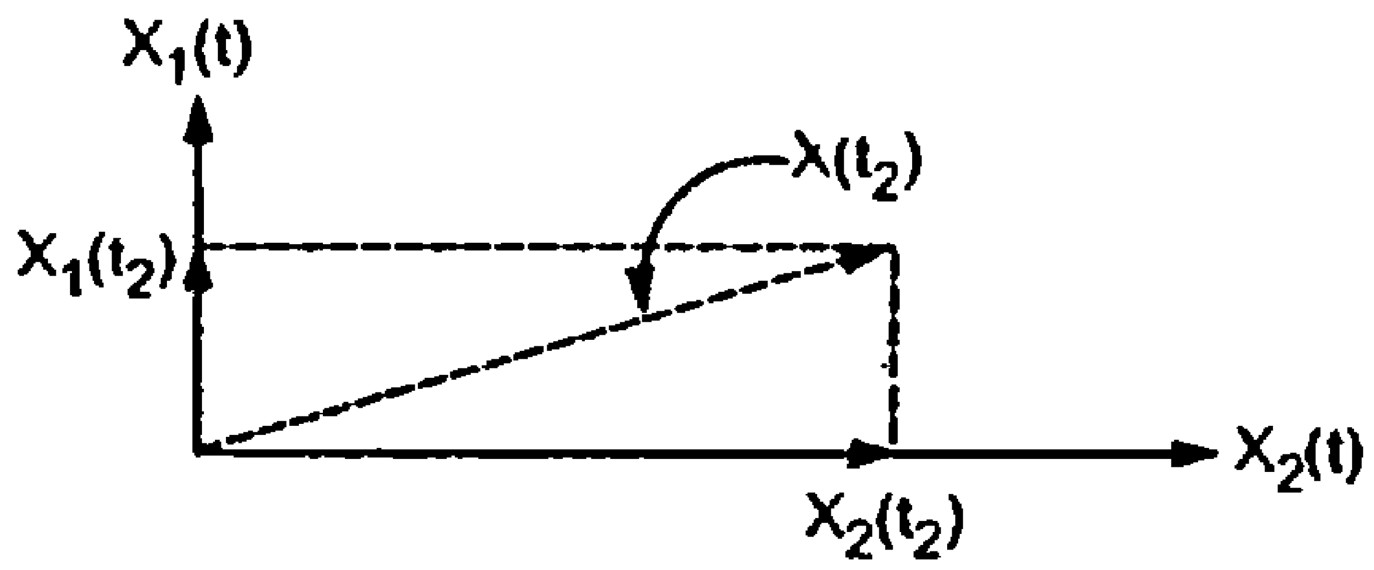


Fig. 15.5

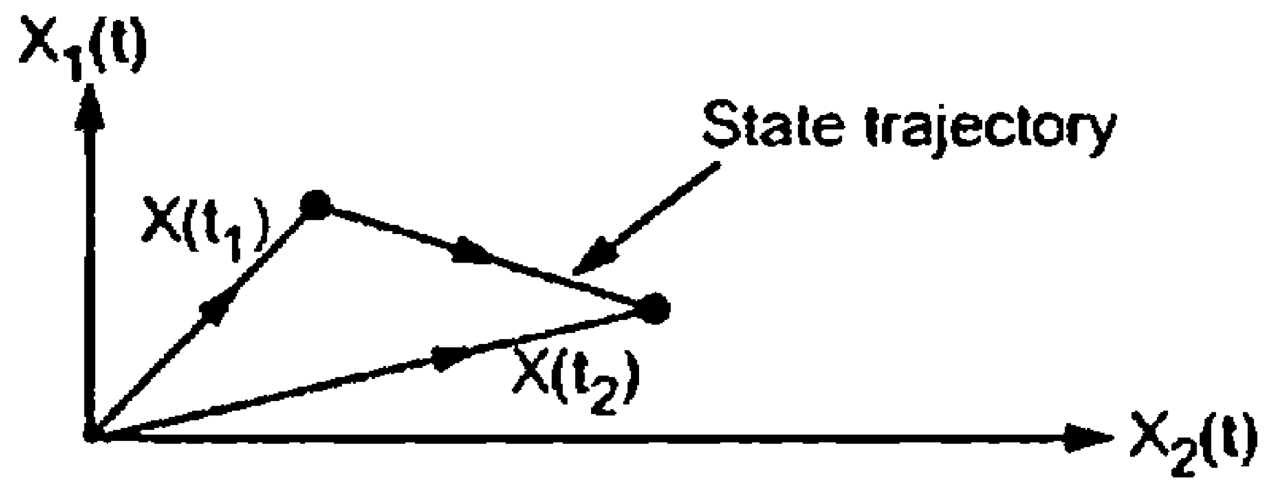


Fig. 15.6

Now consider $t = t_2$

$\therefore$ $$\vec{X(t_2)} = \vec{X_1(t_2)} + \vec{X_2(t_2)}$$

The state trajectory can be shown by joining the tips of the two state vectors i.e. $X(t_1)$ and $X(t_2)$.

15.3 State Model of Linear Systems

Consider Multiple Input Multiple Output, nth order system as shown in the Fig. 15.7.

Number of inputs = m

Number of outputs = p

Where i = 1, 2, ...n.

Thus `n' state variables and hence state vector at any time `t' can be determined uniquely.

Any `n' dimensional time invariant system has state equations in the functional form as,

$$\dot{X}(t) = f(X, U)$$

While outputs of such system are dependent on the state of system and instantaneous inputs.

∴ Functional output equation can be written as,

$$Y(t) = g(X, U) \quad \text{where `g' is the functional operator.}$$

For time variant system, the same equations can be written as,

$$\dot{X}(t) = f(X, U, t) \text{ .... State equation}$$

$$Y(t) = g(X, U, t) \text{ .... Output equation}$$

Diagramatically this can be represented as in the Fig. 15.8.

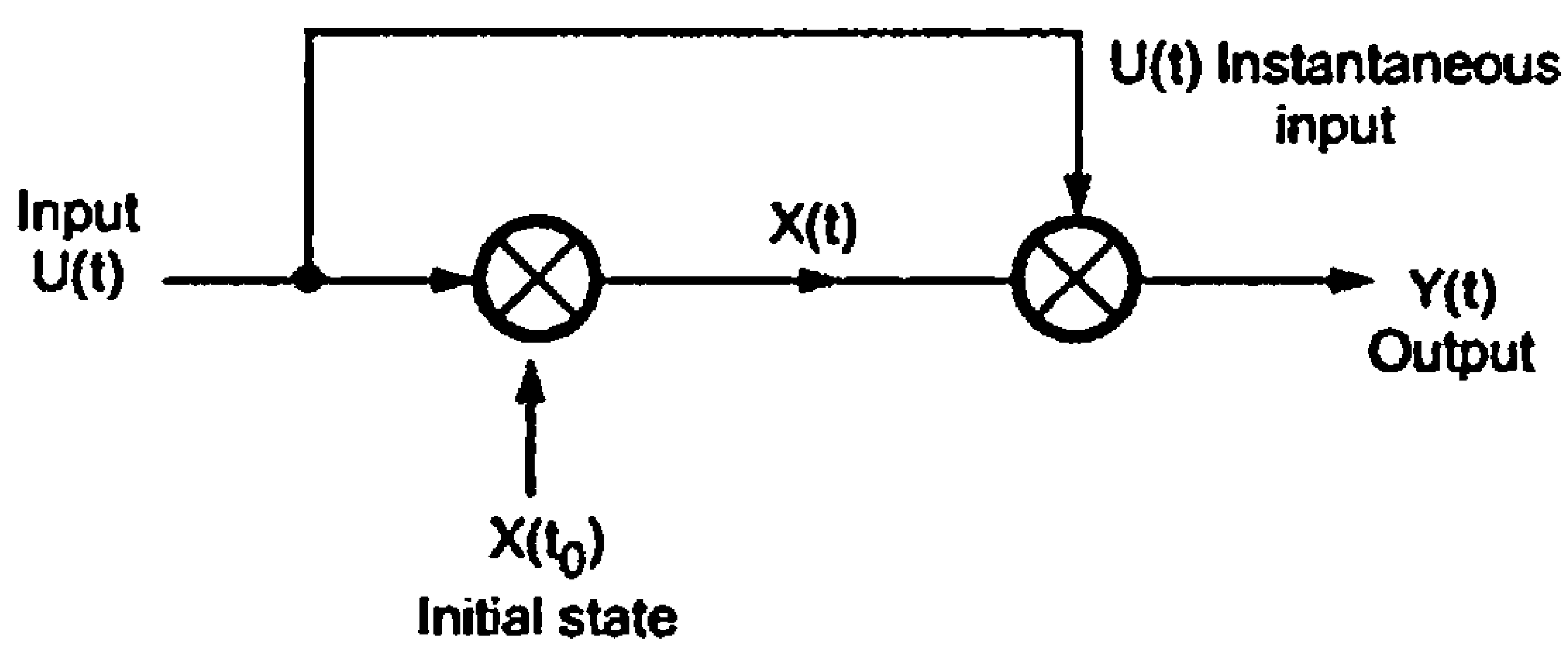


Fig. 15.8 Input-output state description of a system

The functional equations can be expressed interms of linear combination of system states and the input as,

$$\dot{X}_1 = a_{11} X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \ldots + b_{1m} U_m$$

$$\dot{X}_2 = a_{21} X_1 + a_{22}X_2 \ldots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \ldots + b_{2m} U_m$$

$$\vdots$$

$$\dot{X}_n = a_{n1} X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \ldots + b_{nm} U_m$$

15.3.1 State Model of Single Input Single Output System

Consider a single input single output system i.e. m = 1 and p = 1. But its order is n' hence n state variables are required to define state of the system. In such a case, the state model is

$$\dot{X}(t) = A\, x(t) + B\, U(t)$$

$$Y(t) = C\, x(t) + d\, U(t)$$

Where $A = n \times n$ matrix, $B = n \times 1$ matrix

$C = 1 \times n$ matrix, d = constant

and $U(t)$ = single scalar input variable

In general remember the orders of the various matrices.

∴

A = Evolution matrix	$\Rightarrow n \times n$
B = Control matrix	$\Rightarrow n \times m$
C = Observation matrix	$\Rightarrow p \times n$
D = Transmission matrix	$\Rightarrow p \times m$

15.4 State Variable Representation using Physical Variables

The state variables are minimum number of variables which are associated with all the initial conditions of the system. As their sequence is not important, the state model of the system is not unique. But for all the state models it is necessary that the number of state variables is equal and minimal. This number 'n' indicates the **order** of the system. For second order system minimum two state variables are necessary and so on.

To obtain the state model for a given system, it is necessary to select the state variables. Many a times, the various physical quantities of system itself are selected as the state variables.

For the **electrical** systems, the **currents through various inductors** and the **voltage across the various capacitors** are selected to be the state variables. Then by any method of network analysis, the equations must be written interms of the selected state variables, their derivatives and the inputs. The equations must be rearranged in the standard form so as to obtain the required state model.

Key Point: *It is important that the equation for differentiation of one state variable should not involve the differentiation of any other state variable.*

In the mechanical systems the displacements and velocities of energy storing elements such as spring and friction are selected as the state variables.

In general, the physical variables associated with energy storing elements, which are responsible for initial conditions, are selected as the state variables of the given system.

Example 15.1 : *Obtain the state model of the given electrical system.*

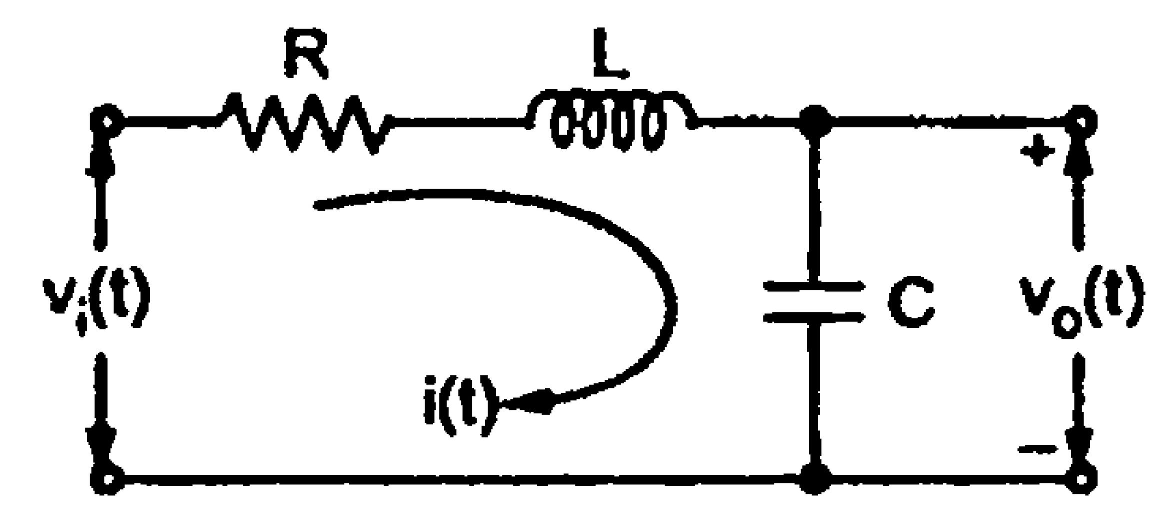


Fig. 15.9

Solution : There are two energy storing elements L and C. So the two state variables are current through inductor i(t) and voltage across capacitor i.e. $v_o(t)$.

$$\therefore \quad X_1(t) = i(t) \quad \text{and} \quad X_2(t) = v_o(t)$$

And $\quad U(t) = v_i(t) = $ Input variable

Applying KVL to the loop,

$$v_i(t) = i(t)\,R + L\,\frac{di(t)}{dt} + v_o(t)$$

Arrange it for di(t)/dt,

$$\therefore \quad \frac{di(t)}{dt} = \frac{1}{L}v_i(t) - \frac{R}{L}i(t) - \frac{1}{L}v_o(t) \quad \text{but} \quad \frac{di(t)}{dt} = \dot{X}_1(t)$$

$$\text{i.e.} \quad \dot{X}_1(t) = -\frac{R}{L}X_1(t) - \frac{1}{L}X_2(t) + \frac{1}{L}U(t) \quad \ldots (1)$$

While $\quad v_o(t) = $ Voltage across capacitor $= \frac{1}{C}\int i(t)\,dt$

$$\therefore \quad \frac{dv_o(t)}{dt} = \frac{1}{C}i(t) \quad \text{but} \quad \frac{dv_o(t)}{dt} = \dot{X}_2(t)$$

$$\text{i.e.} \quad \dot{X}_2(t) = \frac{1}{C}X_1(t) \quad \ldots (2)$$

The equations (1) and (2) give required state equation.

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U(t)$$

$$\text{i.e.} \quad \dot{X}(t) = A\,X(t) + B\,U(t)$$

While the output variable $Y(t) = v_o(t) = X_2(t)$

$$\therefore \quad Y(t) = [0 \ \ 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [0]\ U(t)$$

i.e. $Y(t) = C\ X(t)$ and $D = [0]$

This is the required state model. As n = 2, it is second order system.

Note : The order of the state variables is not important. $X_1(t)$ can be $v_o(t)$ and $X_2(t)$ can be i(t) due to which state model matrices get changed. Hence **state model is not the unique property of the system.**

15.4.1 Advantages

The advantages of using available physical variables as the state variables are,

1. The physical variables which are selected as the state variables are the physical quantities and can be measured.
2. As state variables can be physically measured, the feedback may consists the information about state variables in addition to the output variables. Thus design with state feedback is possible.
3. Once the state equations are solved and solution is obtained, directly the behaviour of various physical variables with time is available.

But the important limitation of this method is that obtaining solution of such state equation with state variables as physical variables is very difficult and time consuming.

15.5 State Diagram Representation

It is the pictorial representation of the state model derived for the given system. It forms a close relationship amongst the state model, differential equations of the system and its solution. It is basically a block diagram type approach which is designed from the view of programming of a computer. The basic advantage of state diagram is when it is impossible to select the state variables as physical variables. When transfer function of system is given then state diagram may be obtained first. And then by assigning mathematical state variables therein, standard state model can be obtained.

State diagram of a linear time invariant continuous system is discussed here for the sake of simplicity. It is a proper interconnection of three basic units.

i) Scalars ii) Adders iii) Integrators

Scalars are nothing but like amplifiers having required gain.

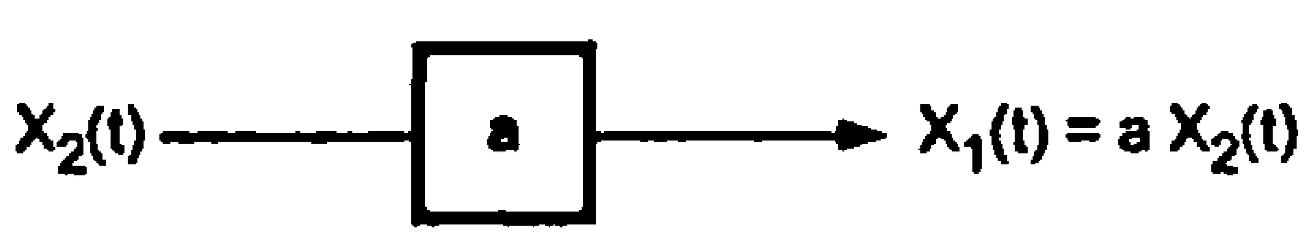


Fig. 15.10 (a)

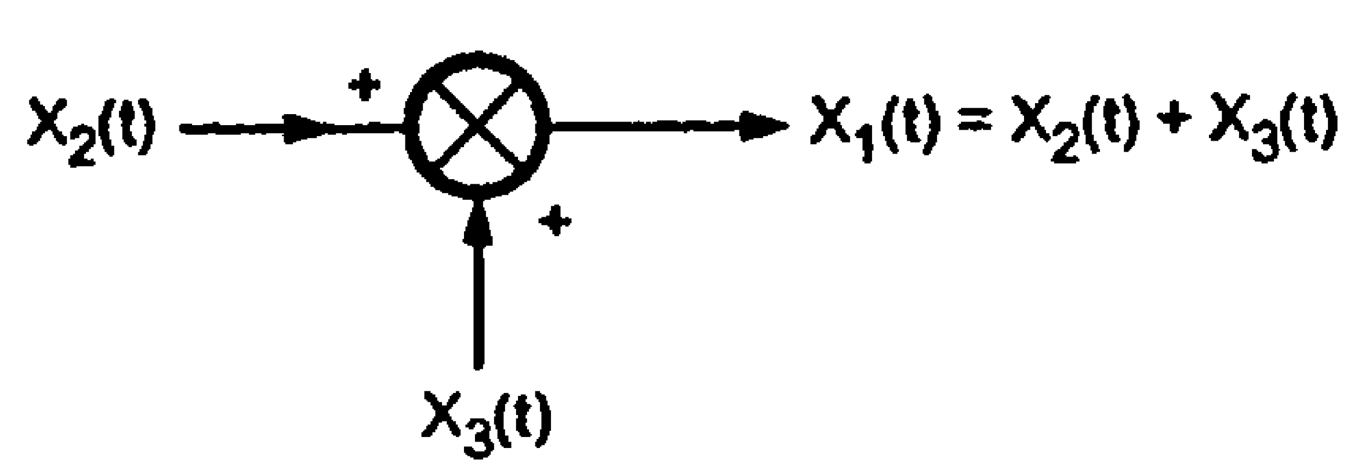


Fig. 15.10 (b)

$$\therefore \quad \dot{Z}(t) = M^{-1}AM\,Z(t) + M^{-1}B\,U(t)$$

where $M^{-1}M = I$ i.e. Identity matrix

$$\therefore \quad \dot{Z}(t) = \hat{A}\,Z(t) + \hat{B}\,U(t) \qquad \dots (5)$$

where $\hat{A} = M^{-1}AM$

$\hat{B} = M^{-1}B$

and $$Y(t) = \hat{C}\,Z(t) + D\,U(t) \qquad \dots (6)$$

Where $\hat{C} = CM$

Equations (5) and (6) forms a new state model of the system.

This shows that state model is not a unique property.

Key Point: *Any linear combinations of the original set of state variables results into a valid new set of state variables.*

15.7 State Space Representation using Phase Variables

Let us study how to obtain state space model using phase variables. The **phase variables** are those state variables which are obtained by assuming one of the system variable as a state variable and other state variables are the derivatives of the selected system variable. Most of the time, the system variable used is the output variable which is used to select the state variable.

Such set of phase variables is easily obtained if the differential equation of the system is known or the system transfer function is available.

15.7.1 State Model from Differential Equation

Consider a linear continuous time system represented by n^{th} order differential equation as,

$$Y^n + a_{n-1}Y^{n-1} + a_{n-2}Y^{n-2} + \dots + a_1\dot{Y} + a_0Y(t) = b_0U + b_1\dot{U} + \dots + b_{m-1}U^{m-1} + b_mU^m \qquad \dots (1)$$

In the equation, $Y^n(t) = \dfrac{dY^n(t)}{dt^n} = n^{th}$ derivative of Y(t).

For time invariant system, the coefficients $a_{n-1}, a_{n-2}, \dots a_0, b_0, b_1, \dots b_m$ are constants.

For the system,

$$Y(t) = \text{Output variable}$$

$$U(t) = \text{Input variable}$$

$Y(0), \dot{Y}(0), \ldots Y(0)^{n-1}$ represent the initial conditions of the system.

Consider the **simple case** of the system in which derivatives of the control force U(t) are absent.

Thus $\dot{U}(t) = \ddot{U}(t) = \ldots = U^{m}(t) = 0$

$$\therefore Y^{n} + a_{n-1} Y^{n-1} + \ldots + a_1 \dot{Y} + a_0 Y(t) = b_0 U(t) \qquad \ldots (2)$$

Choice of state variable is generally output variable Y(t) itself. And other state variables are derivatives of the selected state variable Y(t).

$$\therefore \quad X_1(t) = Y(t)$$

$$\therefore \quad X_2(t) = \dot{Y}(t) = \dot{X}_1(t)$$

$$\therefore \quad X_3(t) = \ddot{Y}(t) = \ddot{X}_1(t) = \dot{X}_2(t)$$

$$\vdots$$

Thus the various state equations are,

$$\dot{X}_1(t) = X_2(t)$$

$$\dot{X}(t) = X_3(t)$$

$$\vdots$$

$$\dot{X}_{n-1}(t) = X_n(t)$$

$$\dot{X}_n(t) = ?$$

Note that only n variables are to be defined to keep their number minimum Thus $\dot{X}_{n-1}(t)$ gives n^{th} state variable $X_n(t)$. But to complete state model $\dot{X}_n(t)$ is necessary.

Important : $\dot{X}_n(t)$ is to be obtained by substituting the selected state variables in the original differential equation (2). We have $Y(t) = X_1$, $\dot{Y}(t) = X_2$, $\ddot{Y}(t) = X_3, \ldots$

$Y^{n-1}(t) = X_n(t)$, $Y^{n}(t) = \dot{X}_n(t)$

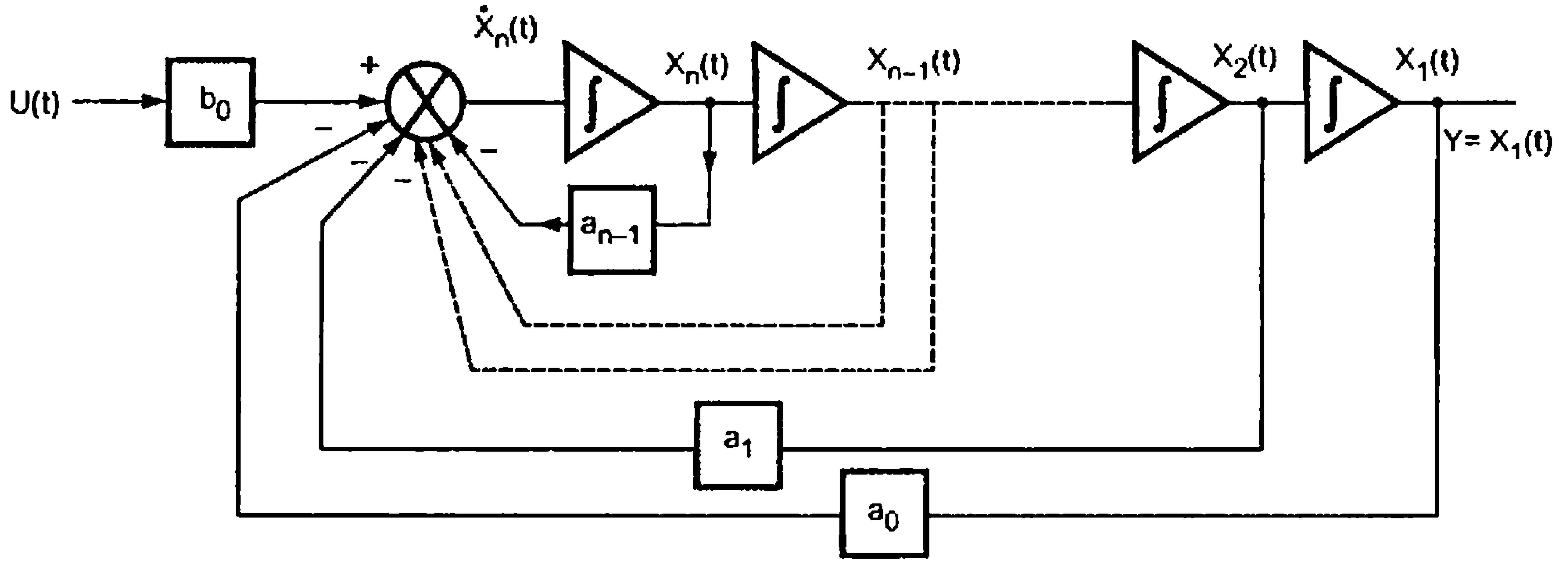


Fig. 15.14 State diagram for phase variable form

> Observe that the transfer function of the blocks in the various feedback paths are the coefficients existing in the original differential equation.

If the differential equation consists of the derivatives of the input control force U(t) then this method is not useful. In such a case, the state model is to be obtained from the transfer function.

Example 15.3 : *Construct the state model using phase variables if the system is described by the differential equation,*

$$\frac{d^2Y(t)}{dt^3}+4\frac{d^2Y(t)}{dt^2}+7\frac{dY(t)}{dt}+2Y(t)=5U(t)$$

Draw the state diagram.

Solution : Choose output Y(t) as the state variable $X_1(t)$ and successive derivatives of it give us remaining state variables. As order of the equation is 3, **only 3 state variables are allowed.**

$$X_1(t) = Y(t)$$

$\therefore$

$$X_2(t) = \dot{X}_1(t) = \dot{Y}(t) = \frac{dY(t)}{dt}$$

and

$$X_3(t) = \dot{X}_2(t) = \ddot{Y}(t) = \frac{d^2Y(t)}{dt^2}$$

Thus

$$\dot{X}_1(t) = X_2(t) \quad \text{... (1)}$$

$$\dot{X}_2(t) = X_3(t) \quad \text{... (2)}$$

To obtain $\dot{X}_3(t)$, substitute state variables obtained in the differential equation.

$$\frac{d^3Y(t)}{dt^3} = \dddot{Y}(t) = \frac{d}{dt}[\ddot{Y}(t)] = \frac{dX_3}{dt} = \dot{X}_3(t)$$

$\therefore \dot{X}_3(t) + 4X_3(t) + 7X_2(t) + 2X_1(t) = 5U(t)$

$$\therefore \quad \dot{X}_3(t) = -2X_1(t) - 7X_2(t) - 4X_3(t) + 5U(t) \quad \text{... (3)}$$

The equations (1), (2) and (3) give us required state equation.

$$\therefore \quad \dot{X}(t) = A\,X(t) + B\,U(t)$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$

The output is, $Y(t) = X_1(t)$

$$\therefore \quad Y(t) = C\,X(t) + D\,U(t)$$

where $C = [1\ 0\ 0]$, $D = 0$

This is the required state model using phase variables.

The state diagram is shown in the Fig. 15.15.

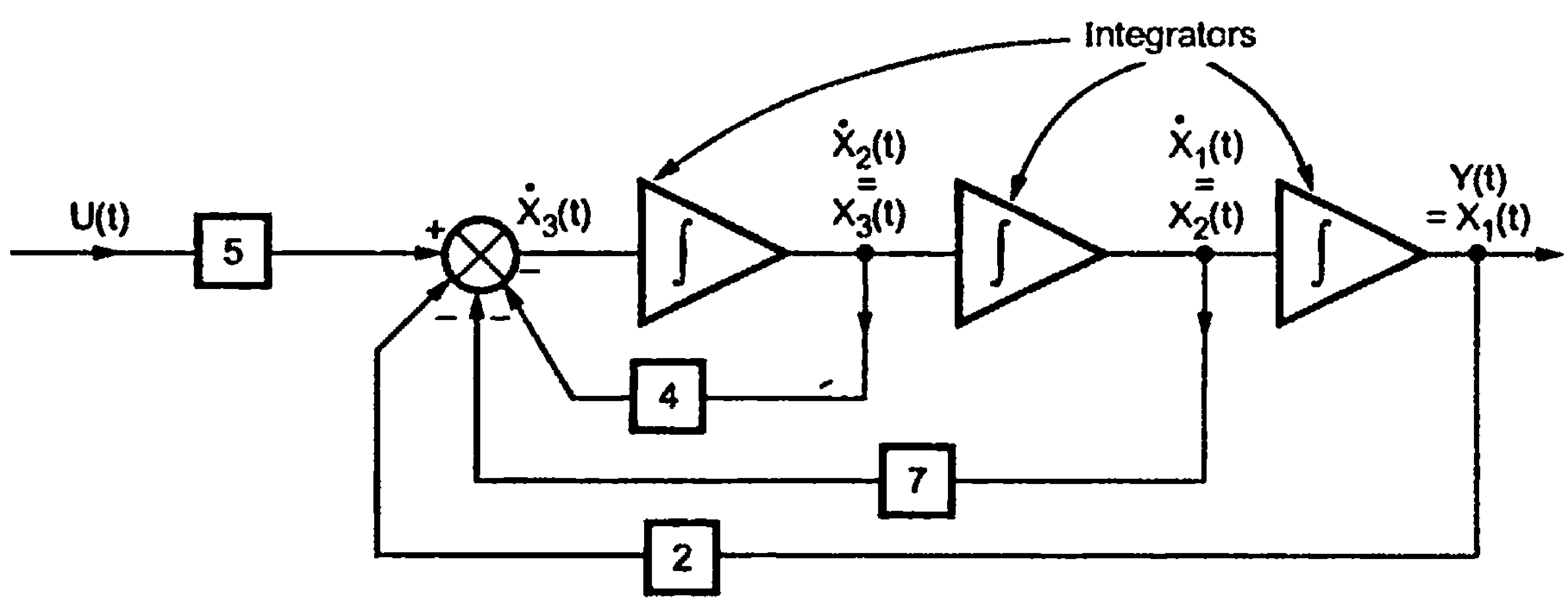


Fig. 15.15

15.7.2 State Model from Transfer Function

Consider a system characterized by the differential equation containing derivatives of the input variable U(t) as,

$$Y^n + a_{n-1}Y^{n-1} + \dots + a_1\dot{Y} + a_0 Y(t) = b_0 U + b_1 \dot{U} + \dots + b_{m-1} U^{m-1} + b_m U^m \quad \text{... (1)}$$

In such a case, it is advantageous to obtain the transfer function, assuming zero initial conditions. Taking Laplace transform of both sides of equation (1) and neglecting initial conditions we get,

$$Y(s)\,[s^n + a_{n-1}\,s^{n-1} + \ldots + a_1\,s + a_0] = [b_0 + sb_1 + \ldots + b_{m-1}\,s^{m-1} + b_m\,s^m]\,U(s)$$

$$\therefore \quad \frac{Y(s)}{U(s)} = T(s) = \frac{b_0 + sb_1 + \ldots + b_{m-1}s^{m-1} + b_m s^m}{a_0 + sa_1 + \ldots + a_{n-1}s^{n-1} + s^n} \quad \ldots (2)$$

Practically in most of the control systems m < n but for general case, let us assume m = n.

$$\therefore \quad T(s) = \frac{b_0 + sb_1 + \ldots + b_{n-1}s^{n-1} + b_n s^n}{a_0 + sa_1 + \ldots + a_{n-1}s^{n-1} + s^n}$$

From such a transfer function, state model can be obtained and then zero initial conditions can be replaced by the practical initial conditions to get required result.

There are two methods of obtaining state model from the transfer function,

1) Using signal flow graph approach

2) Using direct decomposition of transfer function

1) Using Signal Flow Graph Approach

The Mason's gain formula for signal flow graph states that,

$$T(s) = \frac{\Sigma T_k \Delta_k}{\Delta}$$

Where T_k = Gain of k^{th} forward path

Δ = system determinant

$$= 1 - \{\Sigma \text{ all loop gains}\} + \left\{\begin{matrix}\Sigma \text{ Gain} \times \text{Gain product of} \\ \text{all combinations of two} \\ \text{non - touching loops}\end{matrix}\right\} - \ldots$$

Δ_k = Value of Δ elliminating those loop gains and products which are touching to k^{th} forward path

According to this formula, construct the signal flow graph from the transfer function. From the signal flow graph, state model can be obtained. While obtaining signal flow graph, try to get the gains of branches as '1/s' representing the integrators. This helps to obtain the required state model. To clear this idea, let us consider the system having transfer function,

$$T(s) = \frac{b_0 + sb_1 + s^2 b_2 + s^3 b_3}{a_0 + sa_1 + s^2 a_2 + s^3} = \frac{Y(s)}{U(s)}$$

Divide both numerator and denominator by highest power of s,

$$\therefore \quad T(s) = \frac{\frac{b_0}{s^3}+\frac{b_1}{s^2}+\frac{b_2}{s}+b_3}{1+\frac{a_2}{s}+\frac{a_1}{s^2}+\frac{a_0}{s^3}} = \frac{b_3+\frac{b_2}{s}+\frac{b_1}{s^2}+\frac{b_0}{s^3}}{1-\left[-\frac{a_2}{s}-\frac{a_1}{s^2}-\frac{a_0}{s^3}\right]}$$

$$= \frac{T_1\Delta_1+T_2\Delta_2+T_3\Delta_3+T_4\Delta_4}{1-[\Sigma \text{All loop gains}]+[\Sigma \text{Gain products of 2 non-touching loops}]-\cdots}$$

Assuming that there are no combinations of 2 and more non-touching loops.

$\therefore$ Loop gains are, $L_1 = -\frac{a_2}{s}$, $L_2 = -\frac{a_1}{s^2}$, $L_3 = -\frac{a_0}{s^3}$

And let $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$ i.e. all loops are touching to all the forward paths. Hence forward path gains are,

$$T_1 = b_3,\ T_2 = \frac{b_2}{s},\ T_3 = \frac{b_1}{s^2},\ T_4 = \frac{b_0}{s^3}$$

Involving various branches having gains $\frac{1}{s}$, the signal flow graph can be obtained as,

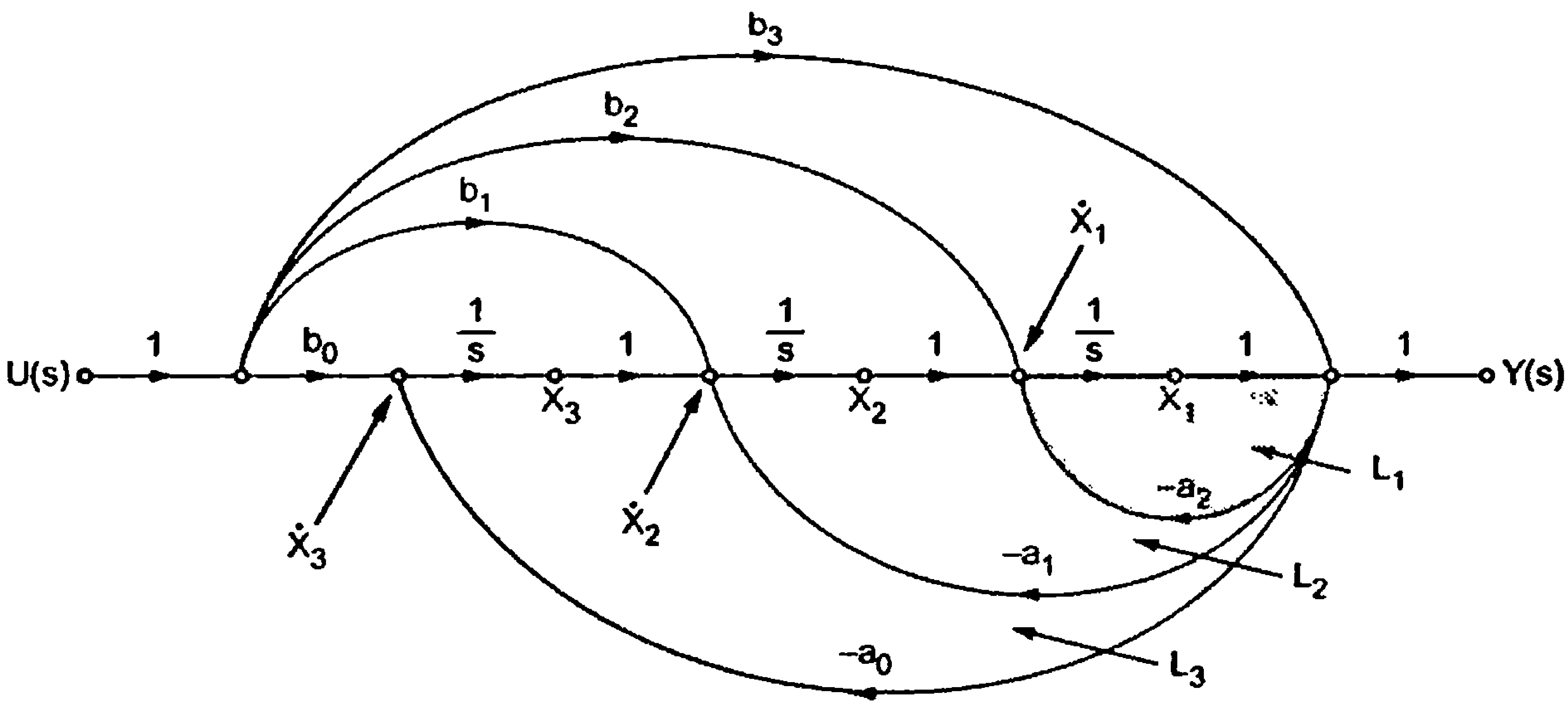


Fig. 15.16

Each branch with gain $\frac{1}{s}$ represents an integrator. **Output of each integrator is a state variable.** According to signal flow graph, value of the variable at the node is an algebraic sum of all the signals entering at that node. Outgoing branches does not affect the value of variable. Hence from signal flow graph,

$$\dot{X}_1 = b_2U + X_2 - a_2 Y$$

There are many signal flow graphs which can be obtained to satisfy above transfer function.

Method 1 : The signal flow graph is as shown in the Fig. 15.17.

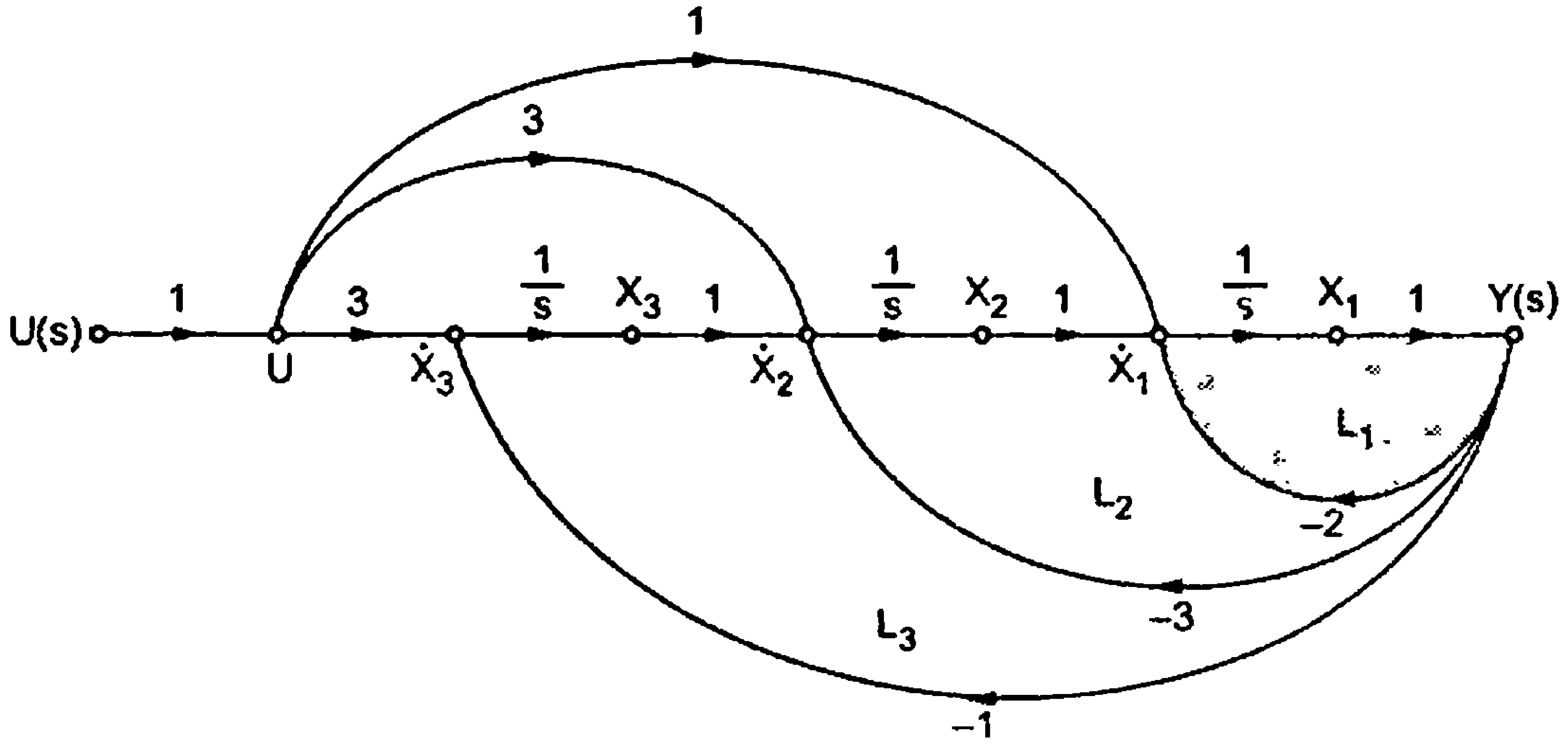


Fig. 15.17

From signal flow graph,

$$Y = X_1$$

$$\dot{X}_1 = X_2 + U - 2Y = -2X_1 + X_2 + U$$

$$\dot{X}_2 = X_3 + 3U - 3Y = -3X_1 + X_3 + 3U$$

$$\dot{X}_3 = 3U - Y = -X_1 + 3U$$

Hence the state model is,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} U(t)$$

i.e. $$\dot{X} = A\,X(t) + B\,U(t)$$

and $$Y(t) = [1 \;\; 0 \;\; 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

i.e. $$Y(t) = X(t) \quad \text{with } D = 0$$

element with transfer function $\frac{1}{s+a}$. From block diagram algebra, the transfer function of minor feedback loop is $\frac{G}{1+GH}$ for negative feedback.

Let $$\frac{1}{s+a} = \frac{\frac{1}{s}}{1+\frac{a}{s}} = \frac{G}{1+GH}$$

where $$G = \frac{1}{s} = \text{integrator and } H = a$$

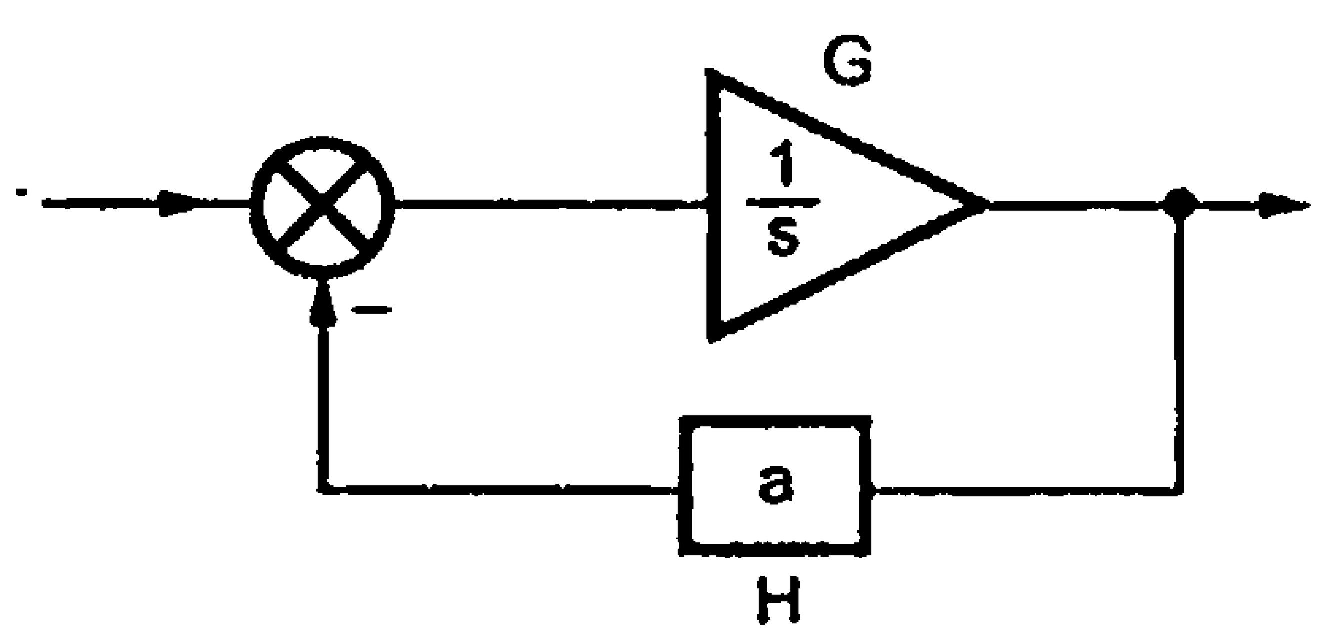


Fig. 15.19

The feedback is negative and the transfer function can be simulated as shown in the Fig. 15.19. with a minor feedback loop.

Now if such a loop is added in the forward path of another such loop then we get the block diagram as shown in the Fig. 15.20.

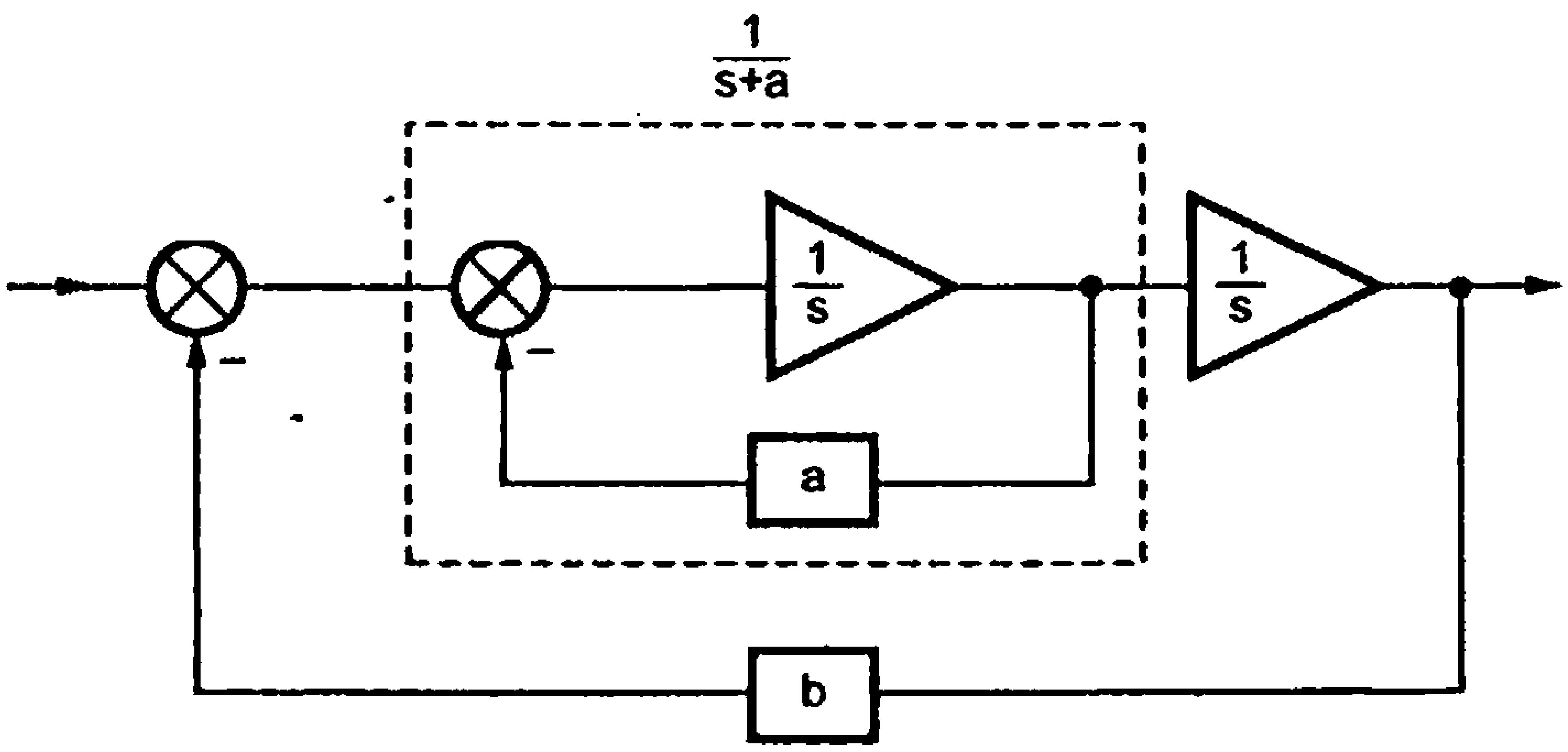


Fig. 15.20

The transfer function now becomes,

$$= \frac{\frac{1}{s(s+a)}}{1+\frac{b}{s(s+a)}} = \frac{1}{s(s+a)+b} = \frac{1}{sX+b}$$

Where $$X = (s+a)$$

If now the entire block shown in the Fig. 15.20 is added in the forward path of another minor loop with an integrator and feedback gain 'c', we get the transfer function as,

$$= \frac{1}{sY+c} \text{ where } Y = s X + b$$

Thus the denominator of transfer function becomes

$s (sX + b) = [s (s \{s + a\} + b)] = s^3 + as^2 + bs$

Thus denominator of any order can be directly programmed as discussed above.

$$s^2 + as + b \Rightarrow \{s (s + a) + b\}$$

$$s^3 + as^2 + bs + c \Rightarrow \{[(s + a) s + b] s + c\}$$

$$s^4 + as^2 + bs^2 + cs + d \Rightarrow \{([(s + a) s + b] s + c) s + d\}$$ and so on.

Now if numerator is $b_1 s + b_0$ simulation is obtained directly as shown in the block diagram in the Fig. 15.21. But $s = \frac{d}{dt}$, which is differentiator and is not used to obtain state model. In such a case, take off point 't' is shifted before the last integrator block.

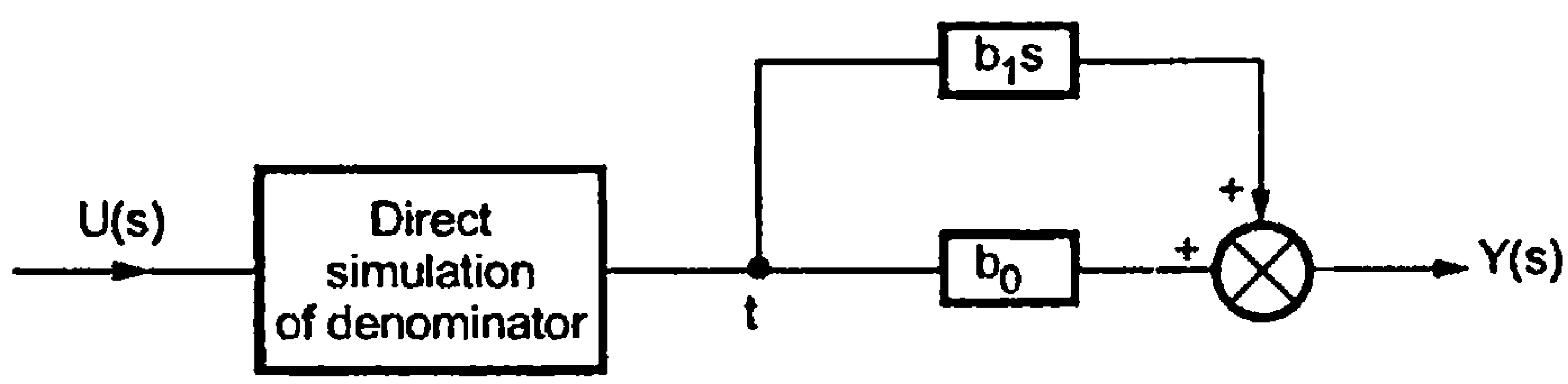


Fig. 15.21 (a)

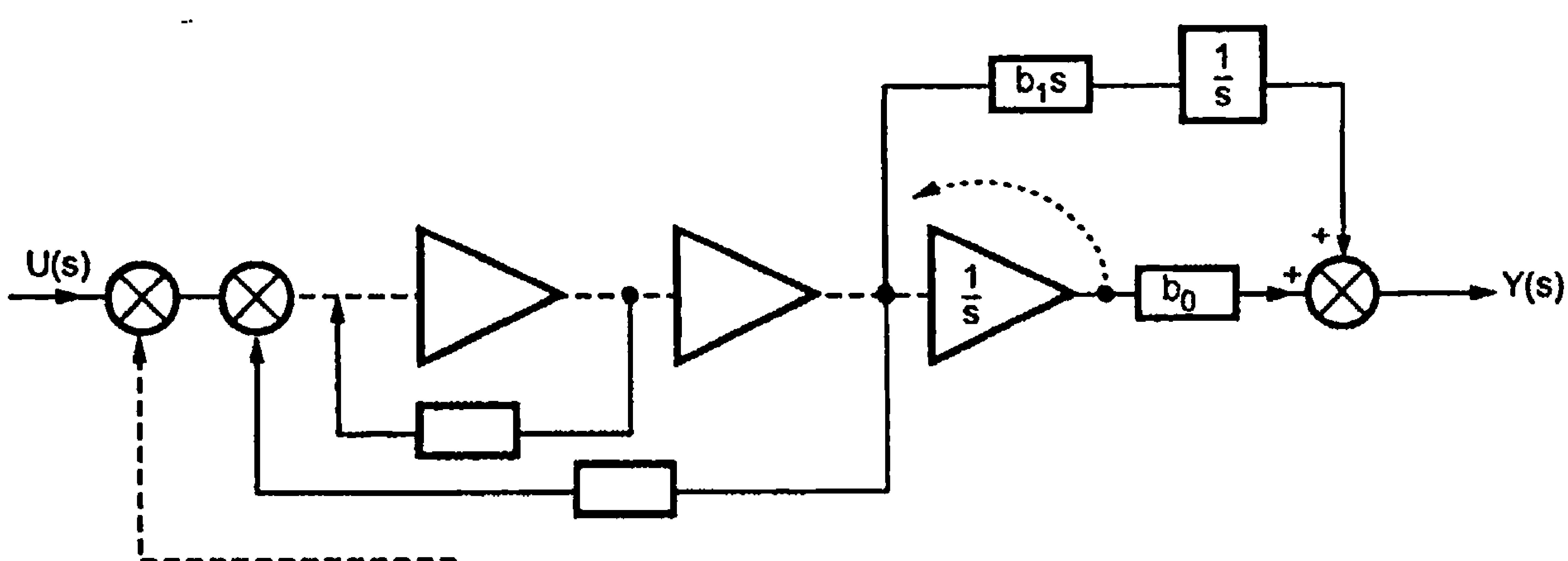


Fig. 15.21 (b)

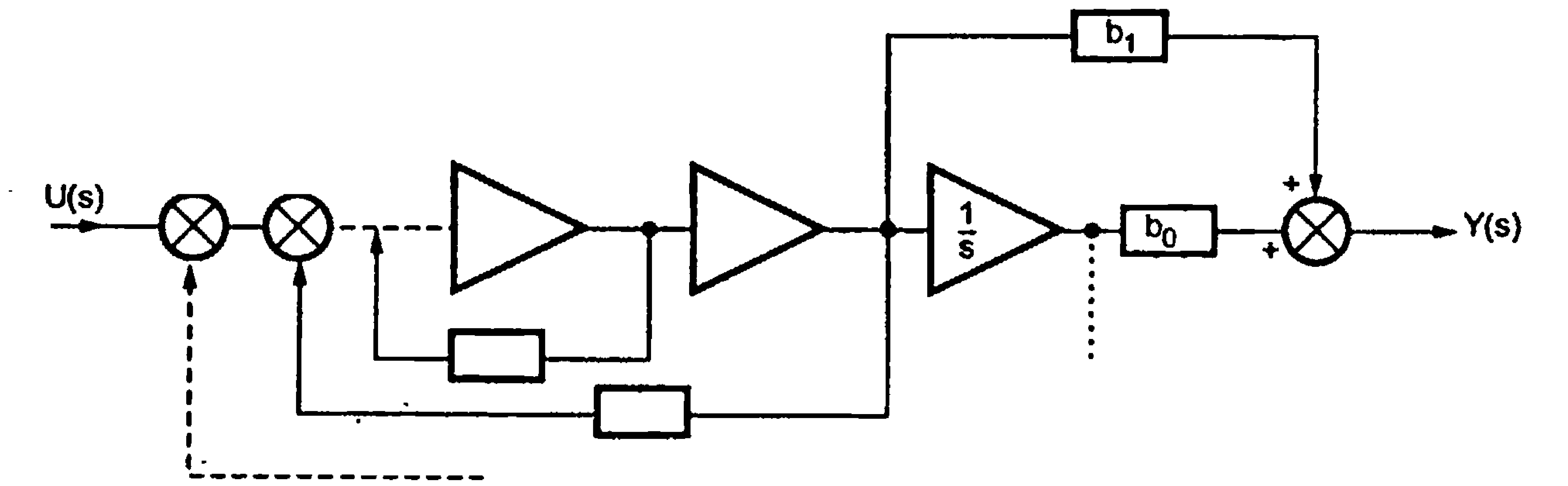


Fig. 15.21 (c)

According to block diagram reduction rule, while shifting takeoff point before the block, the takeoff signal must be multiplied by transfer function of block before which it is to be shifted. Thus we get block of b_1 with take off from input of last integrator.

Similarly if there is a term $b_2 s^2$ in the numerator then shift takeoff point before one more integrator as shown in the Fig. 15.21 (d).

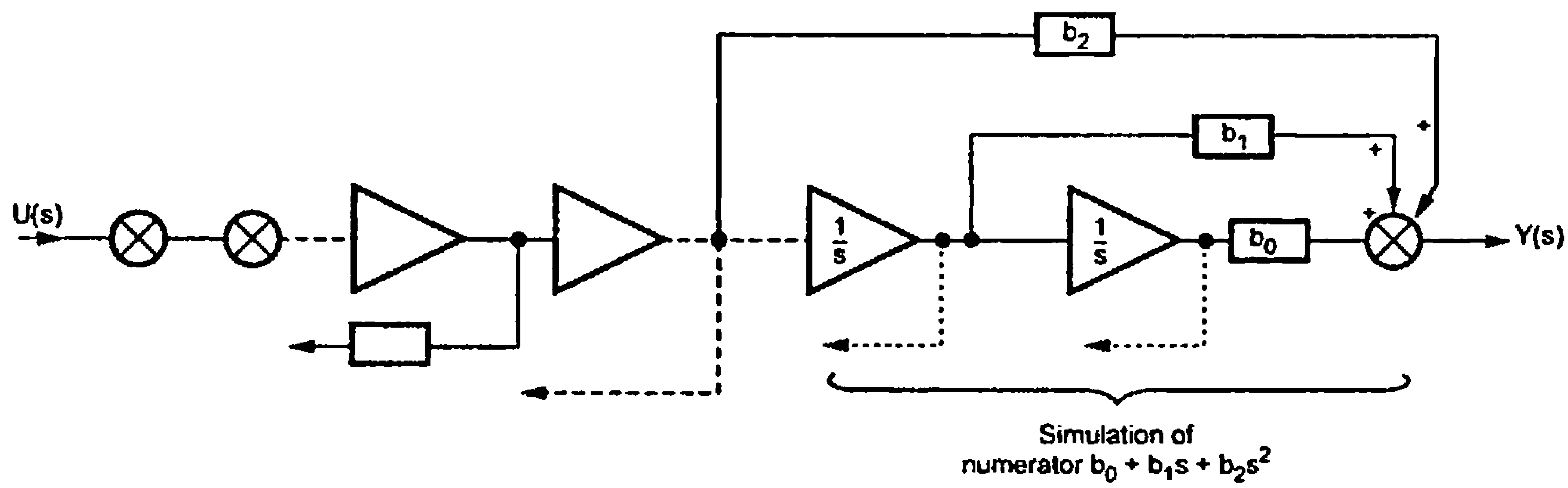


Fig. 15.21 (d)

Thus for any order of numerator, complete simulation of the transfer function can be achieved.

Then assigning output of each integrator as the state variable, state model in the phase variable form can be obtained.

Example 15.5 : *Obtain state model by direct decomposition method of a system whose transfer function is*

$$\frac{Y(s)}{U(s)} = \frac{5s^2+6s+8}{s^3+3s^2+7s+9}$$

Solution : Decompose denominator as below,

$$s^3+3s^2+7s+9 = \{([s+3]s+7)s+9\}$$

Its simulation starts from (s + 3) in denominator

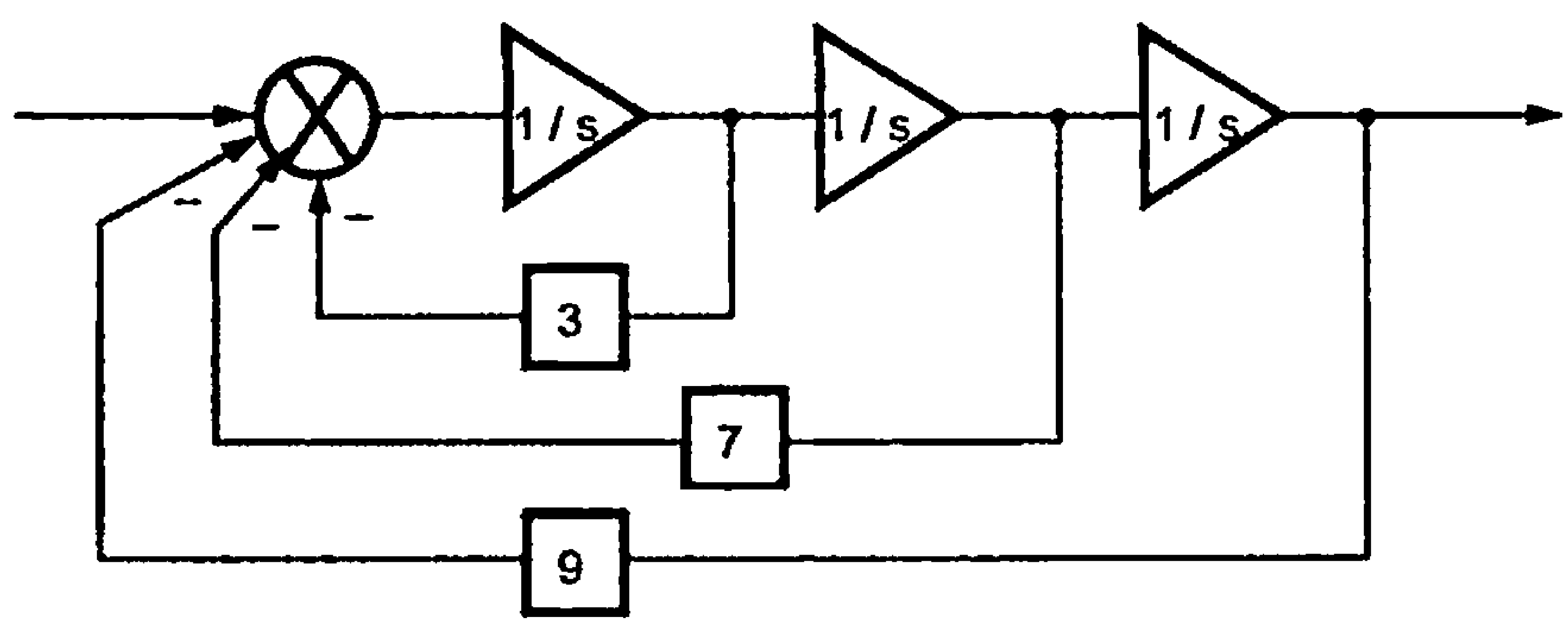


Fig. 15.22

To simulate numerator, shift takeoff point once for 6s and shift twice for $5s^2$.

Therefore complete state diagram can be obtained as follows.

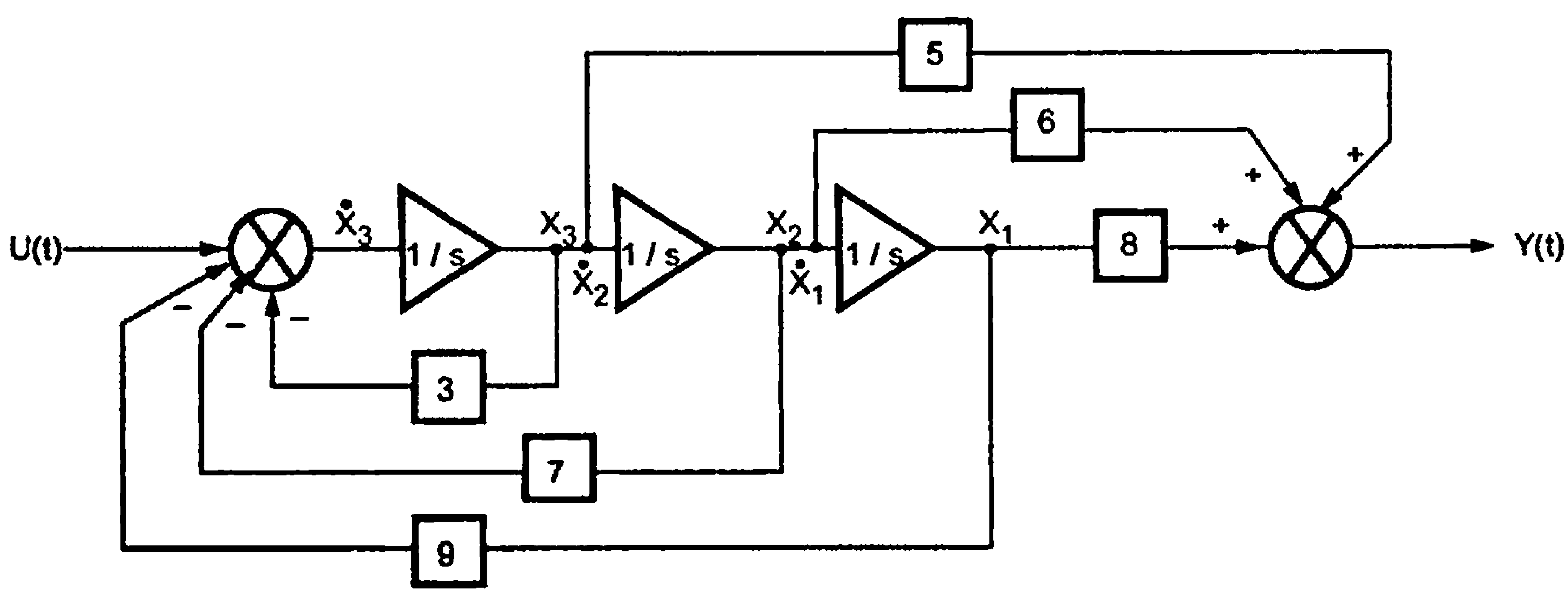


Fig. 15.23

Assign output of each integrator as the state variable.

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = X_3$$

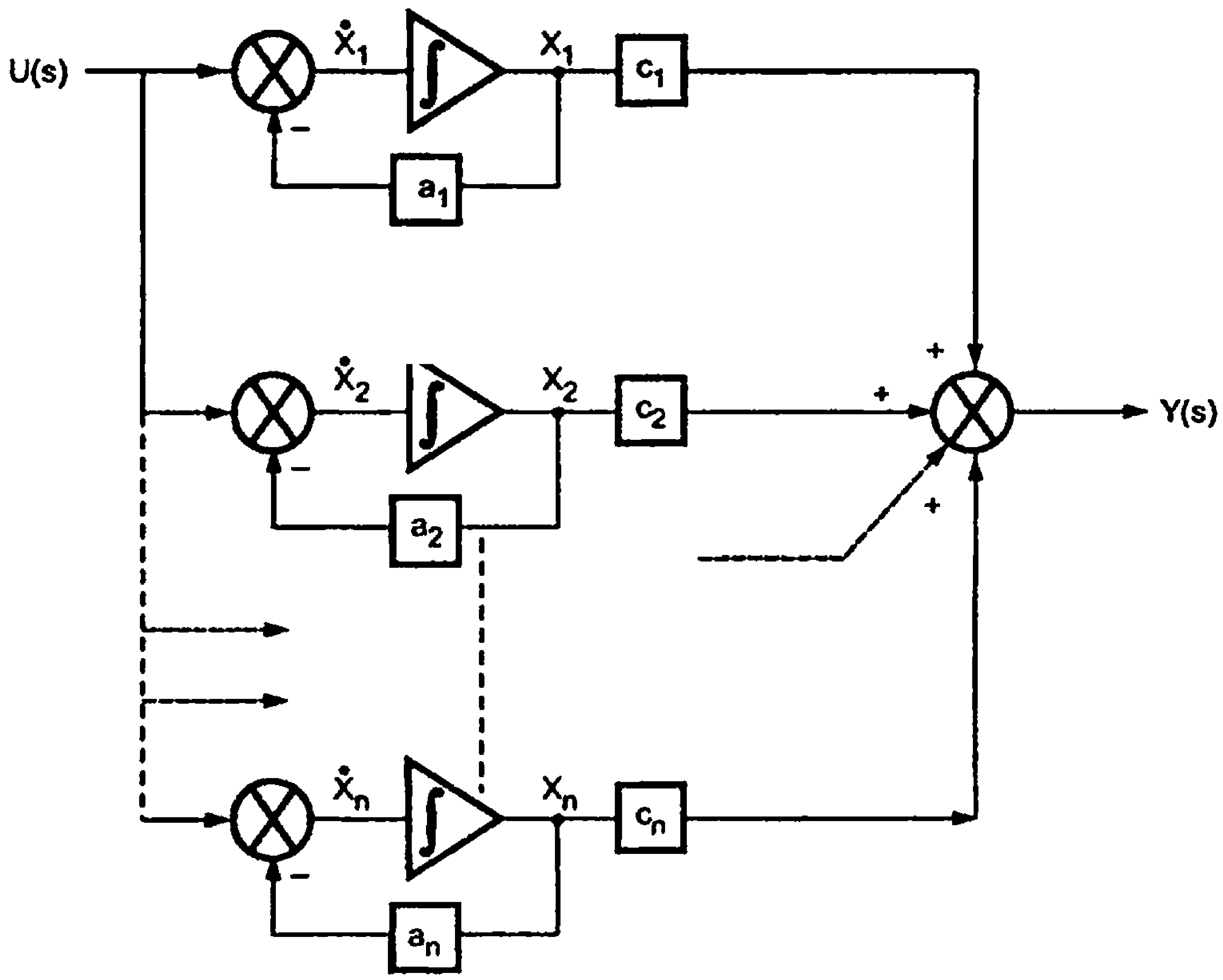


Fig. 15.25 Foster's form simulation

Where

$$A = \begin{bmatrix} -a & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [c_1 \ c_2 \ \ldots\ldots \ c_n] \quad \text{and} \quad D = 0$$

Case 2 : If the degree m = n i.e. numerator and the denominator have same degree then first divide the numerator by denominator and then obtain partial fractions of remaining factor.

$$\therefore \qquad T(s) = \frac{N(s)}{D(s)} = c_0 + \sum_{i=1}^{n} \frac{c_i}{s+a_i}$$

where c_0 = Constant obtained by dividing N(s) by D(s).

In such a case, the state diagram for partial fraction terms remains same as before and in addition to all the outputs, c_0 U(t) gets added to obtain the resultant output as shown in the Fig. 15.26.

The method of obtaining partial fraction for such a case is,

$$T(s) = \frac{c_1}{(s+a_1)^r} + \frac{c_2}{(s+a_1)^{r-1}} + \ldots + \frac{c_r}{(s+a_1)} + \frac{c_{r+1}}{(s+a_2)} + \ldots + \frac{c_n}{(s+a_n)} \qquad \ldots m < n$$

If the degree of N(s) and D(s) is same i.e. m = n we get additional constant c_0 as,

$$T(s) = c_0 + \frac{c_1}{(s+a_1)^r} + \ldots + \frac{c_r}{(s+a_1)} + \frac{c_{r+1}}{(s+a_2)} + \ldots + \frac{c_n}{(s+a_n)} \qquad \ldots m = n$$

This can be mathematically expressed as,

$$T(s) = \sum_{i=1}^{r} \frac{c_i}{(s+a_1)^{r-i+1}} + \sum_{i=r+1}^{n} \frac{c_i}{(s+a_i)} \qquad \ldots m < n$$

and

$$T(s) = c_0 + \sum_{i=1}^{r} \frac{c_i}{(s+a_1)^{r-i+1}} + \sum_{i=r+1}^{n} \frac{c_i}{(s+a_i)} \qquad \ldots m = n$$

Key Point: ***Note that in partial fraction expansion, a separate coefficient is assumed for each power of repeated factor.***

In simulating such an equation by parallel programming, $\frac{1}{(s+a_1)^r}$ is simulated by connecting $\frac{1}{(s+a_1)}$ groups, r times in series first. While all other distinct factors are simulated by parallel programming as before. The components of each power of $\frac{1}{(s+a_1)}$ to be added to get output is to be taken from output of each integrator which are connected in series. This is shown in the Fig. 15.28.

Now assign state variables at the output of each integrator. **For series integrators, assign the state variables from right to left, as shown in the Fig. 15.28.**

For series integrators, the state equations are,

$$\dot{X}_1 = -a_1 X_1 + X_2$$

$$\dot{X}_2 = -a_1 X_2 + X_3$$

$$\vdots$$

$$\dot{X}_{r-1} = -a_1 X_{r-1} + X_r$$

$$\dot{X}_r = -a_1 X_r + U(t)$$

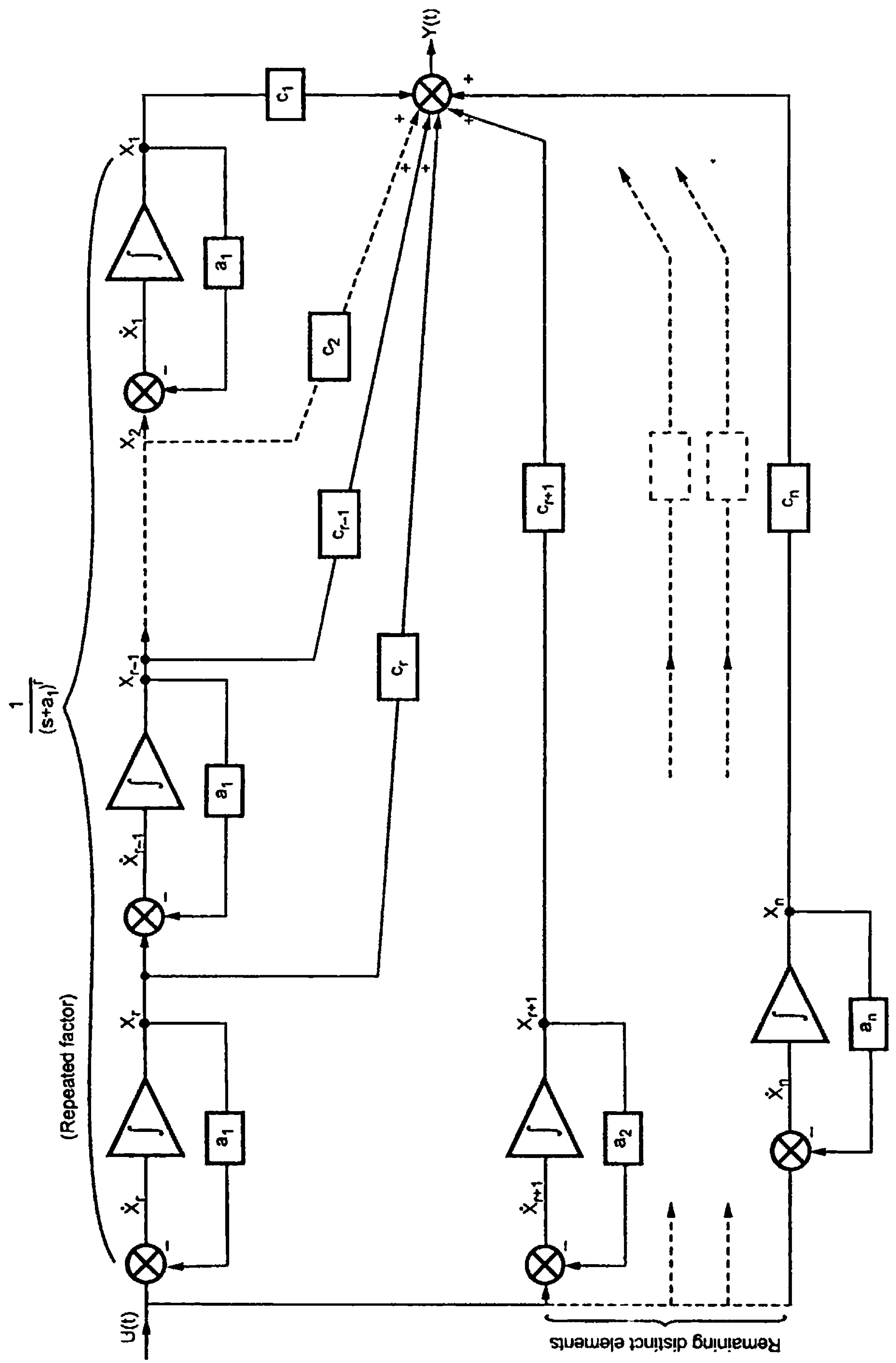


Fig. 15.28 State diagram for Jordan's form with m < n

$$A + 3B + 4C = 0, \quad \text{... from power of s}$$

$$A + 2B + 4C = 1 \quad \text{... from constant term}$$

Solving we get, $A = -1,\ B = -1,\ C = 1$

$$T(s) = \frac{-1}{(s+2)^2} - \frac{1}{(s+2)} + \frac{1}{(s+1)}$$

Simulate first term by series integrators while other nonrepeated terms by parallel integrators.

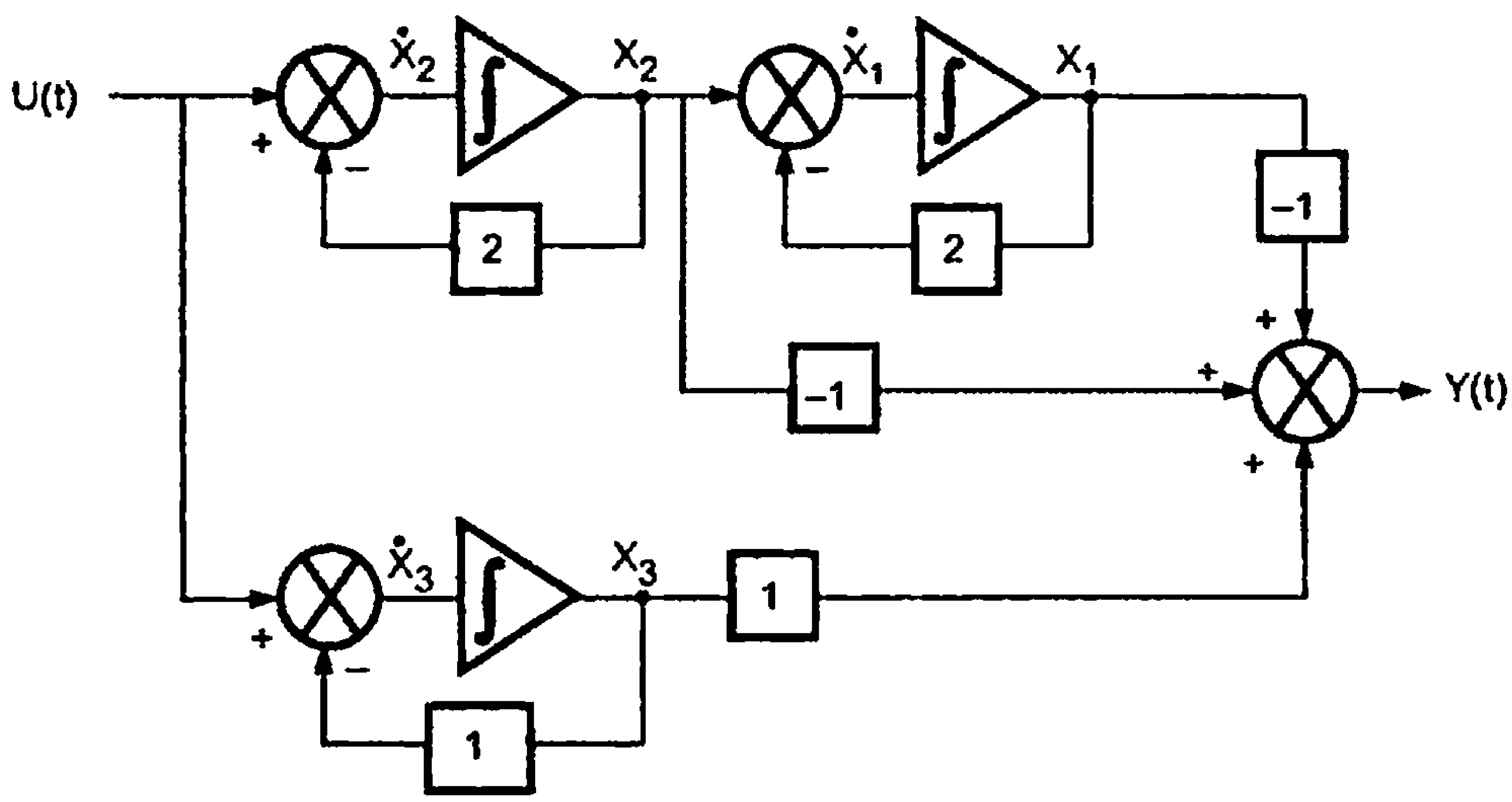


Fig. 15.30

Total simulation is,

$$\dot{X}_1 = -2X_1 + X_2 \qquad \dot{X}_2 = U(t) - 2X_2 \qquad \dot{X}_3 = U(t) - X_3(t)$$

$$Y(t) = -X_1(t) - X_2(t) + X_3(t)$$

$\therefore$ State model is , $\dot{X} = AX + BU$ and $Y = CX + DU$

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}, \quad D = 0$$

The matrix A consists of Jordan block.

Important Note : If some of the poles of T(s) are complex in nature, then a mixed approach can be used. The quadratic or higher order polynomial having complex roots can be simulated by direct decomposition while real distinct roots can be simulated by the parallel programming using canonical variable.

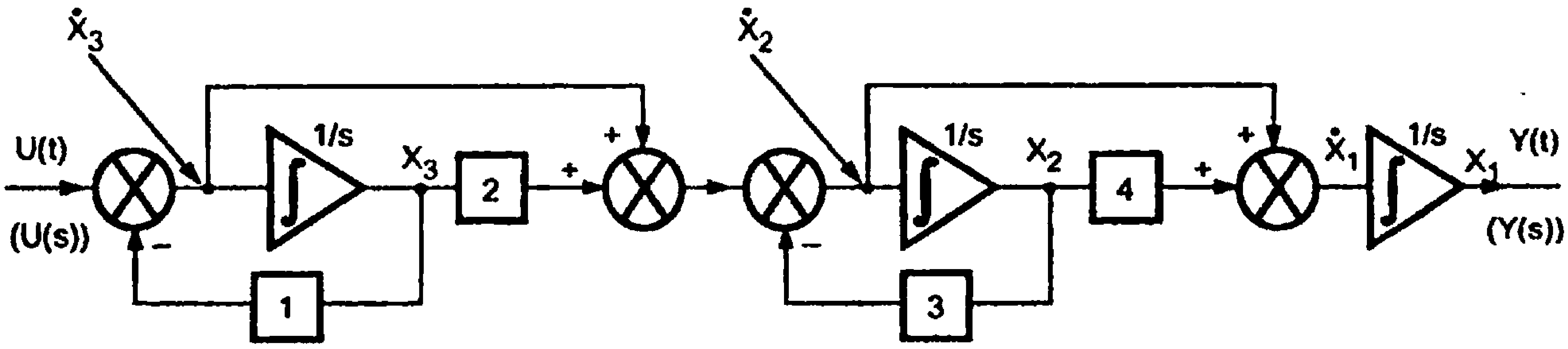


Fig. 15.32

Now $\dot{X}_1 = \dot{X}_2 + 4X_2$, $\dot{X}_2 = -3X_2 + 2X_3 + \dot{X}_3$

and $\dot{X}_3 = U(t) - X_3$

Substituting $\dot{X}_3$ into $\dot{X}_2$ equation,

$$\dot{X}_2 = -3X_2 + 2X_3 + U(t) - X_3 = U(t) - 3X_2 + X_3$$

Substituting $\dot{X}_2$ into $\dot{X}_1$ equation,

$$\dot{X}_1 = U(t) - 3X_2 + X_3 + 4X_2 = U(t) + X_2 + X_3$$

Model becomes,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U(t)$$

and $Y(t) = X_1(t)$

i.e. $$Y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

So $$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [1 \ 0 \ 0] \qquad D = 0$$

15.11.1 Homogeneous Equation

If A is a constant matrix and input control forces are zero then the equation takes the form,

$$\dot{X}(t) = A\,X(t) \qquad ...(1)$$

Such an equation is called **homogeneous equation.** The obvious equation is if input is zero,how output can exist ? In such systems, the driving force is provided by the initial conditions of the system to produce the output. For example, consider a series RC circuit in which capacitor is initially charged to V volts. The current is the output. Now there is no input control force i.e.. external voltage applied to the system. But the initial voltage on the capacitor drives the current through the system and capacitor starts discharging through the resistance R. Such a system which works on the initial conditions without any input applied to it is called homogeneous system.

15.11.2 Nonhomogeneous Equation

If A is a constant matrix and matrix U(t) is non zero vector i.e. the input control forces are applied to the system then the equation takes normal form as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \qquad ...(2)$$

Such an equation is called **nonhomogenous equation.** Most of the practical systems require inputs to drive them. Such systems are nonhomogeneous linear systems.

The solution of the state equation is obtained by considering basic method of finding the solution of homogenous equation.

15.12 Review of Classical Method of Solution

Consider a scalar differential equation as,

$$\frac{dx}{dt} = ax \quad \text{where} \quad x(0) = x_0 \qquad ...(1)$$

This is a homogenous equation without the input vector.

Assume the solution of this equation as,

$$x(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_k\, t^k \qquad ...(2)$$

$$\text{At} \quad t = 0,\ x(0) = x_0 = b_0 \qquad ...(3)$$

The solution has to satisfy the original differential equation hence using (2) in (1),

$$\frac{d}{dt}[b_0 + b_1 t + \ldots + b_k t^k] = a\,[b_0 + b_1 t + \ldots + b_k\, t^k]$$

$$\therefore \quad b_1 + 2b_2 t + \ldots + k b_k t^{k-1} = ab_0 + ab_1 t + \ldots + ab_k\, t^k$$

For validity of this equation, the coefficients of various powers of 't' on both sides, must be equal.

$\therefore \quad b_1 = ab_0,\ 2b_2 = ab_1,\ \ldots\ kb_k = ab_{k-1}$

$\therefore \quad b_1 = a\, b_0$

$$b_2 = \frac{1}{2} a\, b_1 = \frac{1}{2} a^2 b_0 = \frac{1}{2!} a^2\, b_0$$

$$b_3 = \frac{1}{3} a\, b_2 = \frac{1}{3\times 2} a^3 b_0 = \frac{1}{3!} a^3\, b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!} a^k\, b_0 \quad \text{and} \quad x(0) = b_0$$

Using all these values in the assumed solution,

$$x(t) = b_0 + ab_0(t) + \frac{1}{2!} a^2\, b_0 t^2 + \ldots + \frac{1}{k!} a^k\, b_0\, t^k$$

$$= b_0 \left[1 + at + \frac{1}{2!} a^2 t^2 + \ldots + \frac{1}{k!} a^k t^k\right]$$

$$= \left[1 + at + \frac{1}{2!} a^2 t^2 + \ldots + \frac{1}{k!} a^k t^k\right] x(0)$$

But $1 + at + \frac{1}{2!} a^2\, t^2 + \ldots + \frac{1}{k!} a^k\, t^k = e^{at}$

$$\therefore \quad \dot{X}(t) = e^{at}\, x(0) \qquad \ldots (4)$$

This is the required solution of homogeneous equation in scalar form.

Thus if the homogeneous state equation is considered,

$$\dot{X}(t) = A\, X(t)$$

then its solution can be written as,

$$\boxed{X(t) = e^{At}\, X(0)}$$

In this case, e^{At} **is not a scalar but a matrix of order n × n as that of matrix A.**

Observation : It can be observed that without input, initial state X(0) drives the state X(t) at any time t. Thus there is transition of the initial state X(0) from initial time t = 0 to any time t through the matrix e^{At}. As it is an exponential term, the matrix e^{At} is called **matrix exponential.** It is also responsible for the transition of the state X(t) at any time t from initial time hence also called **state transition matrix.** It is denoted as $\phi(t)$.

$$\therefore \quad \phi(t) = e^{At} = \text{State transition matrix}$$

And

$$\boxed{e^{At} = I + At + \frac{1}{2!} A^2\, t^2 + \ldots + \frac{1}{k!} A^k\, t^k + \ldots}$$

The each term of the above equation is a matrix of order n × n.

If instead of initial time t = 0, it is selected as $t = t_0$ then the state transition matrix is,

$$\phi(t) = e^{A(t-t_0)}$$

15.12.1 Zero Input Response

The solution of the homogeneous state equation is under the condition of zero input. Such a response is called **zero input response** and for convenience denoted as ZIR.

Thus the behaviour of X(t) under the initial conditions without any input U(t) is called the **zero input response** of the system. It is also called **free, natural** or **unforced response** of the system.

15.13 Solution of Nonhomogeneous Equation

Consider a nonhomogeneous state equation as,

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$\therefore \quad \dot{X}(t) - A\,X(t) = B\,U(t)$$

Premultiplying both sides by e^{-At},

$$e^{-At}[\dot{X}(t) - A\,X(t)] = e^{-At}\,B\,U(t)$$

$$\text{But } e^{-At}\,\dot{X}(t) - e^{-At}\,A\,X(t) = \frac{d}{dt}[e^{-At}\,X(t)]$$

Substituting in above equation,

$$\frac{d}{dt}[e^{-At}\,X(t)] = e^{-At}\,B\,U(t)$$

Assuming initial time as t = 0 and integrating both sides from t = 0 to t,

$$e^{-At}X(t)\Big|_0^t = \int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

$$\therefore \quad e^{-At}\,X(t) - X(0) = \int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

Premultiplying both sides by e^{At},

$$\therefore e^{At}\,e^{-At}\,X(t) - e^{At}X(0) = e^{At}\int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

$$\therefore \quad X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B\, U(\tau)\, d\tau \qquad \dots (1)$$

This is the complete **solution of the nonhomogeneous equation.**

Observations :

1. The solution is divided into two parts. The first part is $e^{At} X(0)$ which is nothing but the homogeneous solution or zero input response (ZIR).
2. The other part $\int_0^t e^{A(t-\tau)} B\, U(\tau)\, d\tau$ is the part existing only due to application of input U(t) from time 0 to t. It is called **forced solution or zero state response** (ZSR).

Thus the solution is,

$$X(t) = \underbrace{e^{At} X(0)}_{\text{ZIR}} + \underbrace{\int_0^t e^{A(t-\tau)} BU(\tau)\, d\tau}_{\text{ZSR}}$$

If the initial time is considered as $t = t_0$ rather than $t = 0$, the solution is,

$$X(t) = e^{A(t-t_0)} X(t_0) + \int_{t_0}^t e^{A(t-\tau)} B\, U(\tau) d\tau$$

In general, the solution of nonhomogeneous equation consists of both the parts, ZIR (zero input response) and ZSR (zero state response).

15.14 Properties of State Transition Matrix

The various useful properties of the state transition matrix are,

$$\phi(t) = e^{At} = \text{State transition matrix}$$

1. $\phi(0) = e^{A \times 0} = I = \text{Identity matrix}$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

   i.e. $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

   $= \phi(t_1)\, \phi(t_2) = \phi(t_2)\, \phi(t_1)$
4. $e^{A(t+s)} = e^{At} e^{As}$
5. $e^{(A+B)t} = e^{At} e^{Bt}$ only if AB = BA
6. $[\phi(t)]^n = [e^{At}]^n = e^{Ant} = \phi(nt)$
7. $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = \phi(t_2 - t_0)$

This property states that the process of transition of state can be divided into number of sequential transition. Thus t_0 to t_2 can be divided as t_0 to t_1 and t_1 to t_2, as stated in the property.

Interms of $\phi(t)$, the solution is expressed as,

$$X(t) = \phi(t - t_0)\, X(t_0) + \int_{t_0}^{t} \phi(t-\tau)\, B\, U(\tau)\, d\tau$$

where $\phi(t - t_0) = e^{A(t-t_0)}$

and $\phi(t - \tau) = e^{A(t-\tau)}$

8. $\phi(t)$ is a **nonsingular matrix** for all finite values of t.

15.15 Solution of State Equation by Laplace Transform Method

The Laplace transform method converts integro differential equations to simple algebraic equations. Due to this important property, it is very convenient to use Laplace transform method to obtain the solution of state equation.

Consider the nonhomogeneous state equation as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{... (1)}$$

Taking Laplace transform of both sides,

$$s\,X(s) - X(0) = A\,X(s) + B\,U(s)$$

$$\therefore \quad s\,X(s) - A\,X(s) = X(0) + B\,U(s)$$

As s is operator, multiplying it by Identify matrix of order n × n,

$$[sI - A]\,X(s) = X(0) + B\,U(s)$$

Premultiplying both sides by $[sI - A]^{-1}$,

$$\therefore [sI - A]^{-1}[sI - A]\,X(s) = [sI - A]^{-1}\{X(0) + B\,U(s)\}$$

$$\therefore \quad X(s) = [sI - A]^{-1}X(0) + [sI - A]^{-1}B\,U(s) \quad \text{... (2)}$$

$$= ZIR + ZSR$$

Comparing zero input response obtained earlier,

$$[sI - A]^{-1} = \phi(s)$$

and

$$\boxed{L^{-1}[sI - A]^{-1} = \phi(t) = e^{At}} \quad \text{... (3)}$$

The matrix $\phi(s) = [sI - A]^{-1}$ is called **resolvant matrix** of A. All the elements of this matrix are rational functions of s.

Taking inverse Laplace transform of (2),

$$X(t) = L^{-1}\{X(s)\} = L^{-1}\{[sI - A]^{-1}\} X(0) + L^{-1}\{[sI - A]^{-1} BU(s)\}$$

Using $[sI - A]^{-1} = \phi(s)$,

$$X(t) = L^{-1}[\phi(s)] X(0) + L^{-1}[\phi(s) B U(s)] \quad \ldots (4)$$

The term $L^{-1}[\phi(s) B U(s)]$ is called zero state response.

From the convolution theorem in Laplace transform it is known that,

$$L^{-1}\{F_1(s) F_2(s)\} = f_1(t) * f_2(t) = \text{convolution}$$

$$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$$

Thus if $F_1(s) = \phi(s)$ and $F_2(s) = BU(s)$ then,

$$L^{-1}\{\phi(s) B U(s)\} = \int_0^t \phi(t-\tau) BU(\tau) d\tau \quad \ldots (5)$$

Thus equation (5) shows that $L^{-1}\{\phi(s) B U(s)\}$ is the zero state response, as obtained earlier by classical approach.

$\therefore$

$$X(t) = L^{-1}[\phi(s)] X(0) + L^{-1}[\phi(s) B U(s)]$$

$$\phi(s) = [sI - A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|}$$

$$L^{-1}[\phi(s)] = \phi(t) = e^{At}$$

The **advantage** of this method is without carrying out actual integration which is time consuming, the zero state response can be easily obtained.

15.16 Computation of State Transition Matrix

While obtaining the solution of state equation, the computation of state transition matrix e^{At} plays an important role. There are various methods of obtaining state transition matrix from the state model. These methods are,

1. Laplace transform method
2. Power series method
3. Cayley Hamilton method
4. Similarity transformation method

Let us discuss these methods of obtaining e^{At} in detail.

$$\therefore \quad e^{At} = L^{-1}[sI - A]^{-1} = L^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Using partial fraction expansion for all the elements.

$$e^{At} = L^{-1}\begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{-1}{s+1}+\frac{1}{s+2} \\ \frac{2}{s+1}-\frac{2}{s+2} & \frac{-1}{s+1}+\frac{2}{s+2} \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix} = \phi(t)$$

This is the required state transition matrix.

Examples with Solutions

Example 15.12 : *Considering V_C and I_i as state variables and I_x as the output variables in the circuit shown below, obtain the state model.*

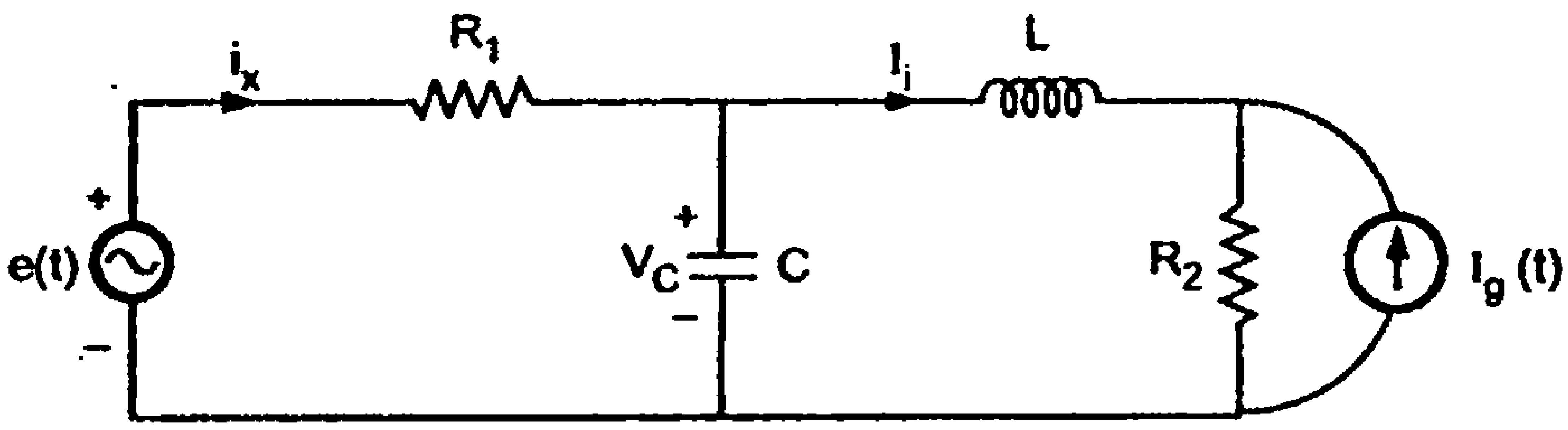


Fig. 15.33

Solution : Convert the current source to voltage source as shown.

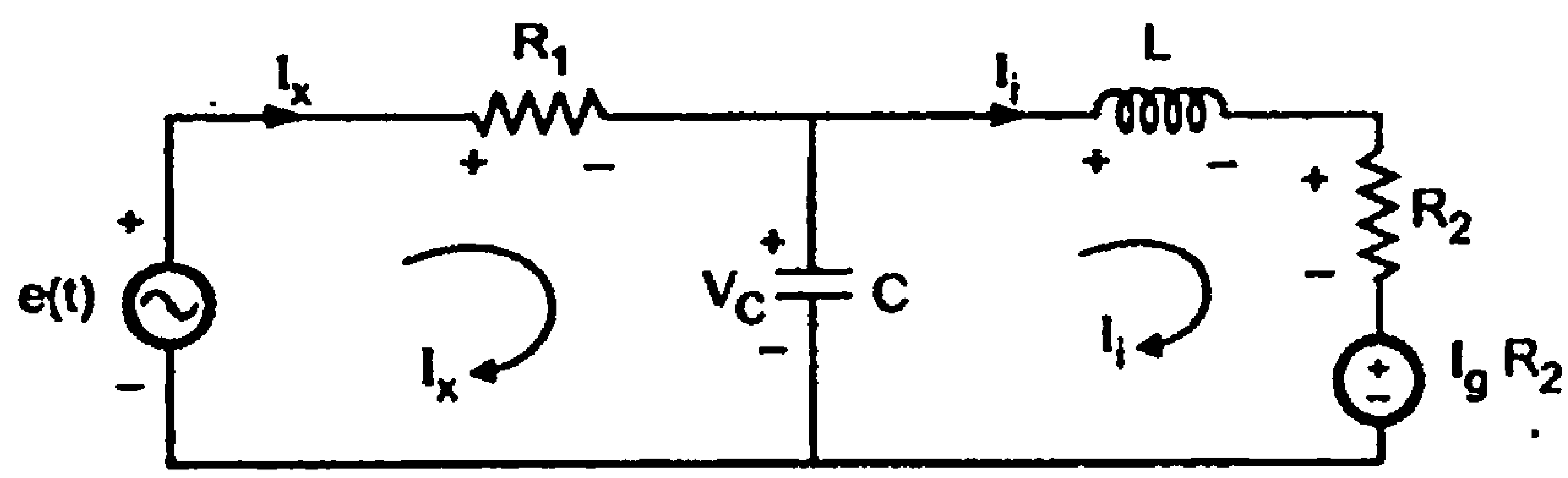


Fig. 15.33 (a)

Two inputs, e(t) and $I_g(t)$ i.e. $U_1 = e(t)$ and $U_2 = I_g(t)$

One output, $I_x(t)$ i.e. $Y(t) = I_x(t)$

Variables, $V_C = X_1(t)$ and $I_i = X_2(t)$

Let $I_x(t)$ and $I_i(t)$ be the loop currents.

Applying KVL to the two loops,

Loop 1, $-I_x R_1 - V_C + e(t) = 0$

$$\therefore \quad I_x R_1 = e(t) - V_C$$

$$\therefore \quad I_x = \frac{1}{R_1} e(t) - \frac{1}{R_1} v_C(t)$$

$$\therefore \quad Y(t) = \frac{1}{R_1} U_1 - \frac{1}{R_1} X_1(t) \qquad \text{... Output equation}$$

Loop 2, $-L\frac{dI_i}{dt} - I_i R_2 - I_g R_2 + V_C = 0$

$$\therefore \quad \frac{dI_i}{dt} = \frac{1}{L} v_C(t) - \frac{R_2}{L} I_i - \frac{R_2}{L} I_g$$

$$\therefore \quad \dot{X}_2 = \frac{1}{L} X_1(t) - \frac{R_2}{L} X_2(t) - \frac{R_2}{L} U_2 \qquad \text{... State equation}$$

and current through capacitor,

$$I_x - I_i = C\frac{dV_C}{dt}$$

$$\therefore \quad \frac{dV_C}{dt} = \frac{1}{C}[I_x - I_i]$$

Substituting I_x from output equation,

$$\dot{X}_1 = \frac{1}{CR_1} U_1 - \frac{1}{CR_1} X_1(t) - \frac{1}{C} X_2(t) \qquad \text{... State equation}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{CR_1} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Example 15.13 : *For a certain system, when*

$$X(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ then } X(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

$$\text{while } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ then } X(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

Determine the system matrix A.

Solution : The solution of the equation is,

$$X(t) = e^{At}\, X(0)$$

Let $$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

and the equation is,

$$\dot{X}(t) = A\, X(t)$$

Now $$X(t) = \begin{bmatrix} e^{-3t} \\ -3\,e^{-3t} \end{bmatrix} \quad \text{hence} \quad \dot{X}(t) = \begin{bmatrix} -3e^{-3t} \\ 9\,e^{-3t} \end{bmatrix}$$

and $$X(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

Now $$\dot{X}(0) = A\, X(0)$$

$$\therefore \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\therefore \quad A_1 - 3\,A_2 = -3 \qquad \text{... (1)}$$

$$A_3 - 3\,A_4 = 9 \qquad \text{... (2)}$$

Similarly $$X(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

$$\therefore \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad \dot{X}(0) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} X(0)$$

$$\therefore \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad A_1 + A_2 = 1 \qquad \text{... (3)}$$

$$\therefore \quad A_3 + A_4 = 1 \qquad \text{... (4)}$$

Subtracting (3) from (1),

$$-4\,A_2 = -4$$

$$\therefore \quad A_2 = 1$$

$$\therefore \quad A_1 = 0$$

Subtracting (4) from (2),

$$-4A_4 = 8$$

$$\therefore \quad A_4 = -2$$

$$\therefore \quad A_3 = 3$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

Example 15.14 : *A linear time invariant system is characterised by the homogeneous state equation :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Compute the solution of homogeneous equation, assume the initial state vector :

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution : From the given model,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}^T = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|sI - A| = (s-1)^2$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$e^{At} = L^{-1}[sI - A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\therefore \quad X(t) = e^{At} X(0) = \text{zero input response}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

This is the required solution.

Example 15.15 : *Determine the state variable matrix for the circuit shown .*

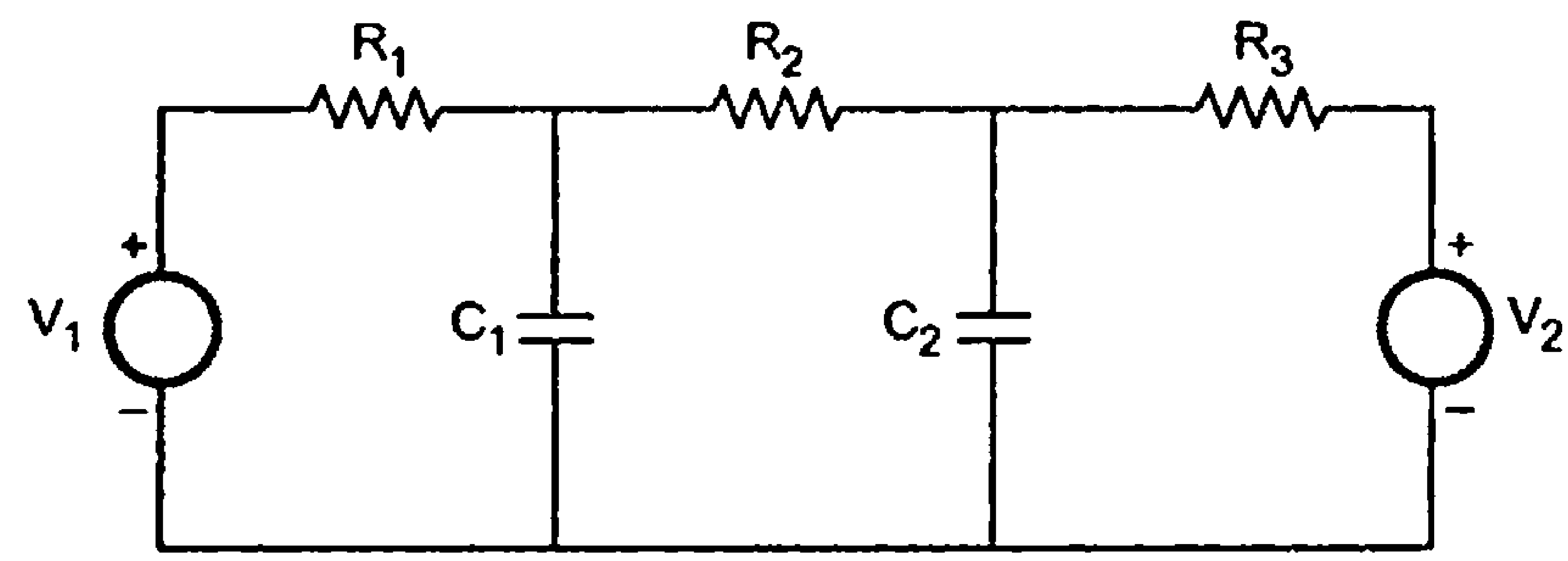


Fig. 15.34

Solution : The various currents are shown in the Fig. 15.35.

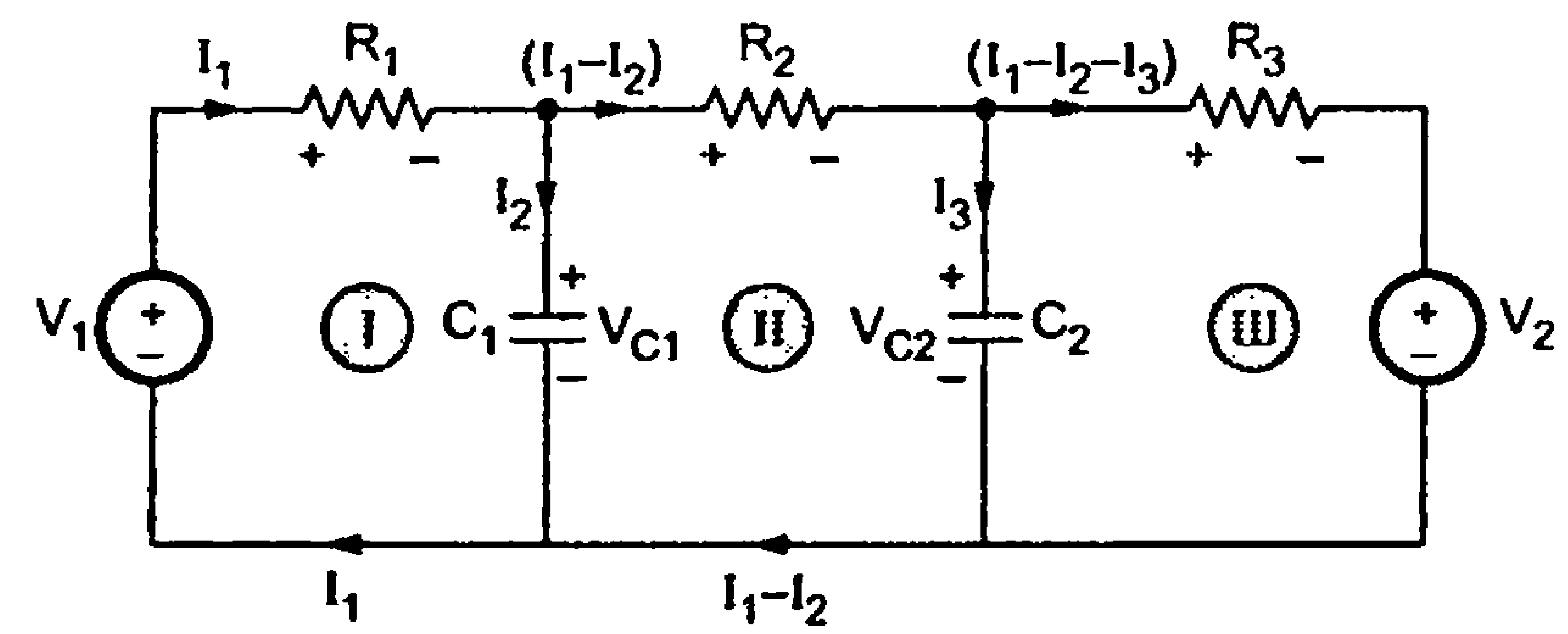


Fig. 15.35

The KVL equations for three loops are,

$$-I_1R_1 - V_{C1} + V_1 = 0 \qquad ...(1)$$

$$-(I_1 - I_2) R_2 - V_{C2} + V_{C1} = 0 \qquad ...(2)$$

$$-(I_1 - I_2 - I_3) R_3 - V_2 + V_{C2} = 0 \qquad ...(3)$$

Let $X_1 = V_{C1}$, $X_2 = V_{C2}$, $U_1 = V_1$, $U_2 = V_2$

Now current through C_1 is I_2 and C_2 is I_3.

$$\therefore \quad I_2 = C_1 \frac{dV_{C1}}{dt} = C_1\dot{X}_1 \qquad ...(4)$$

$$I_3 = C_2 \frac{dV_{C2}}{dt} = C_2 \dot{X}_2 \qquad \text{...(5)}$$

Using (4) and (5) in (1), (2) and (3) we get,

$$-I_1R_1 - X_1 + U_1 = 0 \qquad \text{i.e.} \qquad I_1 = -\frac{1}{R_1}X_1 + \frac{1}{R_1}U_1$$

$$-I_1R_2 - C_1\dot{X}_1R_2 - X_2 + X_1 = 0 \qquad \text{and use } I_1 \text{ obtained,}$$

$$\therefore \quad \frac{R_2}{R_1}X_1 - \frac{R_2}{R_1}U_1 - C_1\dot{X}_1R_2 - X_2 + X_1 = 0$$

$$\therefore \quad \dot{X}_1 = \left[\frac{R_1+R_2}{C_1R_1R_2}\right]X_1 - \frac{1}{C_1R_2}X_2 - \frac{1}{R_1C_1}U_1 \qquad \text{...(A)}$$

$$-I_1R_3 - C_1\dot{X}_1R_3 - C_2\dot{X}_2R_3 - U_2 + X_2 = 0 \qquad \text{Use } I_1 \text{ and (A)}$$

$$+\frac{R_3}{R_1}X_1 - \frac{R_3}{R_1}U_1 - R_3C_1\left[\frac{R_1+R_2}{C_1R_1R_2}X_1 - \frac{1}{C_1R_2}X_2 - \frac{1}{R_1C_1}U_1\right] - C_2\dot{X}_2R_3 - U_2 + X_2 = 0$$

$$\therefore \quad \dot{X}_2 = -\frac{1}{R_2C_2}X_1 + \left[\frac{R_3+R_2}{C_2R_2R_3}\right]X_2 - \frac{1}{R_3C_2}U_2 \qquad \text{...(B)}$$

The equations (A) and (B) give the state model, $\dot{X} = AX + BU$ hence state variable matrix is,

$$A = \begin{bmatrix} \frac{R_1+R_2}{C_1R_1R_2} & -\frac{1}{C_1R_2} \\ -\frac{1}{R_2C_2} & \frac{R_2+R_3}{C_2R_2R_3} \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_3C_2} \end{bmatrix}$$

➡ **Example 15.16 :** *Obtain the state equation and output equation of the electric network as shown in figure.*

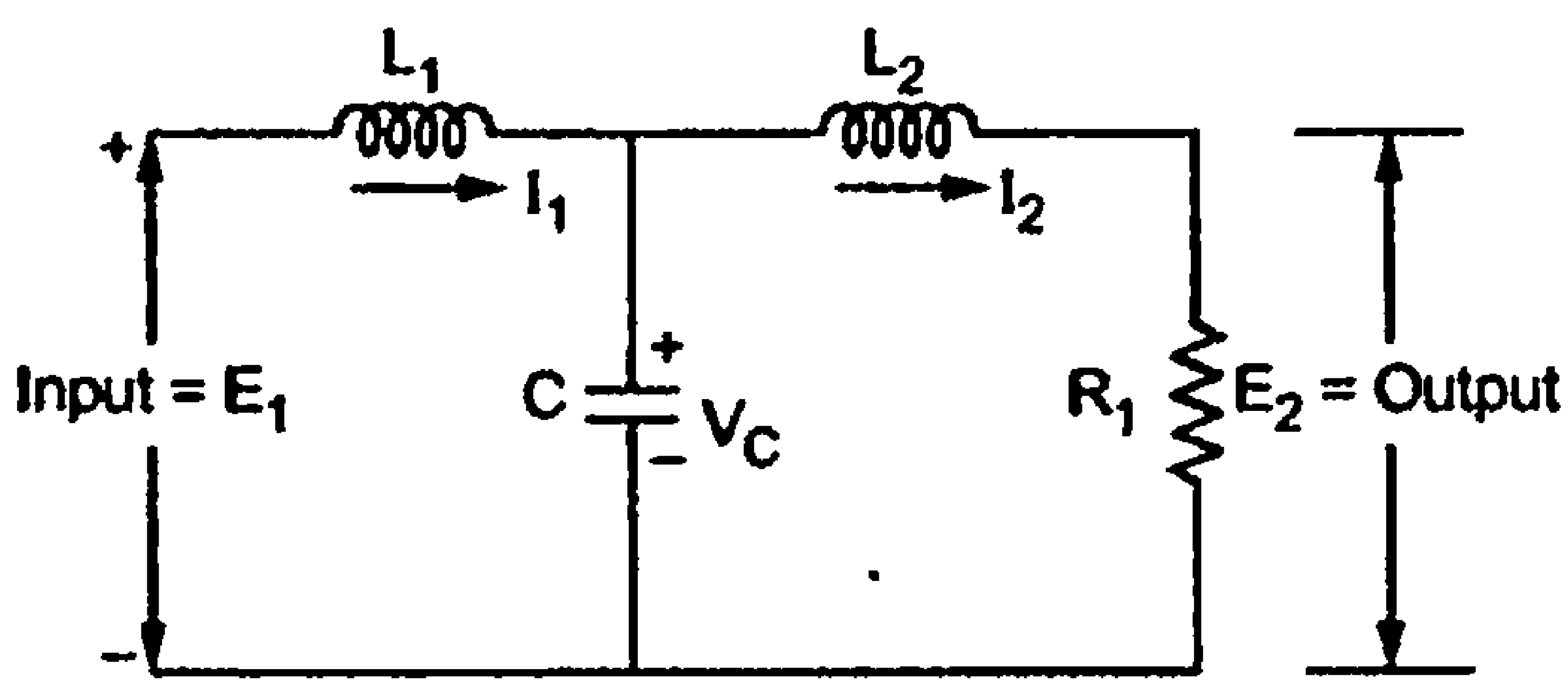


Fig. 15.36

Solution : The various currents are as shown in the Fig. 15.37.

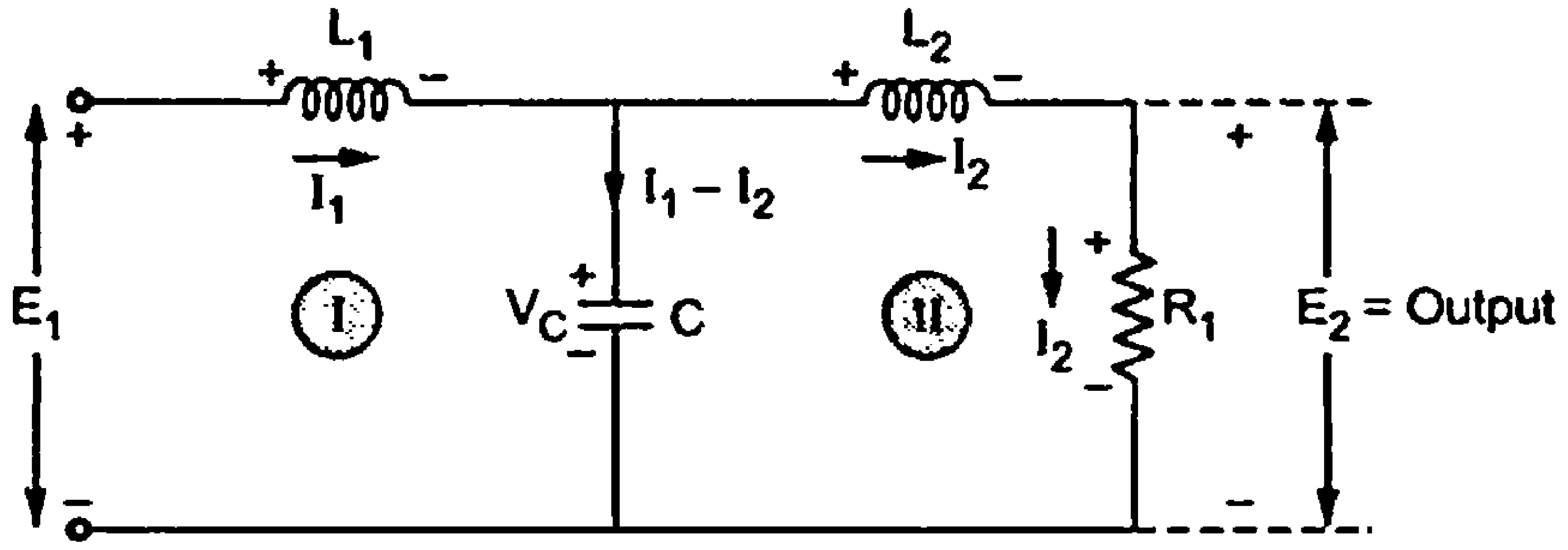


Fig. 15.37

Applying KVL to loop I and II,

$$-L_1\frac{dI_1}{dt}-V_C+E_1 = 0$$

$$\therefore \quad \frac{dI_1}{dt} = -\frac{1}{L_1}V_C+\frac{1}{L_1}E_1 \qquad ...(1)$$

$$-L_2\frac{dI_2}{dt}-I_2R_1+V_C = 0$$

$$\therefore \quad \frac{dI_2}{dt} = +\frac{1}{L_2}V_C+\frac{R_1}{L_2}I_2 \qquad ...(2)$$

$$I_1 - I_2 = C\frac{dV_C}{dt}$$

$$\therefore \quad \frac{dV_C}{dt} = \frac{1}{C}I_1-\frac{1}{C}I_2 \qquad ...(3)$$

And $\quad E_2 = I_2R_1$

Select state variables as, $I_1 = X_1$, $I_2 = X_2$, $V_C = X_3$ and input $E_1 = U$ and output $E_2 = Y$. Using in the equations (1), (2), (3) and (4).

$$\dot{X}_1 = -\frac{1}{L_1}X_3+\frac{1}{L_1}U$$

$$\dot{X}_2 = -\frac{R_2}{L_2}X_2+\frac{1}{L_2}X_3$$

$$\dot{X}_3 = \frac{1}{C}X_1-\frac{1}{C}X_2$$

and $\quad Y = X_2 R_2$

$$\therefore \quad \dot{X} = AX + BU \qquad \text{and} \qquad Y = CX+DU$$

$$\text{Where } A = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_1}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 & R_1 & 0 \end{bmatrix}, D = [0]$$

Example 15.17 : *Given* $\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$

Find the unit step response when, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution : $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj } [sI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2 = (s+1)(s+2)$$

$$[sI - A]^{-1} = \frac{\text{Adj } [sI - A]}{|sI - A|} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \phi(s)$$

$$\therefore \quad \phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[\phi(s)] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\text{ZIR} = e^{At}X(0) = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} + e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$\text{ZSR} = L^{-1}\{\phi(s)BU(s)\} \qquad \text{where } U(s) = \frac{1}{s} \text{ due to unit step input}$$

$$= L^{-1}\left\{\begin{bmatrix} \dfrac{(s+3)}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\left[\dfrac{1}{s}\right]\right\}$$

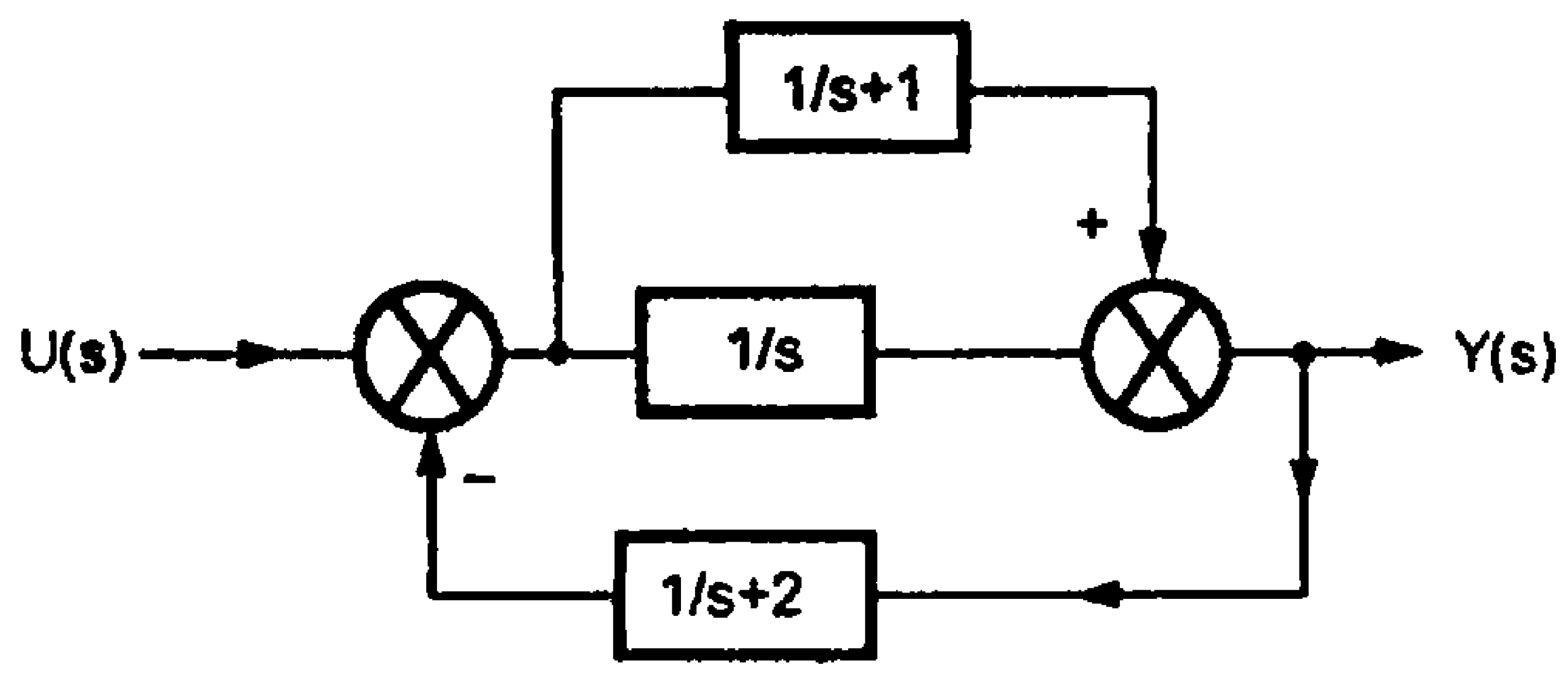
$$= L^{-1}\left\{\begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}\right\} = L^{-1}\begin{bmatrix} \dfrac{0.5}{s}-\dfrac{1}{s+1}+\dfrac{0.5}{s+2} \\ \dfrac{1}{s+1}-\dfrac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5-e^{-t}+0.5\,e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix} s$$

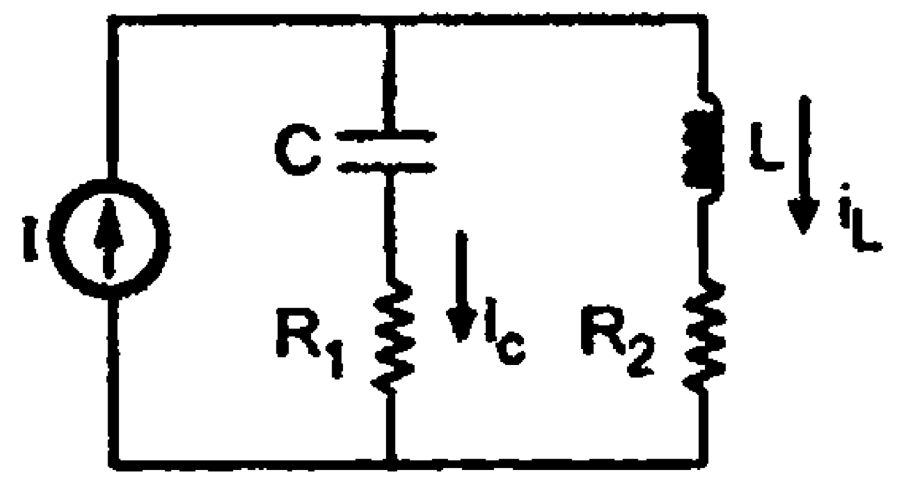
$$\therefore\ Y(t) = ZIR + ZSR = \begin{bmatrix} 3e^{-t}-2e^{-2t} \\ -3e^{-t}+4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5-e^{-t}+0.5e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix} = \begin{bmatrix} 0.5+2e^{-t}-1.5e^{-2t} \\ -2e^{-t}+3e^{-2t} \end{bmatrix}$$

Review Questions

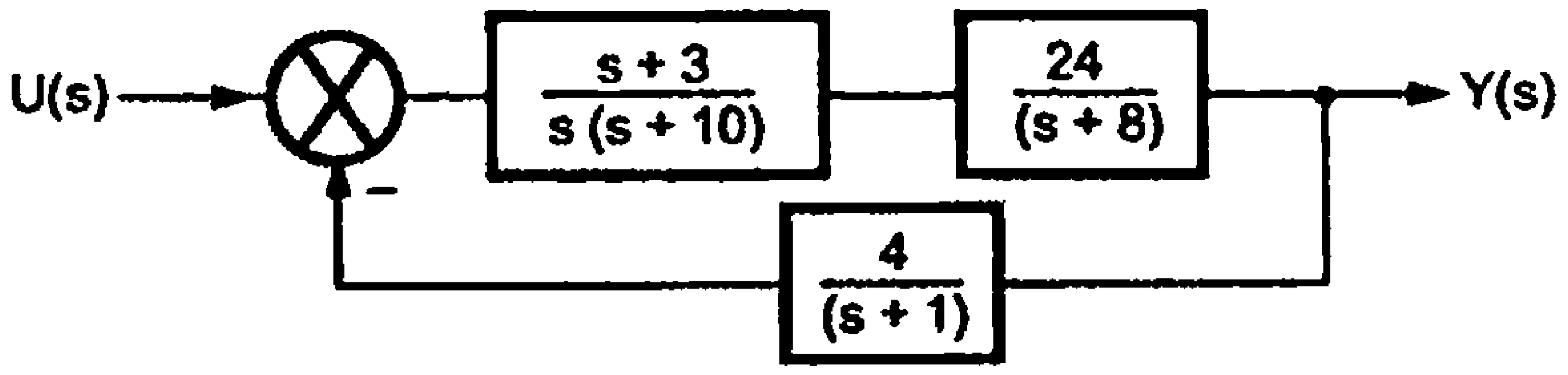
1. *Explain the concept of state.*
2. *Define and explain the following terms,*
   a. State variables *b. State vector*
   c. State trajectory *d. State*
   e. State space
3. *Explain advantages of state variable method over conventional one.*
4. *Discuss the derivation of the state model using following methods of programming.*
   i) Bush form *ii) Gullemin's form*
   iii) Foster's form *iv) Jordan's form*
5. *Elaborate upon the basis of selecting suitable state variables for a system.*
6. *Obtain the state model for the block diagram shown.*



7. *Voltage across capacitor is V_C. With V_C and i_L as a set of state variables derive the state model.*



8. *Obtain the state model in standard Bush form of a system shown in figure.*



Ans. : $\dot{X} = AX + BU, \quad Y = CX$

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -288 & -176 & -98 & -19 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [72 \;\; 96 \;\; 24 \;\; 0]$$

9. *State whether the following statement is True or False with reasons :*
   The state space approach of the system analysis is a frequency domain analysis.
10. *State whether the following statement is True or False with reason.*
    The state space approach of the system analysis is a time domain analysis.
11. *Write a short note on advantages and limitations of state variable approach.*
12. *Derive the transfer function from state model.*
13. *What is characteristic equation of a system matrix A ?*
14. *Obtain the T.F. of the system having state model,*

$$\dot{X}(t) = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} X(t) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} U(t)$$

$$Y(t) = [1 \;\; 2] \, X(t)$$

Ans. : $\dfrac{12s + 59}{(s+2)(s+4)}$

15. *What is homogenous and nonhomogeneous state equation ?*
16. *Define state transition matrix using classical method of obtaining solution*
17. *What is zero input response and zero state response ?*
18. *Obtain the complete solution of nonhomogeneous state equation using time domain method.*
19. *State the importance of state transition matrix.*
20. *State the various properties of state transition matrix.*
21. *Obtain the solution of nonhomogeneous state equation using Laplace transform method*
22. *Explain Laplace transform method of obtaining e^{At}*

23. *Obtain the homogeneous solution of the equation* $\dot{X}(t) = A\, X(t)$ *where*

a. $A = \begin{bmatrix} -9 & 1 \\ -14 & 0 \end{bmatrix}$ *and* $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans. : $\begin{bmatrix} -\frac{1}{5}e^{-2t} + \frac{6}{5}e^{-7t} \\ -\frac{7}{5}e^{-2t} + \frac{12}{5}e^{-7t} \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}$ *and* $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ans. : $\begin{bmatrix} 1.11\, e^{0.56t} - 0.11\, e^{-3.56t} \\ 0.48\, e^{0.56t} - 0.48\, e^{-3.56t} \end{bmatrix}$

16 Control Components and Controllers

Note : The discussion of d.c. and a.c. servomotors is included in the chapter 4.

16.1 Introduction to Stepper Motors

The stepper motor is electromechanical device which actuates a train of step movements in response to train of input pulses. The step movement may be angular or linear. Each pulse input actuates one step movement. The device plays an important role in incremental motion control systems such as printers, tape drives etc.

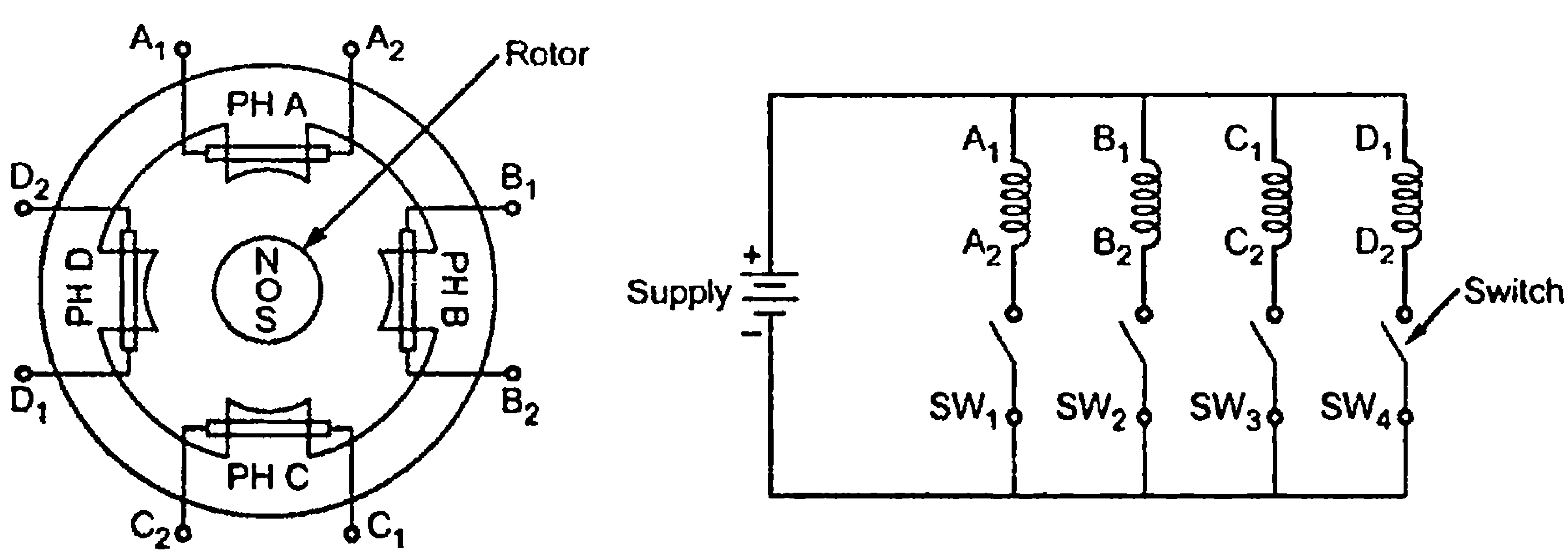
The two most widely used types of stepper motors are

i) Permanent magnet motor

ii) Variable reluctance motor

16.2 Permanent Magnet Stepper Motor

Construction and basic drive circuit of four phase permanent magnet stepper motor is shown in the Fig. 16.1 (a) and Fig. 16.1 (b) respectively.



(a) Four phase permanent magnet stepper motor

(b) Basic drive circuit

Fig. 16.1

The stator of this type may be multipolar. As shown, the stator has four poles. Around the poles, exciting coils are wound. The number of slots per pole per phase is usually choosen as one in such multipolar machines. The rotor may be salient or smooth cylindrical. For the motor shown, it is smooth cylindrical type. It is made out of ferrite material permanently magnetised.

As soon as the voltage pulses are applied to various phases with the help of driving circuit, a rotor makes 90° revolution called as step for each input voltage pulse. The various steps in a four phase permanent magnet stepper motor are shown in Fig. 16.2.

i) At first, switch SW_1 is closed exciting the phase A. We have a N pole in phase A due to its excitation as shown in the Fig. 16.2 (a). Due to electromechanical torque developed, rotor rotates to adjust its magnetic axis with the magnetic axis of the stator.

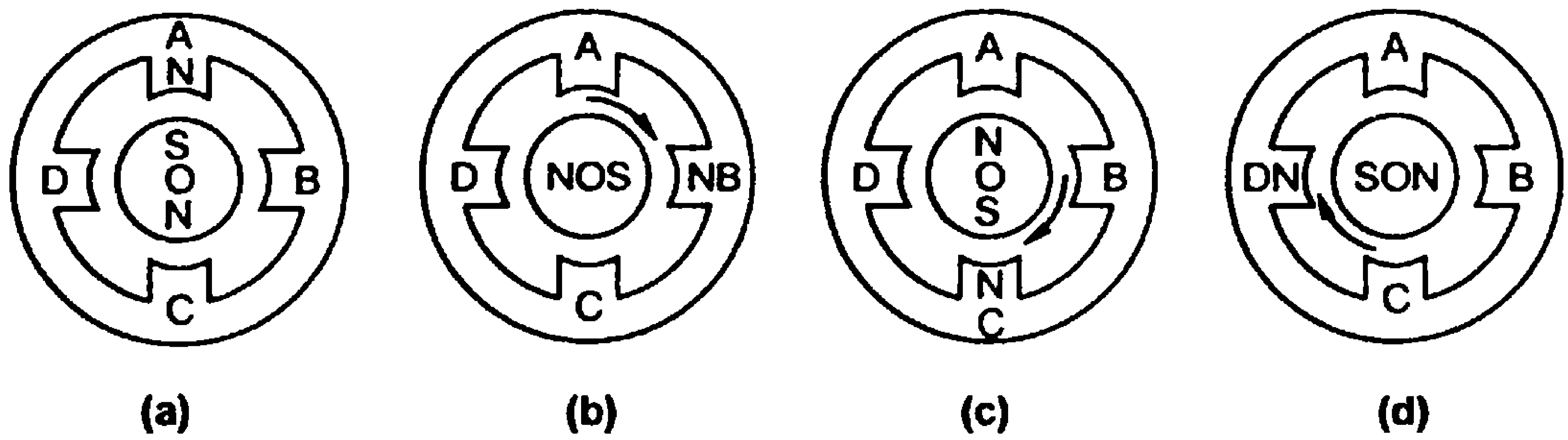


Fig. 16.2 Steps in four phase P.M. stepper motor

ii) Next phase B is excited with switch SW_2, disconnecting phase A. Due to this, rotor further rotates to adjust it is magnetic axis with N pole of phase B. Hence it rotates through 90° called as step, as shown in Fig. 16.2 (b).

Similarly when phase C and phase D are sequentially excited the rotor tends to rotate through 90°, everytime. When such sequence is repeated, it results into a step motion of a permanent magnet stepper motor.

Further reduction in step angle can be achieved by increasing number of rotor poles. Infact the step angle α_s is given by

$$\alpha_s = \frac{360°}{2Pm}$$

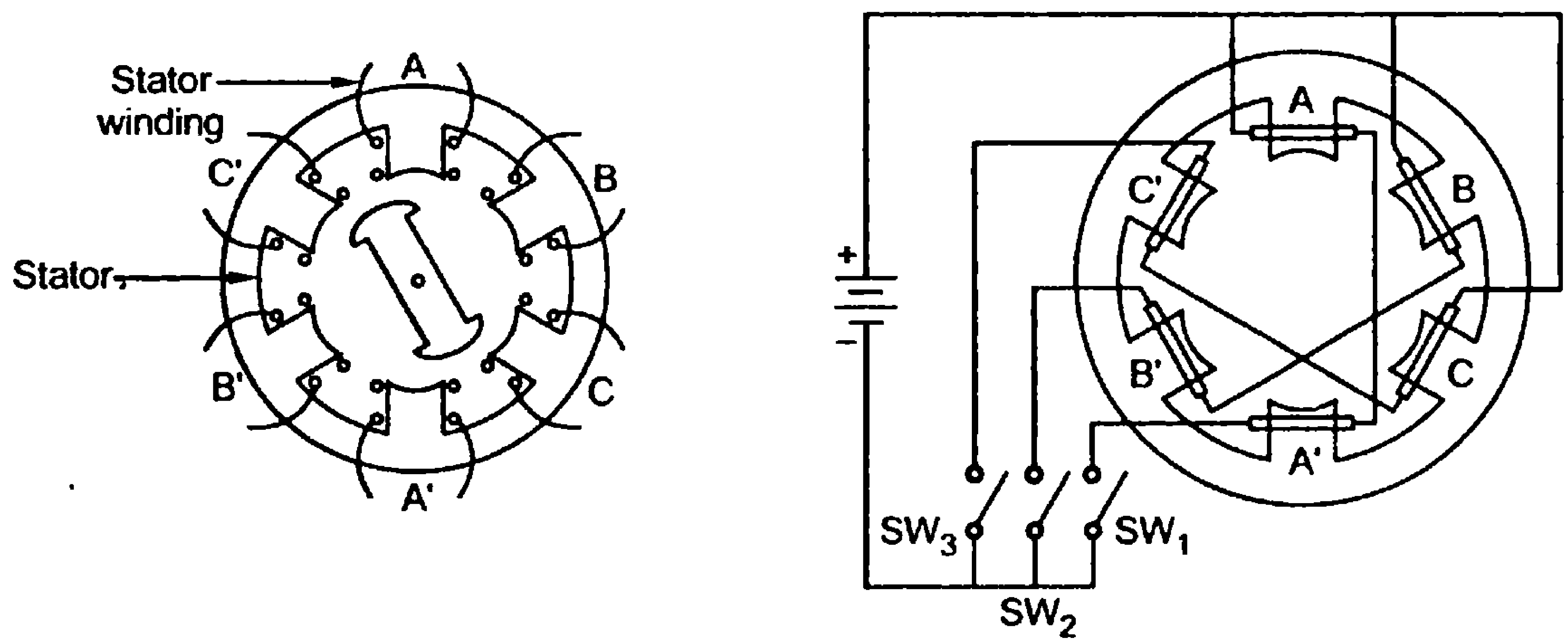
where $2P$ = Number of poles

m = Number of phases in the exciting winding

The stepper motors with permanent magnet rotors with large number of poles cannot be manufactured in small size. Hence small steps are not possible. This is the biggest disadvantage of permanent magnet stepper motor. This is overcome by the use of variable reluctance type stepper motor.

16.3 Variable Reluctance Stepper Motor

The stator in this type is usually wound for three phases. The stator has six salient poles (teeth) with concentrated exciting windings around each one of them. The rotor is made out of slotted steel laminations and has two salient poles (or teeth) without any exciting winding as shown in Fig. 16.3 (a). The basic driving circuit is as shown in the Fig.16.3 (b).



(a) Schematic representation of variable reluctance

(b) Driving circuit

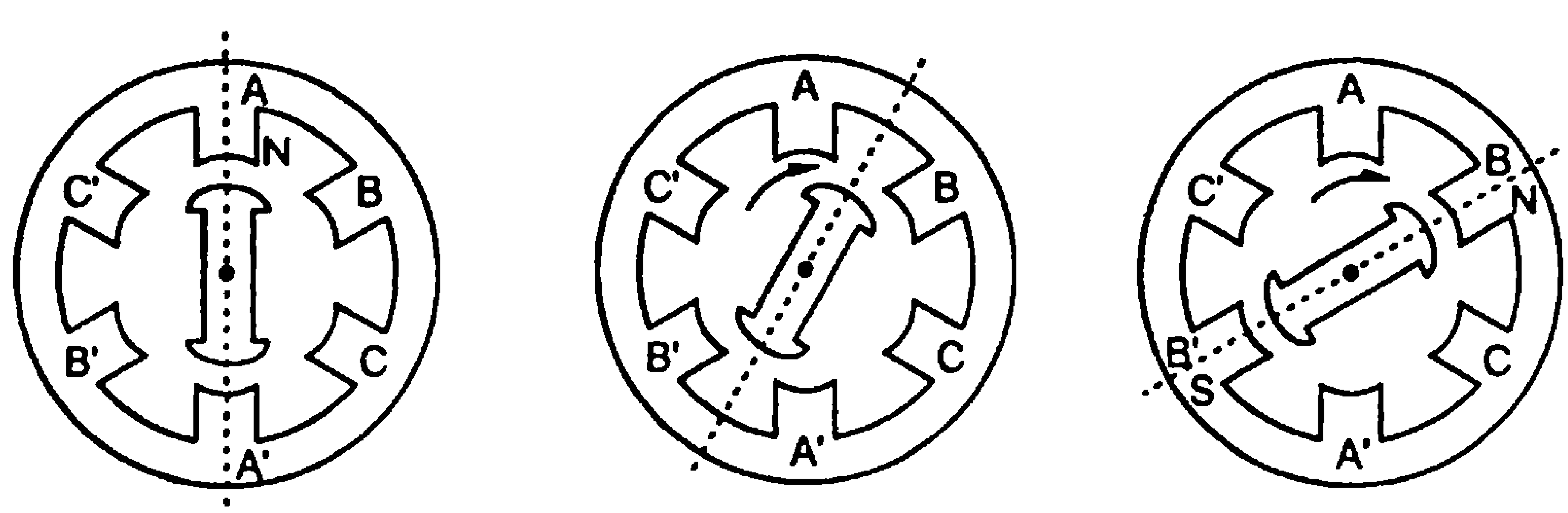
Fig. 16.3

The coils wound around diametrically opposite poles are connected in series and the three phases are energized from a d.c. source with the help of switches.

i) When the phase A-A′ is excited with switch SW_1 closed (with A forming N pole and A′ as S pole), the rotor tries to adjust itself in a minimum reluctance position between stator and rotor. This position is shown in Fig. 16.4 (a).

ii) Next, when the phase B-B′ is also excited with switch SW_2 closed, keeping AA′ energized the magnetic axis of stator moves 30° in clockwise direction and hence rotor also rotates through 30° step in clockwise direction to attain new minimum reluctance position. This is shown in the Fig. 16.4 (b).

iii) After that the excitation of AA′ is disconnected and only BB′ is kept energised. Rotor further moves through 30° step to adjust itself in new minimum reluctance position. This is shown in the Fig. 16.4 (c).



(a) (b) (c)

Fig. 16.4 Steps in variable reluctance motor

By successively exciting the three phases in the specific sequence, the motor takes twelve steps to make one complete revolution. If the rotor were to have four salient projections instead of two, the successive voltage pulses would cause a step of 15° and the number of steps would increase to 24 in one complete revolution. A further reduction in step angle can be achieved by increasing the number of poles of the stator and rotor or by adopting different constructions such as,

i) Using reduction gear stepper motor ii) Using multistack type stepper motor.

16.3.1 Reduction Gear Stepper Motor

Fig. 16.5 shows a reduction gear stepper motor. The stator has 8 salient poles and four phase for use as exciting winding. The rotor has 18 teeth and 18 slots uniformly distributed around. Each salient pole of the stator consists of two teeth, forming an intervening slot of the same angular periphery as the rotor teeth or slots. When coil A-A′ is excited, the resulting electromechanical torque brings the rotor to the position as shown in the Fig. 16.5.

With this combination, the step angle reduces to 5°. By successive excitation of coils A-A′, B-B′, C-C′, D-D′ the rotor makes 72 steps to complete one revolution.

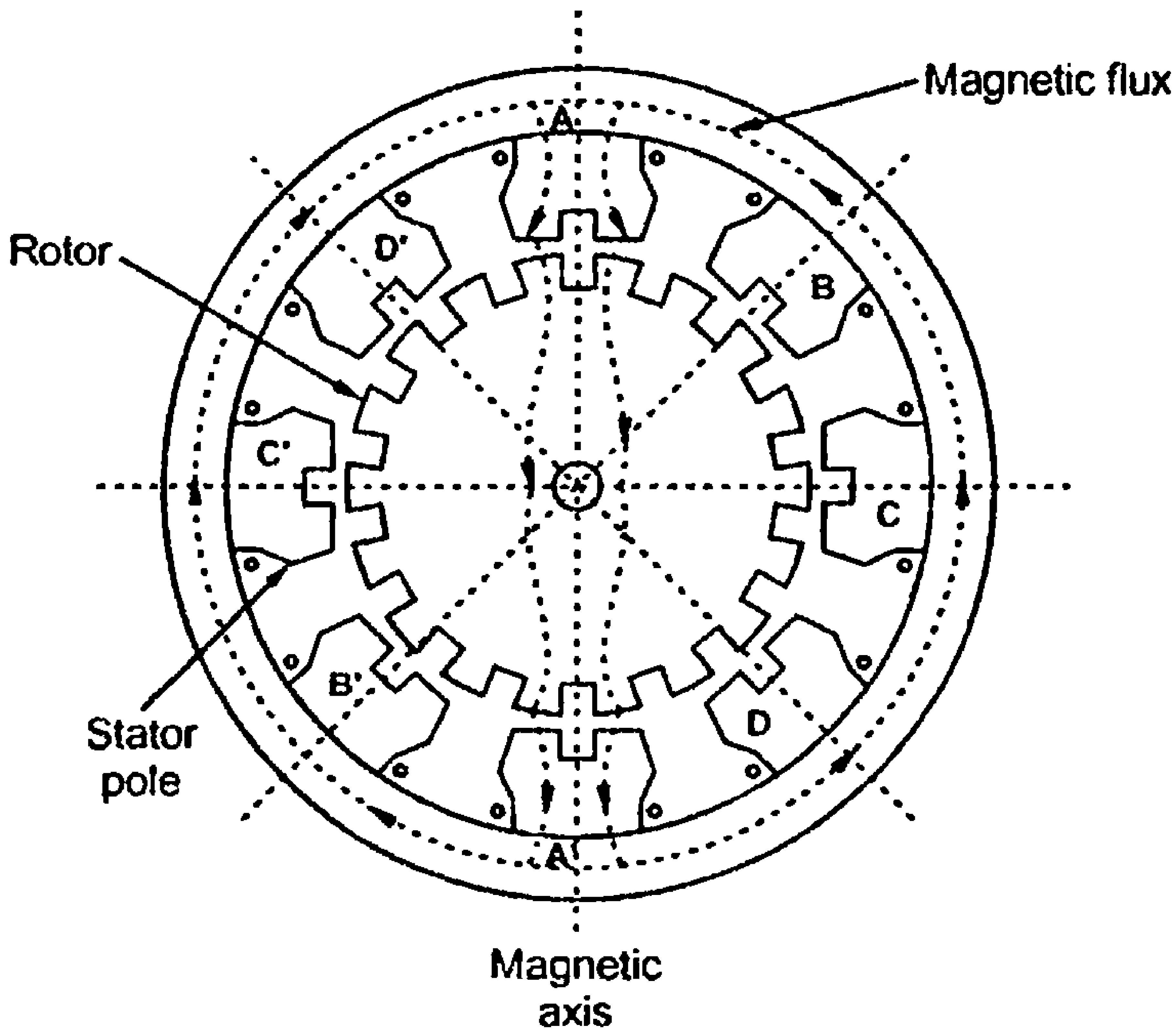


Fig. 16.5 Reduction gear stepper motor

The general relationship between steps angle α_s, number of stator phases m and rotor teeth N_r is given by

$$\alpha_s = \frac{360°}{m N_r}$$

By choosing different combinations of number of rotor teeth and stator phases, any desired step angle can be achieved.

16.3.2 Multistack Stepper Motor

The stepper motors discussed above are single stack type i.e. the windings for all the phases are in same plane. But in multistack type, such windings are arranged in different stacks.

The three stacks of stator have a common frame while the rotors have a common shaft as shown in the Fig. 16.6. Both stator stacks and rotors have toothed structure with same teeth size. The stators are pulse excited while rotors are unexcited, when the stator is excited, the rotor gets pulled into the nearest minimum reluctance position where the stator and rotor teeth are aligned.

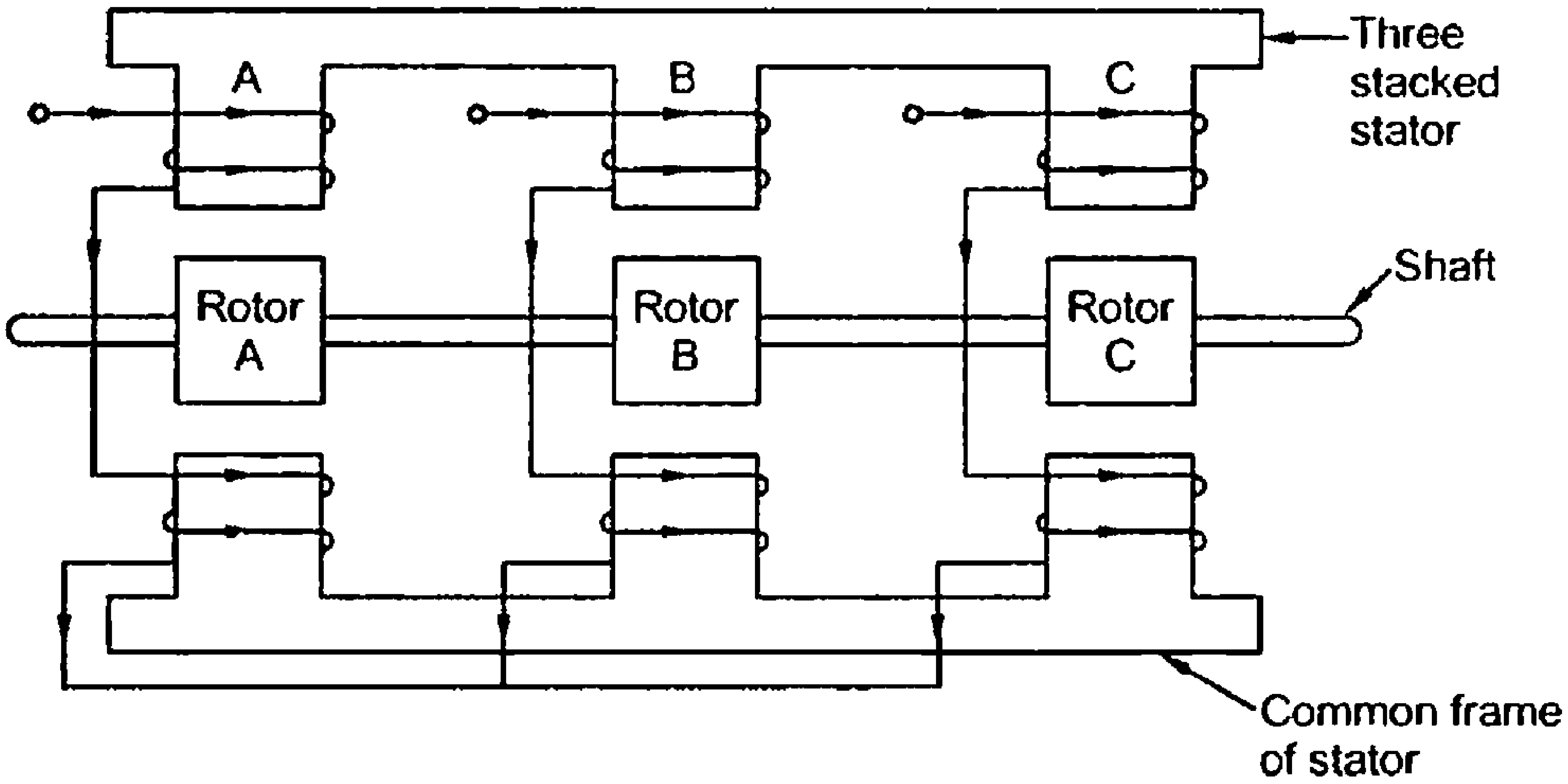


Fig. 16.6 Cross-sectional view of 3 stack variable reluctance torque

The stator teeth of various stacks are arranged to have a progressive angular displacement of

$$\alpha = \frac{360^\circ}{q\,T}$$

where q = Number of stacks

T = Number of teeth

16.4 Important Definitions Related to Stepper Motor

i) **Holding torque** : It is defined as the maximum static torque that can be applied to the shaft of an excited motor without causing a continuous rotation.

ii) **Detent torque** : It is defined as the maximum static torque that can be applied to the shaft of an unexcited motor without causing a continuous rotation.

iii) **Step angle** : It is defined as the angular displacement of the rotor in response to each pulse.

iv) **Critical torque** : It is defined as the maximum load torque at which rotor does not move when an exciting winding is energized. This is also called as pull-out torque.

v) **Limiting torque** : It is defined for a given pulsing rate or stepping rate measured in pulses per second, as the maximum load torque at which motor follows the control pulses without missing any step. This is also called as pull-in torque.

vi) **Synchronous stepping rate** : It is defined as the maximum rate at which the motor can step without missing steps. The motor can start, stop or reverse at this rate.

vii) **Slewing rate :** It is defined as the maximum rate at which the motor can step unidirectionally. The slewing rate is much higher than the synchronous stepping rate. Motor will not be able to stop or reverse without missing steps at this rate.

16.5 Stepper Motor Characteristics

The characteristics are classified as

i) Static characteristics and

ii) Dynamic characteristics.

The static are at the stationary position of the motor while the dynamic are under running conditions of the motor.

16.5.1 Static Characteristics

These characteristics include

i) Torque-displacement characteristics

ii) Torque-current characteristics

i) **Torque-displacement characteristics :** This gives the relationship between an electromagnetic torque developed and displacement angle θ from steady state position as shown in the Fig. 16.7.

Initially static torque increases with angular displacement reaching a maximum value at $\theta = \theta_m$, and then it falls to zero. The maximum static torque which occurs at $\theta = \theta_m$ is called as holding torque.

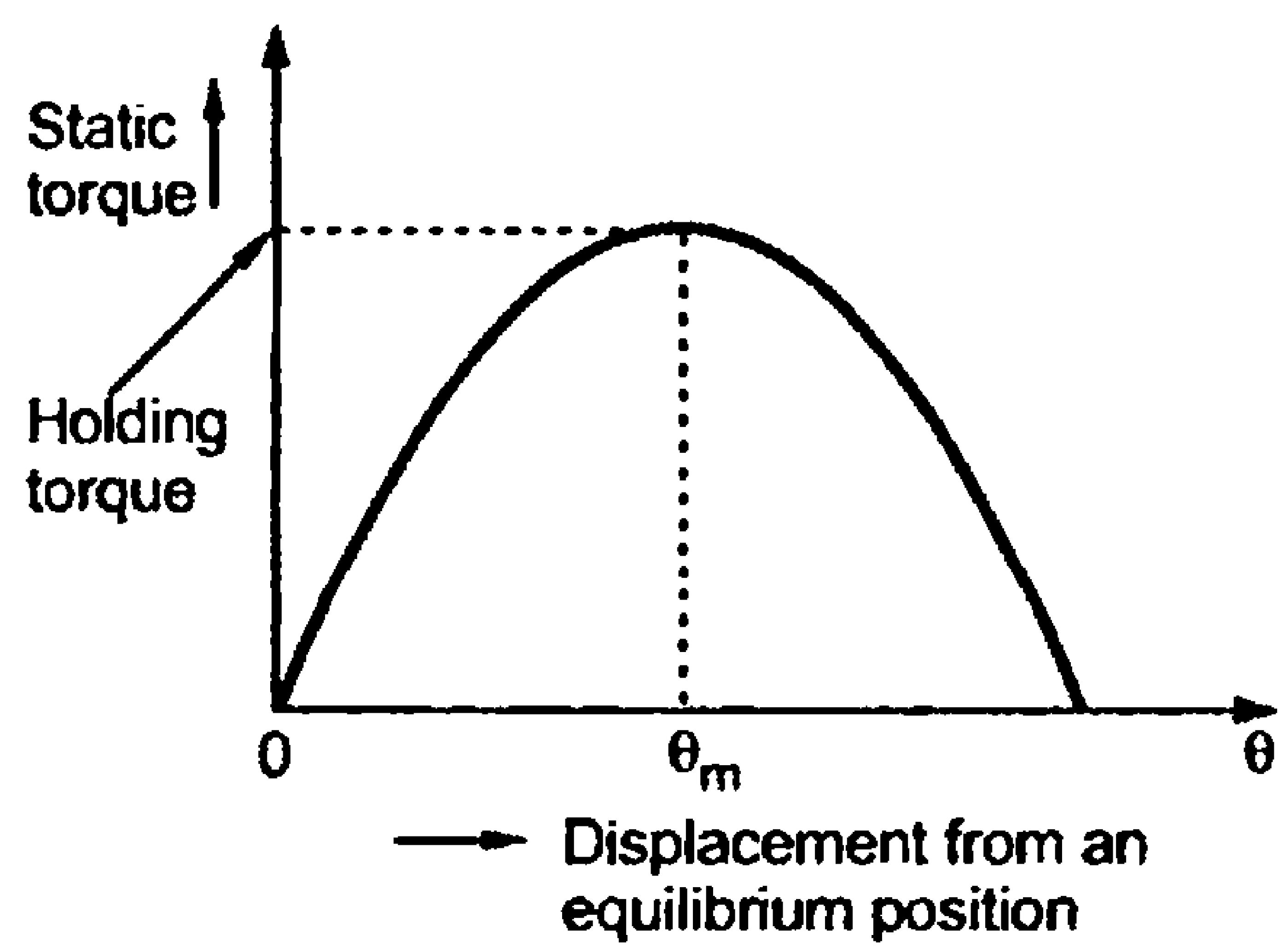


Fig. 16.7 Torque-displacement characteristics

ii) **Torque-current characteristics :** The holding torque of the motor increases with the exciting current. The relation between the holding torque and the current is called as torque-current characteristics which is shown in the Fig. 16.8.

In variable reluctance motor, torque is zero when current is zero. Initially it increases according to parabolic nature and later on becomes linear. In permanent magnet motor, the static torque occurs though motor is unexcited called as detent torque. Further it increases linearly with the current.

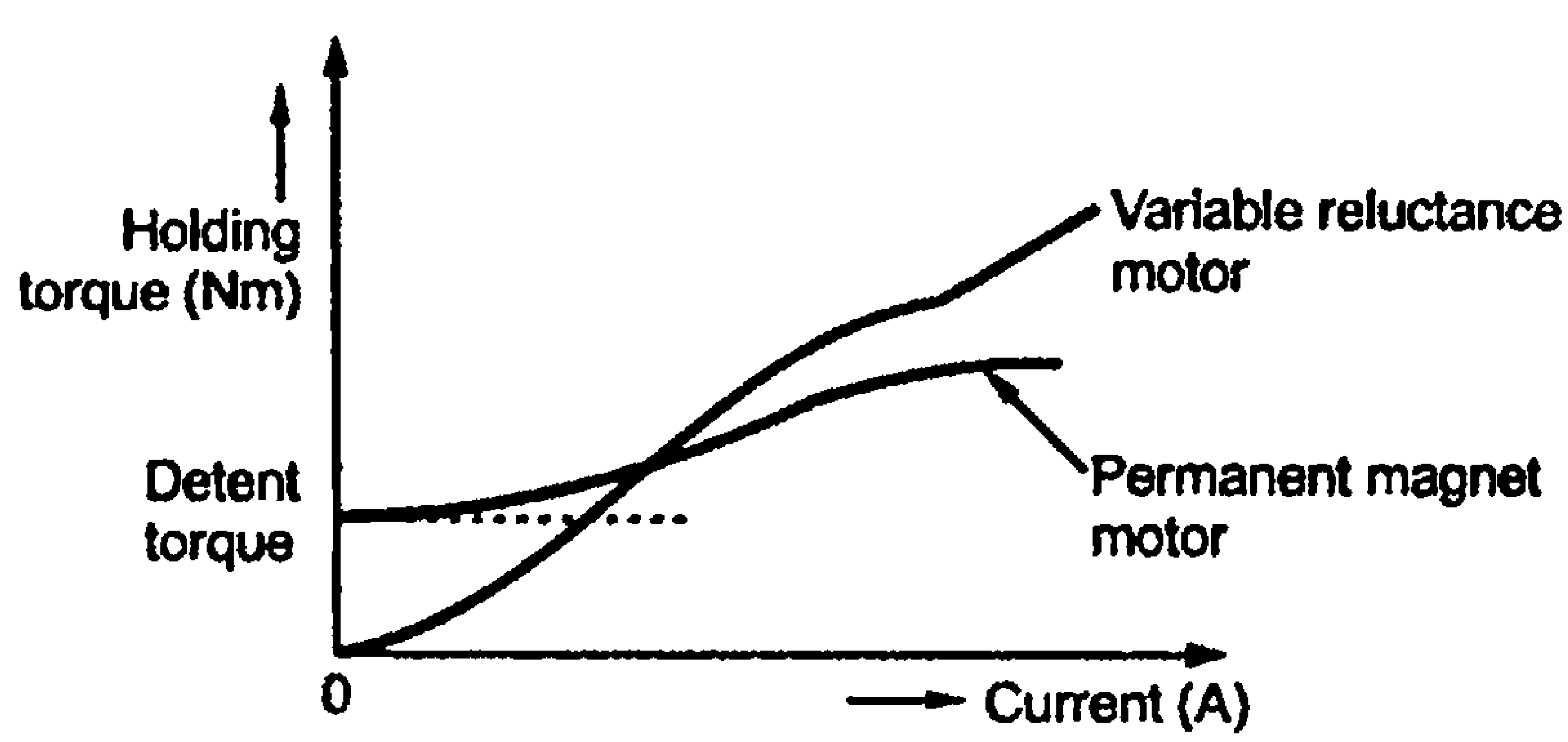


Fig. 16.8 Torque-current characteristics

16.7 Applications of Stepper Motor

Stepper motors are used in computer peripheral systems such as printers, tape drives, capstan drives, memory access mechanisms, machine tool and process control systems, robotic control systems, actuators, printers paper drive mechanism, spacecrafts, watches etc.

16.8 Synchros

Synchros are used widely in control systems as detectors and encoders because of their rigidness in construction and high reliability. Synchro is basically a rotary device, an electromagnetic transducer which operates on same principle as that of transformer. It converts angular position of shaft into an electric signal.

16.8.1 Synchro Transmitter

This is a basic synchro unit. It's construction is similar to that of 3 phase alternator.

The stator which is stationary part is made up of laminated steel. This part is slotted to accommodate a balanced three phase winding. The stator windings are star connected which are usually of concentric coil type structure.

The rotor which is rotating part is a salient pole, dumb-bell shaped magnet with a single winding. Schematic diagram is as shown in Fig. 16.10.

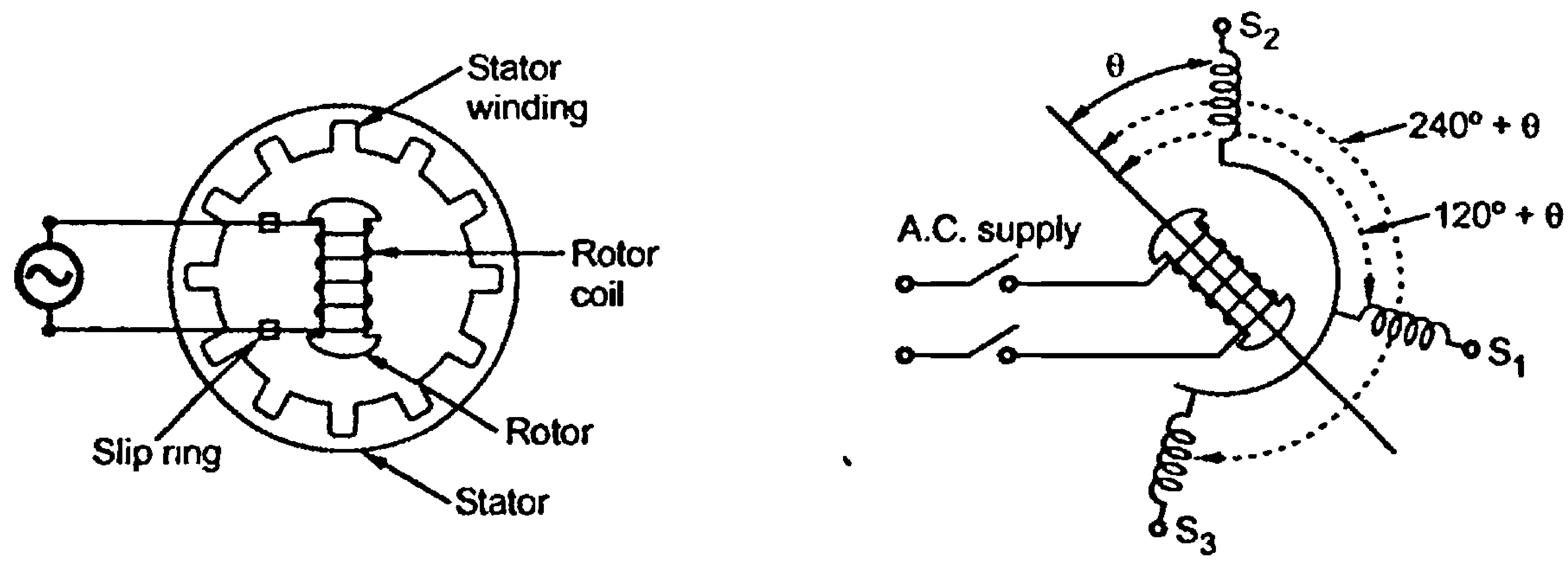


Fig. 16.10 Synchro Unit

A single phase a.c. voltage is applied to the rotor through slip rings. The symbol 'G' is used to denote synchro transmitter. Also known as synchro generator.

Let a.c. voltage applied to rotor is

$$e_r(t) = E_r \sin \omega_0 t$$

When $\theta = 0°$ with reference to schematic diagram, voltage induced in S_2 winding will be maximum and position is called as *electric zero.*

v) **Loading error :** When the load is connected across the potentiometer the resistance of the potentiometer gets affected this causes error in its output which is called as the loading error.

16.9.4 Loading in Potentiometers

As mentioned earlier, there can be loading error in the potentiometer.

Let R_L be the resistance of the load connected to be potentiometer and α be the setting ratio as shown in the Fig. 16.22.

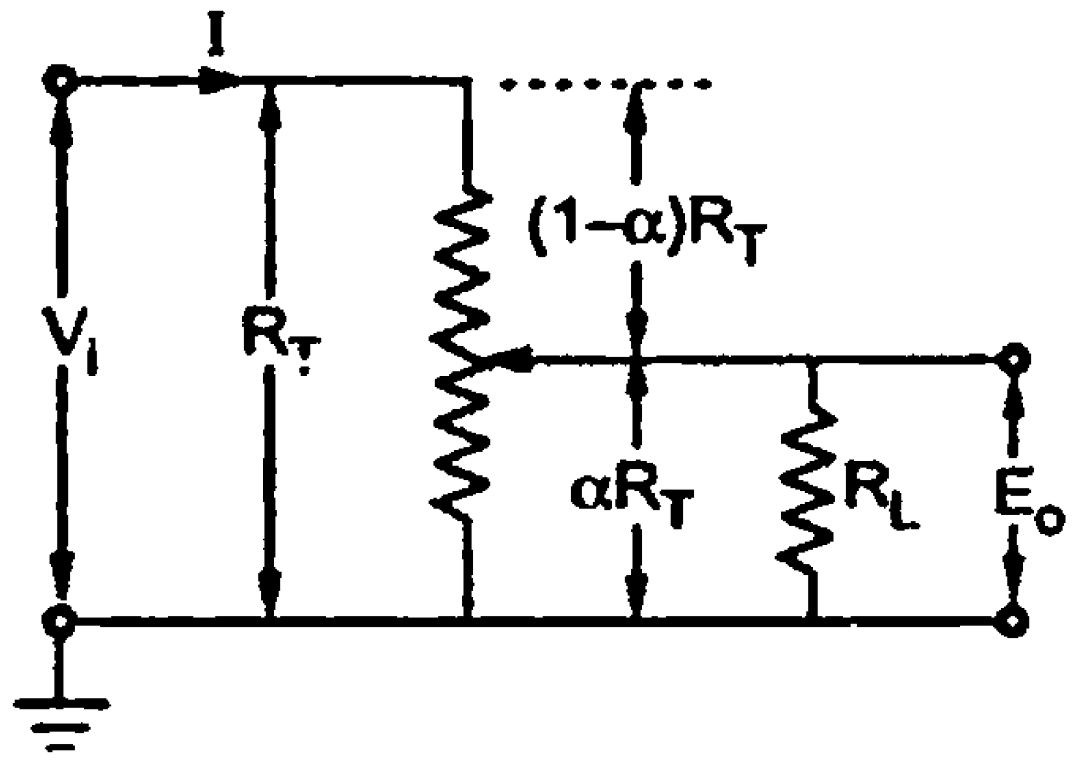


Fig. 16.22

The current, $$I = \frac{V_i}{R_{equi}}$$

Now $$R_{equi} = (1-\alpha)R_T + (\alpha R_T || R_L) = (1-\alpha)R_T + \frac{\alpha R_T R_L}{(\alpha R_T + R_L)}$$

$$= \frac{(\alpha R_T + R_L)(1-\alpha)\, R_T + \alpha R_T R_L}{(\alpha R_T + R_L)}$$

$$\therefore \quad I = \frac{V_i(\alpha R_T + R_L)}{(\alpha R_T + R_L)(1-\alpha)R_T + \alpha R_T R_L}$$

The output voltage E_o is,

$$E_o = I \times [\alpha R_T || R_L]$$

$$= \frac{V_i(\alpha R_T + R_L)(\alpha R_T R_L)}{[(\alpha R_T + R_L)(1-\alpha)R_T + \alpha R_T R_L](\alpha R_T + R_L)}$$

$$= \frac{V_i \alpha R_T R_L}{(\alpha R_T + R_L)(1-\alpha)R_T + \alpha R_T R_L}$$

Dividing numerator and denominator by $R_T R_L$,

$$E_o = \frac{V_i \alpha}{\frac{(\alpha R_T + R_L)(1-\alpha)}{R_L} + \alpha}$$

$$= \frac{V_i \alpha}{\left[\frac{\alpha R_T + R_L - \alpha^2 R_T - \alpha R_L + \alpha R_L}{R_L}\right]} = \frac{V_i \alpha}{\frac{R_L + \alpha R_T(1-\alpha)}{R_L}}$$

$$\therefore \quad E_o = \frac{\alpha V_i}{1 + \frac{\alpha(1-\alpha)R_T}{R_L}}$$

voltage $e_2(t)$ is generated at the brushes BB′ which are place perpendicular to the direction of flux ϕ_2. This voltage $e_2(t)$ is used to supply the power to the d.c. motor and the connected load in the system. A small charge in field current changes a large voltages at brushes BB′ which is output of amplidyne.

In practice brushes AA are shorted through a series field coil which enables the required value of ϕ_2. The compensating winding in output circuit through the brushes BB′ prevents the reduction in armature reaction flux ϕ_2 due to presence of flux ϕ_1.

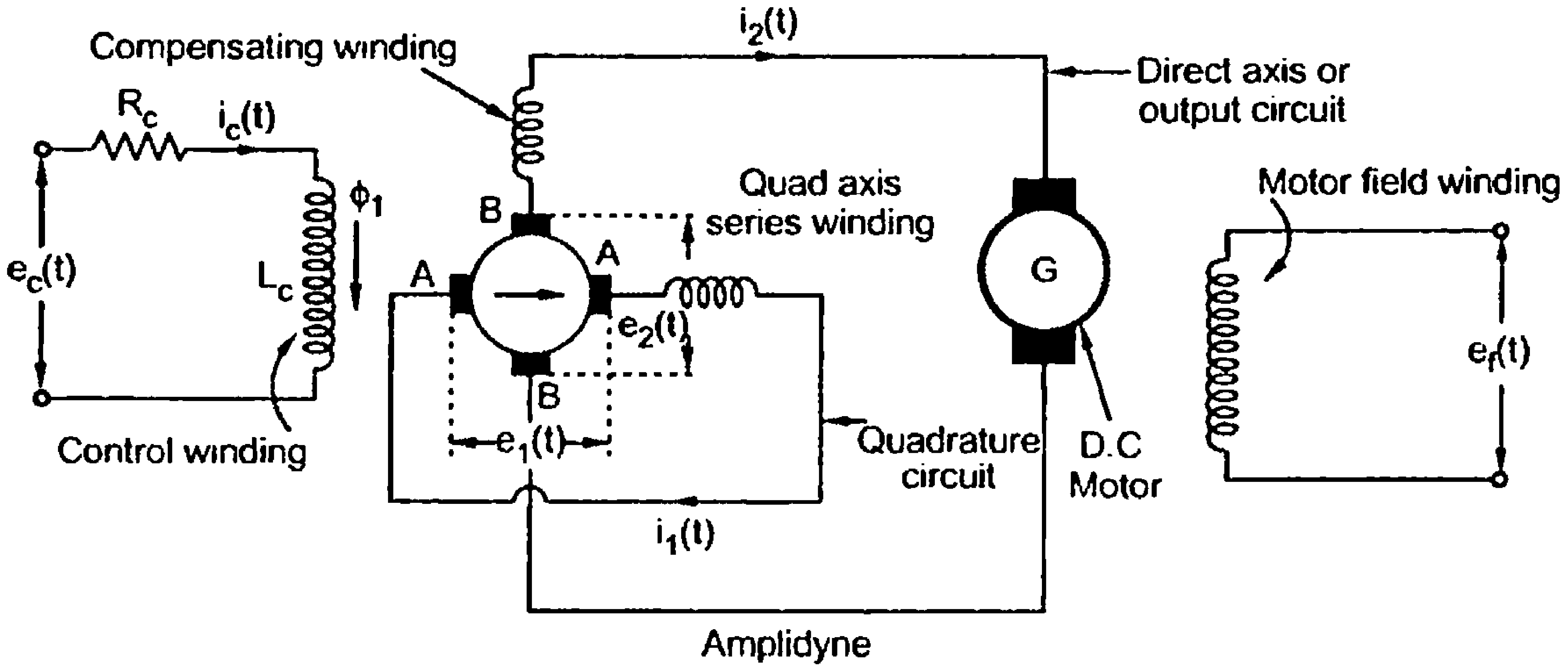


Fig. 16.23 Amplidyne

16.10.3 Transfer Function

Applying KVL to control winding

$$e_c(t) = i_c(t)\,R_c + L_c\frac{di_c(t)}{dt}$$

Taking Laplace transform,

$$E_c(s) = I_c(s)[R_c + s\,L_c]$$

$$E_c(s) = \frac{E_c(s)}{[R_c + sL_c]} \qquad \text{... (1)}$$

Now $$I_2(s) = \frac{K_1\,I_c(s)}{(R_1 + sL_q)} \qquad \text{... (2)}$$

Substituting $I_c(s)$, $$I_2(s) = \frac{K_q E_c(s)}{(R_q + sL_q)(R_c + sL_c)}$$

and $$E_f(s) = E_d(s) - (R_d + sL_d)\,I_1(s)$$

$$= K_d \cdot I_2(s) - (R_a + sL_d)\,I_1(s) \qquad \text{... (3)}$$

On no load, $I_1(s) \approx 0$

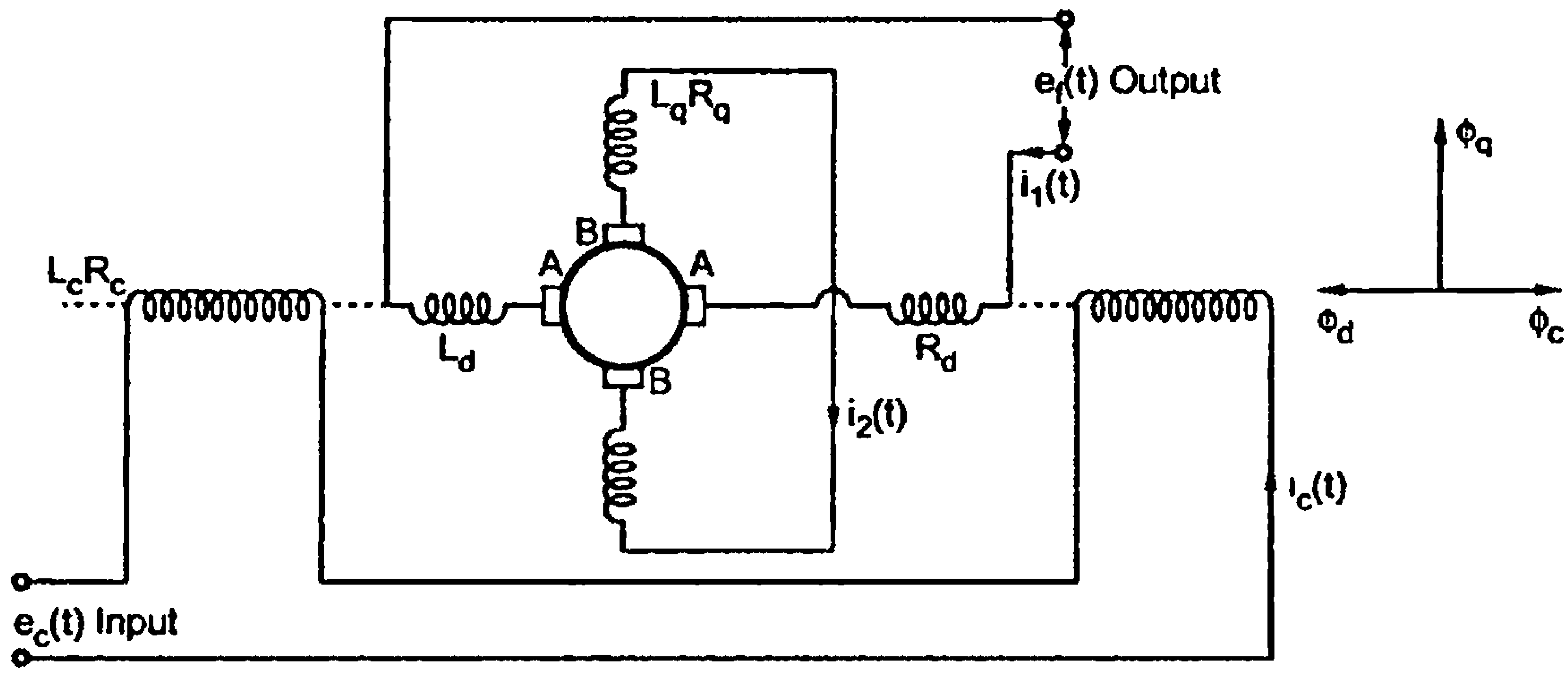


Fig. 16.24

$$\therefore \quad E_f(s) = \frac{K_d K_q E_c(s)}{(R_q + sL_q)(R_c + sL_c)}$$

$$\therefore \quad \boxed{\frac{E_f(s)}{E_c(s)} = \frac{K_d K_q}{(R_q + sL_q)(R_c + sL_c)}}$$

This is the required transfer function.

16.11 Magnetic Amplifier

Magnetic amplifiers are power amplifiers using the principle of magnetic saturation. They consist of a saturable reactor and rectifiers which cause self-saturation.

A balanced saturable core reactor is shown in Fig. 16.25. It consists of a three-limbed core with output windings as gate windings GW_1 and GW_2 on the outer limbs and control windings (CW) on the central limb. GW_1 and GW_2 are connected in series opposition.

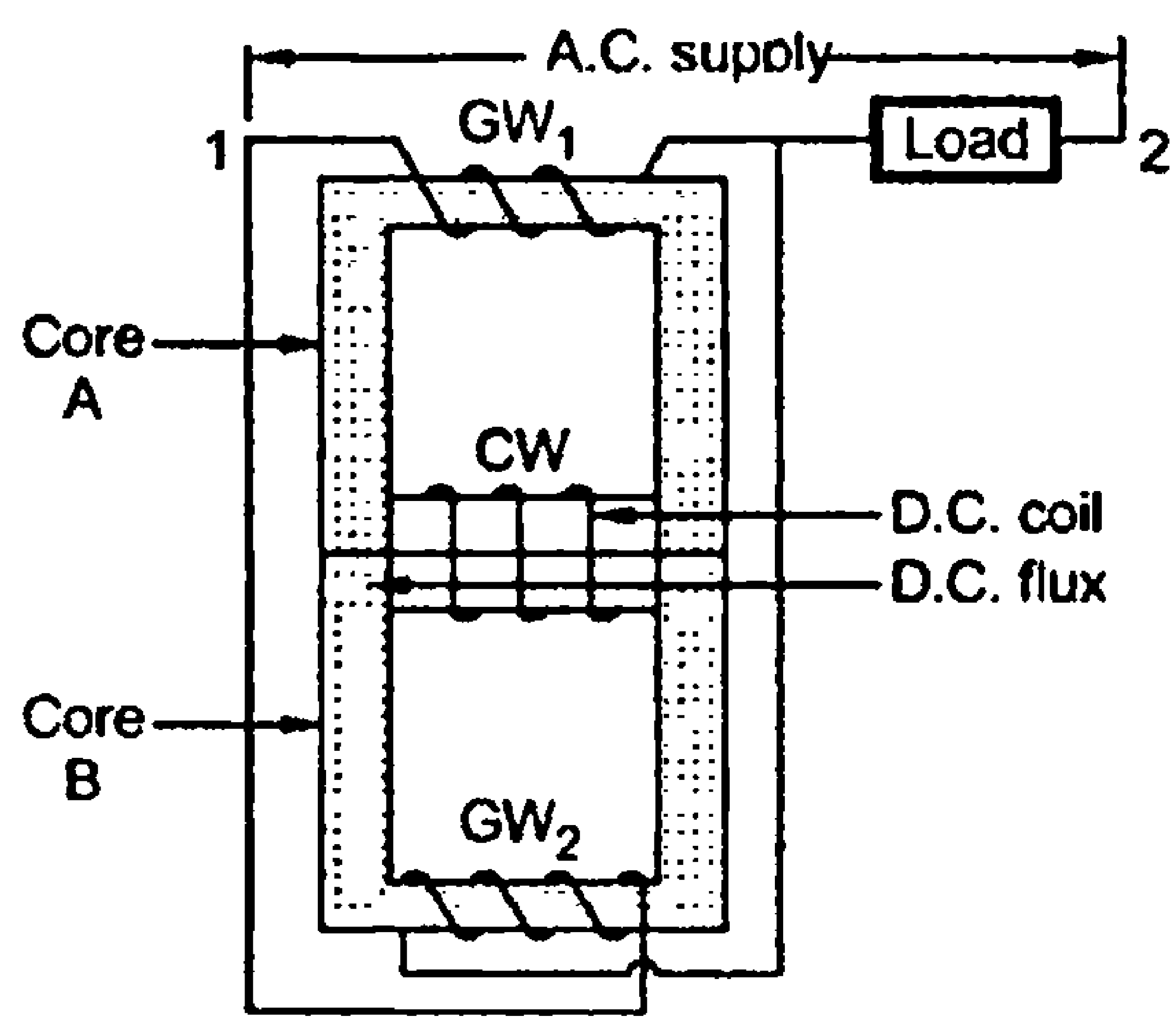


Fig. 16.25 Magnetic amplifier

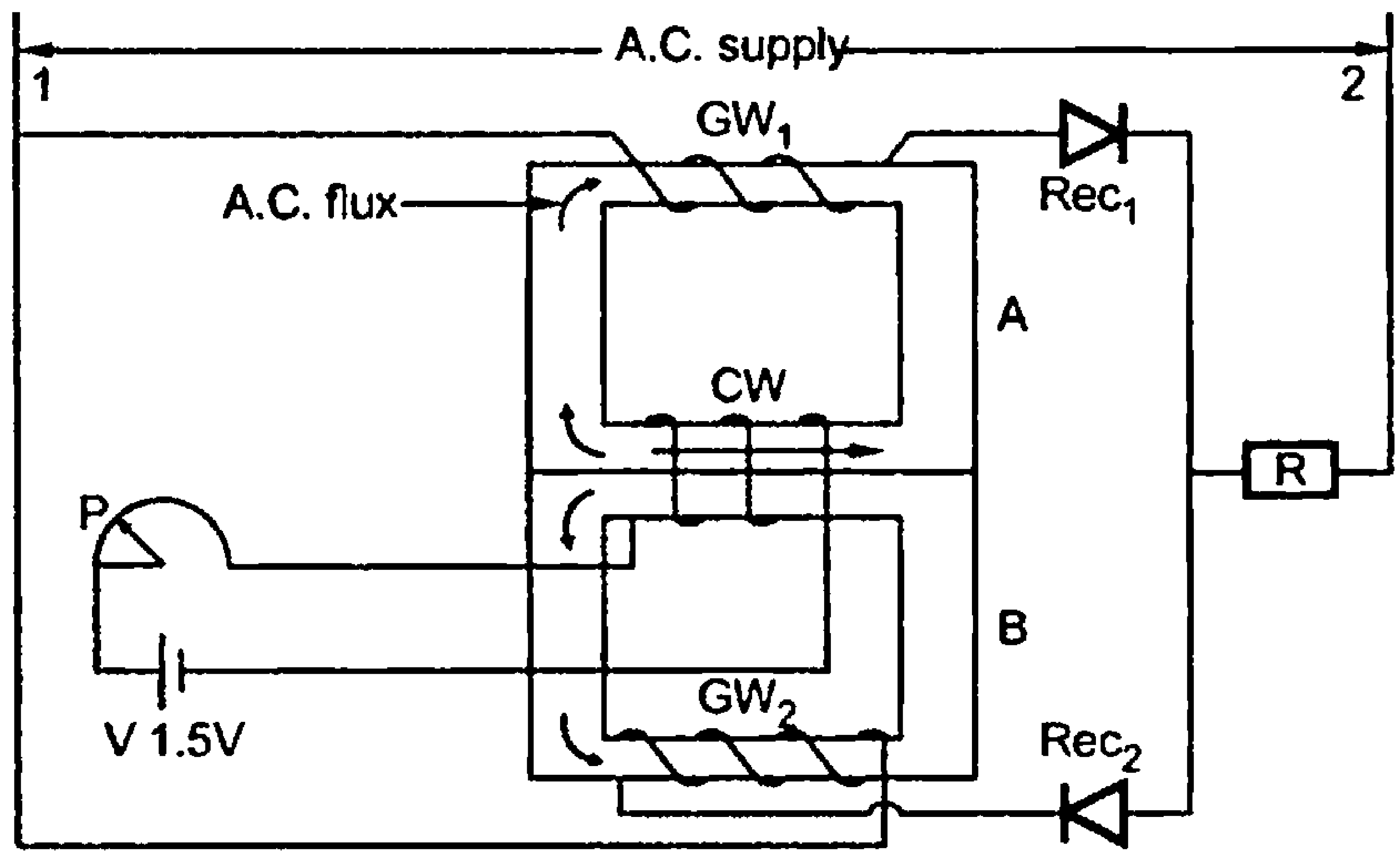


Fig. 16.26 (d)

On upper core A, current can flow in the GW_1 and through the load only when supply terminal 1 is (+). Current flows in the core B gate winding GW_2 and the load only when terminal 2 is (+). Thus the load receives both half cycles of alternating current, but each core is magnetized by only a half cycle of current.

The magnetization curve is shown in Fig. 16.26 (e). But only upper part of this curve concerns the operation of upper core A, during that half cycle when current flows in its a.c. winding; the other half of the magnetization curve can shown similar action in core B, but occurs a half cycle later.

When first a.c. power is connected to the circuit in Fig. 16.26 (d), then terminal 1 is (+ve) a small magnetizing current flows in the upper a.c. winding. At number 1 in Fig. 16.26 (e) these ampere-turns are shown.

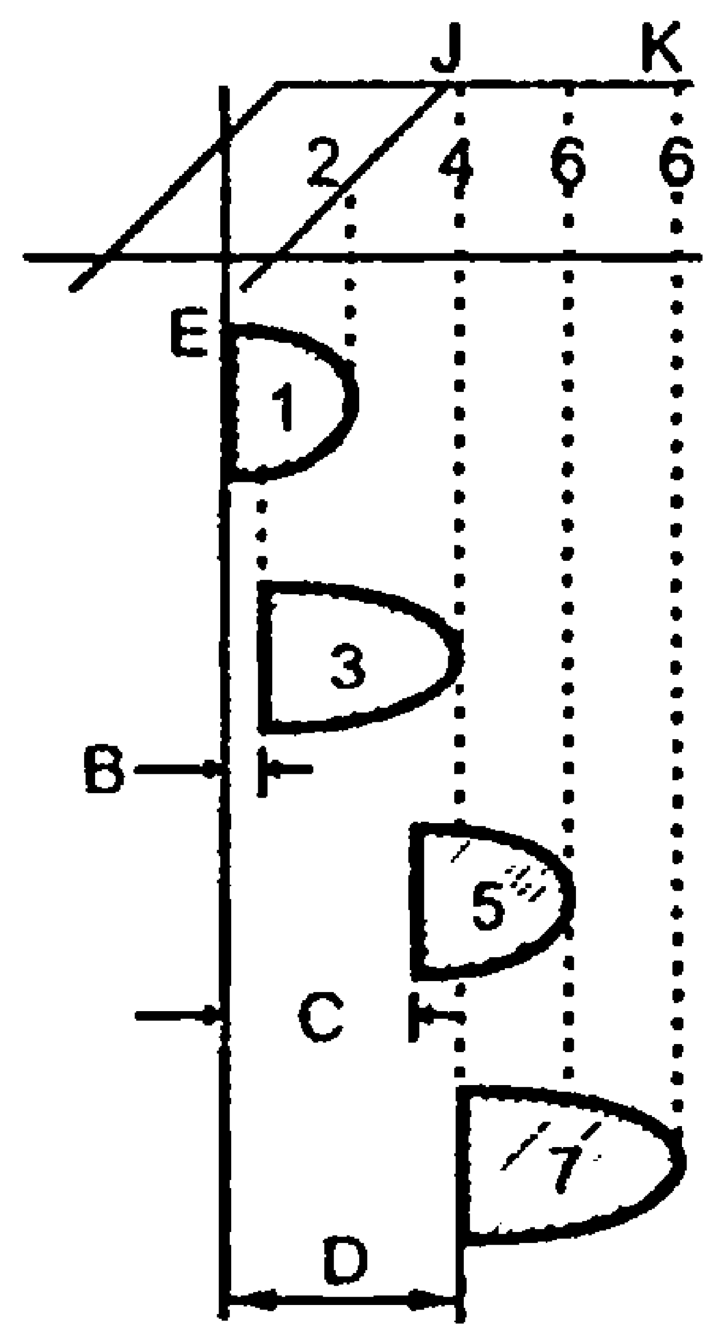


Fig. 16.26 (e) Progressive steps in self saturation

They produce an initial flux in core A, as shown at 2. During the following half cycle this flux is not reset and some of it remains, because diode Rec1 blocks the reverse current. The d.c. or average value of this number 1 half wave is shown at B. From the a.c. supply the next half wave (numbered 3) combines with B and raises the core flux to point 4. After reverse current again is blocked by Rec 1, the flux remaining in the core is greater than before and the average value of m.m.f. has increased to 'C'.

When half wave number 5 raises the total core flux above the knee of magnetization curve, at J, core A lose some of its inductance so that it permits greater current to flow in the a.c. winding; to total flux rises to 6 and the average m.m.f. has increased to D. At half wave 7 it is seen that the flux is driven to point 8, so that the core operators along the flat portion of the magnetization curve throughout the entire half cycle. Here core A is saturated so that coil inductance has decreased greatly; the saturable reactor now has such

Example 16.1 : *A 5 K potentiometer is operated with a grounded centre rate as shown in the Fig. 16.27 If the potentiometer has a rotation angle of 320°, what is the gain constant of potentiometer ?*

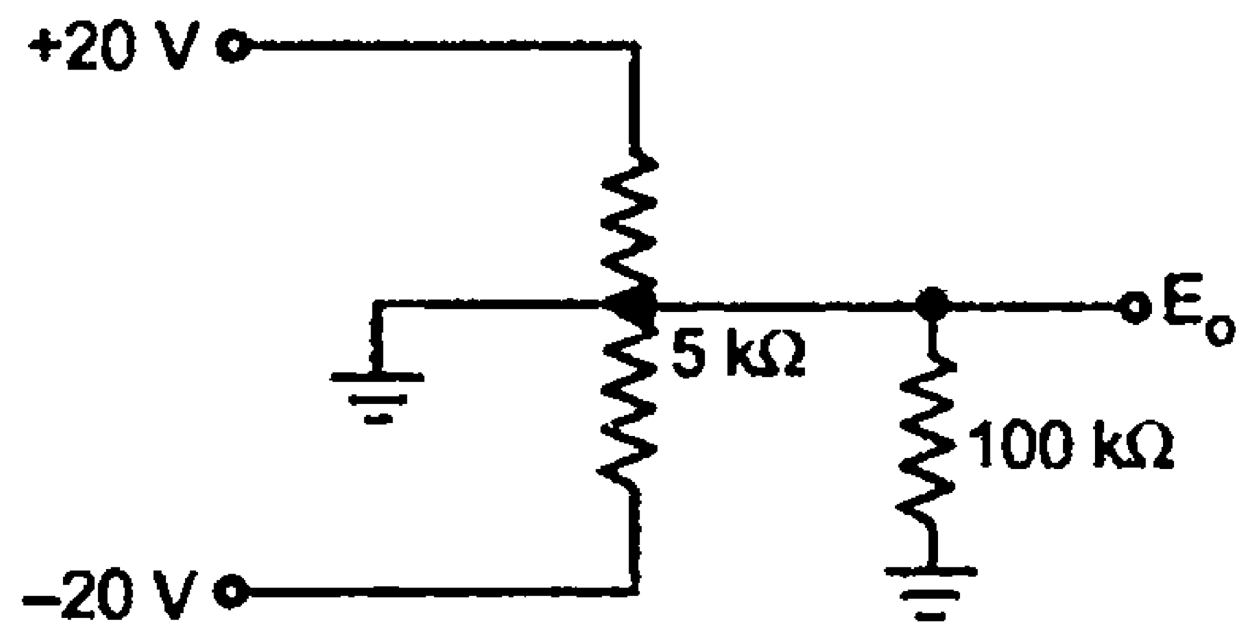


Fig. 16.27

Solution : The reference voltage magnitude is

$$V = 20 - (-20) = 40 \text{ V}$$

$$\theta_{max} = 320° = 320 \times \frac{\pi}{180} \text{ radians} = 5.585 \text{ rad}$$

$$\therefore \quad K_s = \frac{V}{\theta_{max}} = \frac{40}{5.585} = \mathbf{7.161\ V/rad}$$

Example 16.2 : *A helical 10 turn potentiometer has a resistance of 20 kΩ.*

a) If the midpoint setting is 10100 Ω, what is its linearity ?

b) If the quarter point setting is 5100 Ω, what is its linearity ?

c) What is its constant if the reference voltage is 80 V ?

Solution : The linearity is the maximum deviation of actual value expressed as a percentage of actual value.

a) $$\% \text{ Linearity} = \frac{\text{Deviation in resistance}}{\text{Actual resistance}} \times 100 = \frac{10100 - \left(\frac{20 \times 10^3}{2}\right)}{20 \times 10^3} \times 100$$

$$= \mathbf{0.5\ \%}$$

b) $$\% \text{ Linearity} = \frac{5100 - \left(\frac{20 \times 10^3}{4}\right)}{20 \times 10^3} \times 100 = \mathbf{0.5\ \%}$$

c) $$K_s = \frac{V}{2\pi N} = \frac{80}{2\pi \times 10} = \mathbf{1.273\ V/rad}$$

Example 16.3 : *A 50 K, 8 turn potentiometer uses ± 10 V supply.*

a) Find its gain constant in V/rad.

b) Find the output voltage if shaft is rotated through 80° from midpoint towards +10 V.

$\therefore \quad 0.02 = \frac{100}{N}$

$\therefore \quad N = \mathbf{5000}$

Example 16.6 : *The voltage applied to the rotor of a synchro control transmitter is 30 V r.m.s. The rotor shaft is moved 60° from the zero position. Determine the stator voltages with respect to the common point for K = 1. Also determine the voltages between terminals s_1 and s_2, s_2 and s_3 , s_3 and s_1.*

Solution : Let s_1 be the rèference stator winding.

$$\text{Rotor voltage} = E_R = K E_r \sin \omega_0 t$$

$$E_R = 30 \text{ V (r.m.s.)}$$

$$\therefore \quad E_{s2} = E_R \cos\theta \quad \text{as } s_1 \text{ is reference} = 30 \times \cos(60)$$

$$= \mathbf{15 \text{ V (r.m.s.)}}$$

$$\therefore \quad E_{s1} = E_R \cos(\theta - 240) = 30 \times \cos(-180)$$

$$= \mathbf{-30 \text{ V (r.m.s.)}}$$

$$\therefore \quad E_{s3} = E_R \cos(\theta - 120) = 30 \times \cos(-60)$$

$$= \mathbf{15 \text{ V (r.m.s.)}}$$

Now

$$E_{s1\,s2} = \sqrt{3}\ E_R \sin(\theta + 240) = \sqrt{3} \times 30 \times \sin(300)$$

$$= \mathbf{-45 \text{ V}}$$

$$E_{s2\,s3} = \sqrt{3}\ E_R \sin(\theta + 120) = \sqrt{3} \times 30 \times \sin(180)$$

$$= \mathbf{0 \text{ V}}$$

$$E_{s3\,s1} = \sqrt{3}\ E_R \sin\theta = \sqrt{3} \times 30 \times \sin(60)$$

$$= \mathbf{+45 \text{ V}}$$

16.13 Introduction to Controllers

The concept of a control system is to sense deviation of the output from the desired value and correct it, till the desired output is achieved. The deviation of the actual output form its desired value is called an error. The measurement of error is possible because of feedback. The feedback allows us to compare the actual output with its desired value to generate the error. The error is denoted as e(t). The desired value of the output is also called **reference input or a set point.** The error obtained is required to be analyzed to take the proper corrective action.

The **controller** is an element which accepts the error in some form and decides the proper corrective action. The output of the controller is then applied to the process or final control element. This brings the output back to its desired set point value. The controller is the heart of a control system. The accuracy of the entire system depends on how sensitive is the controller to the error detected and how it is manipulating such an error. The controller has its own logic to handle the error. Now a days for better accuracy, the digital controllers such as microprocessors, microcontrollers, computers are used. Such controllers execute certain algorithm to calculate the manipulating signal.

This chapter explains the basic discontinuous controllers such as on-off controller, continuous controllers such as proportional, integral etc. and composite controllers such as proportional plus integral, proportional plus derivative etc. Let us study first the general properties of the controller.

16.14 Properties of Controller

Consider a control system shown in the Fig. 16.29 which includes a controller.

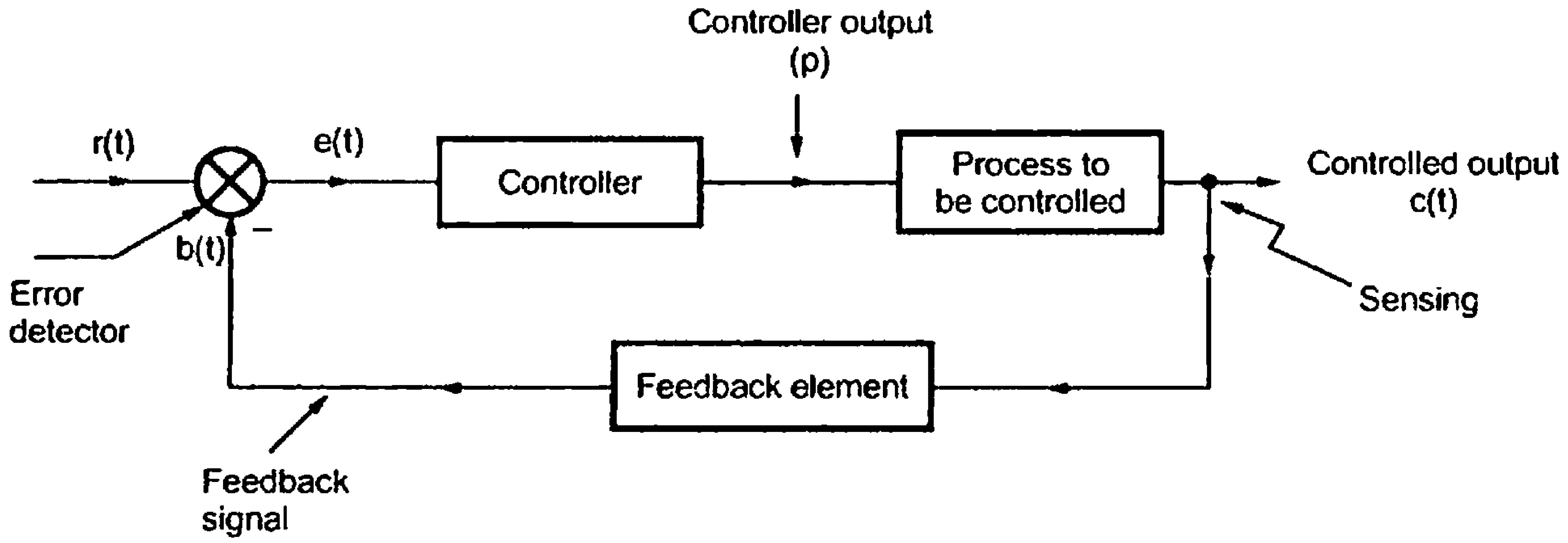


Fig. 16.29 Basic control system

The actual output is sensed by a sensor and converted to a proper feedback signal b(t) using a feedback element. The set point value is the reference input r(t). For example the actual output variable may be temperature but using the thermocouple as the feedback element, the feedback signal b(t) is an electrical voltage. This is then compared with reference input which is also an electrical voltage. The thermocouple senses the output temperature and produces the corresponding electrical e.m.f. as the feedback signal. Hence actual output variable sensed and the feedback signal may be having different forms.

16.14.1 Error

The error detector compares the feedback signal b(t) with the reference input r(t) to generate an error.

$$\therefore \quad \boxed{e(t) = r(t) - b(t)}$$

This gives an absolute indication of an error.

For example if the set point for a range of 5 mV to 20 mV is 12 mV and the feedback signal is 11.8 mV then error is 0.2 mV. But actual variable to be controlled may be different such as temperature, pressure etc. Hence to obtain correct information from the error, it is expressed in percentage form related to the controller operation. It is expressed as the percentage of the measured variable range. The range of the measured variable b(t) is also called **span**.

Thus $\text{span} = b_{max} - b_{min}$

Hence error can be expressed as percent of span as,

$$e_p = \frac{r-b}{b_{max}-b_{min}} \times 100$$

where e_p = Error as % of span

Example 16.7 : *The range of measured variable for a certain control system is 2 mV to 12 mV and a setpoint of 7 mV. Find the error as percent of span when the measured variable is 6.5 mV.*

Solution : $b_{max} = 12$ mV, $b_{min} = 2$ mV, $b = 6.5$ mV, $r = 7$ mV

$$\therefore \quad e_p = \frac{r-b}{b_{max}-b_{min}} \times 100 = \frac{7-6.5}{12-2} \times 100$$

$$= 5\ \%$$

16.14.2 Variable Range

In practical systems, the controlled variable has a range of values within which the control is required to be maintained. This range is specified as the maximum and minimum values allowed for the controlled variable. It can be specified as some nominal values and plus-minus tolerance allowed about this value. Such range is important for the design of controllers.

16.14.3 Controller Output Range

Similar to the controlled variable, a range is associated with a controller output variable. It is also specified interms of the maximum and minimum values.

But often the controller output is expressed as a percentage where minimum controller output is 0 % and maximum controller output is 100 %. But 0 % controller output does not mean, zero output. For example it is necessary requirement of the system that a steam flow corresponding to $\frac{1}{4}^{th}$ opening of the valves should be minimum. Thus 0 % controller output in such case corresponds to the $\frac{1}{4}^{th}$ opening of the valve.

The controller output as a percent of full scale when the output changes within the specified range is expressed as,

$$p = \frac{u - u_{min}}{u_{max} - u_{min}} \times 100$$

where p = Controller output as a percent of full scale

u = Value of the output

u_{max} = Maximum value of controlling variable

u_{min} = Minimum value of controlling variable

16.14.4 Control Lag

The control system can have a lag associated with it. The control lag is the time required by the process and controller loop to make the necessary changes to obtain the output at its setpoint. The control lag must be compared with the process lag while designing the controllers. For example in a process a valve is required to be open or closed for controlling the output variable. Physically the action of opening or closing of the valve is very slow and is the part of the process lag. In such a case there is no point in designing a fast controller than the process lag.

16.14.5 Dead Zone

Many a times a dead zone is associated with a process control loop. The time corresponding to dead zone is called dead time. The time elapsed between the instant when error occurs and the instant when first corrective action occurs is called dead time. Nothing happens in the system, during this time though the error occurs. This part is also called dead band. The effect of such dead time must be considered while the design of the controllers.

16.15 Classification of Controllers

The classification of the controllers is based on the response of the controller and mode of operation of the controller. For example in a simple temperature control of a room, the heater is to be controlled. It should be switched on or off by the controller when temperature crosses its setpoint. Such an operation of the controller is called discontinuous operation and the mode of operation is called **discontinuous mode** of controller. But in some process control systems, simple on/off decision is not sufficient. For example, controlling the steam flow by opening or closing the valve. In such case a smooth opening or closing of valve is necessary. The controller in such a case is said to be operating in a **continuous mode.**

Thus the controllers are basically classified as discontinuous controllers and continuous controllers.

The discontinuous mode controllers are further classified as ON-OFF controllers and multiposition controllers.

controller and also reduces the undershoot and overshoot which is common in two position controller.

Mathematically multiposition mode is expressed as,

$$\boxed{p = p_i,} \qquad e_p > |e_i| \quad \text{where } i = 1, 2, \ldots\ldots n$$

As the error e_p exceeds the set limits $\pm e_i$ then the output of the controller is adjusted to preset value p_i.

For example a three position controller in which,

$$p = 100\ \% \qquad e_p > +e_1$$

$$p = 75\ \% \qquad -e_1 < e_p < +e_1$$

$$p = 0\ \% \qquad e_p < -e_1$$

So as long as error is within $\pm e_1$ of the setpoint, the controller output is 75 %. If error increases the setpoint by e_1 or more, the output is 100 %. If error is less than the setpoint by $-e_1$ or more, the output of the controller is 0 %.

The Fig. 16.31 shows the graph of error and controller output against time, without time lag, for a three position controller.

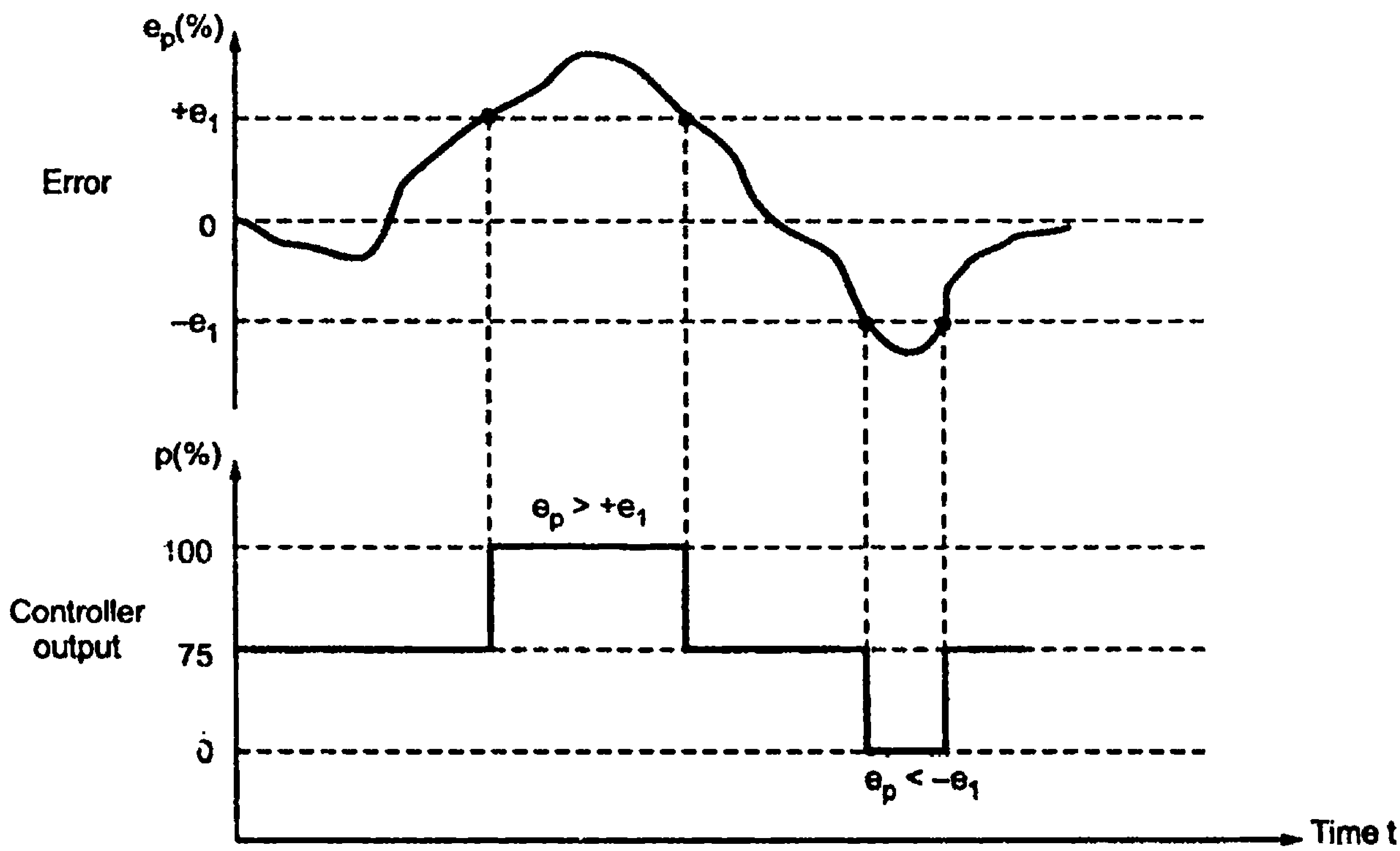


Fig. 16.31 Behaviour of three position controller

Similar to the three position, multiposition mode controllers also can be designed.

16.17 Continuous Controller Modes

In the discontinuous controller mode, the output of the controller is discontinuous and not smoothly varying. But in the continuous controller mode, the controller output varies smoothly proportional to the error or proportional to some form of the error. Depending upon which form of the error is used as the input to the controller to produce the continuous controller output, these controllers are classified as,

1. Proportional control mode
2. Integral control mode
3. Derivative control mode

Let us discuss these control modes in detail.

16.18 Proportional Control Mode

In this control mode, the output of the controller is simple proportional to the error e(t). The relation between the error e(t) and the controller output p is determined by constant called **proportional gain constant** denoted as K_p. The output of the controller is a linear function of the error e(t). Thus each value of the error has a unique value of the controller output. The range of the error which covers 0 % to 100 % controller output is called **proportional band.**

Now though there exists linear relation between controller output and the error, for a zero error the controller output should not be zero, otherwise the process will come to halt. Hence there exists some controller output p_o for the zero error. Hence mathematically the proportional control mode is expressed as,

$$\boxed{p(t) = K_p e(t) + p_o} \quad \dots (1)$$

where K_p = Proportional gain constant

p_o = Controller output with zero error

The direct and reverse action is possible in the proportional control mode. The error may be positive or negative because error is r–b and b can be less or greater than reference setpoint r.

If the controlled variable i.e. input to the controller increases, causing increase in the controller output, the action is called **direct action.** For example the output valve is to be controlled to maintain the liquid level in a tank. So if the level increases, the valve should be opened more to maintain the level.

16.19 Integral Control Mode

In the proportional control mode, error reduces but cannot go to zero. It finally produces an offset error. It cannot adapt with the changing load conditions. To avoid this, another control mode is oftenly used in the control systems which is based on the history of the errors. This mode is called **integral mode or reset action controller.**

In such a controller, the value of the controller output p(t) is changed at a rate which is proportional to the actuating error signal e(t). Mathematically it is expressed as,

$$\frac{dp(t)}{dt} = K_i e(t)$$

where K_i = Constant relating error and rate

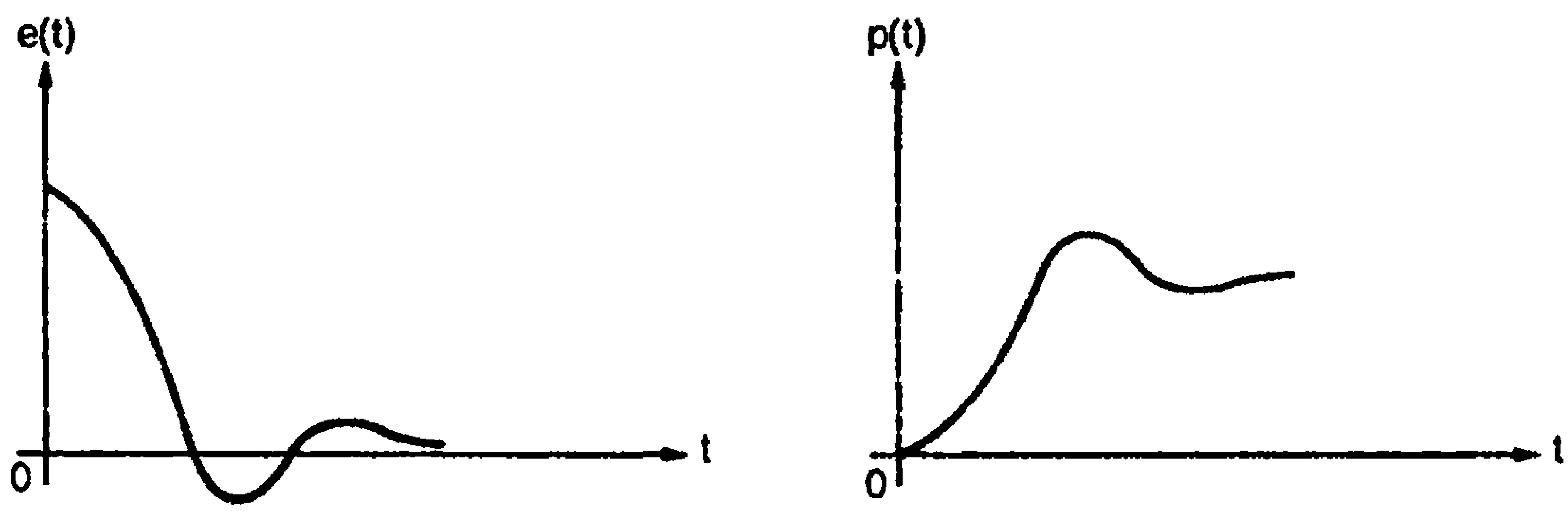
The constant K_i is also called integral constant. Integrating the above equation, actual controller output at any time t can be obtained as,

$$p = K_i \int e(t)dt + p(0) \qquad \text{... (2)}$$

where p(0) = Controller output when integral action starts i.e. at t = 0.

The output signal from the controller, at any instant is the area under the actuating error signal curve upto that instant. If the value of the error is doubled, the value of p(t) varies twice as fast i.e. rate of the controller output change also doubles.

If the error is zero, the controller output is not changed. The control signal p(t) can have nonzero value when the error signal e(t) is zero. This is because the output depends on the history of the error and not on the instantaneous value of the error. This is shown in the Fig. 16.34.



(a) Error (b) Controller output

Fig. 16.34 Integral mode

For example, let us assume that the integral controller is used to control the armature current of a d.c. motor and to keep its value constant at 500 A. As long as the armature current is less than 500 A, the armature voltage, controlled by the controller, will increase. Thus the output of the controller will increase and will continue to do so till the error becomes zero i.e. armature current becomes 500 A. Then the controller output will remain at that value reached. This is possible because the output of the controller can remain at any value within its range, if the input is zero. The controller must not be overdriven as it will not then be effective.

Thus for an integral mode,

1. If error is zero, the output remains at a fixed value equal to what is was, when the error became zero.
2. If the error is not zero, then the output begins to increase or decrease, at a rate K_i % per second for every ± 1 % of error.

In some cases, the inverse of K_i called **integral time** is specified, denoted as T_i.

$$T_i = \frac{1}{K_i} = \text{Integral time}$$

It is expressed in minutes instead of seconds.

16.19.3 Applications

The comparison of proportional and integral mode behaviour at the time of occurrence of an error signal is tabulated below.

Controller	Initial behaviour	Steady state behaviour
P	Acts immediately. Action according to K_p.	Offset error always present. Larger the K_p smaller the error.
I	Acts slowly. It is the time integral of the error signal.	Error signal always becomes zero.

Table 16.1

It can be seen that proportional mode is more favourable at the start while the integral is better for steady state response. In pure integral mode, error can oscillate about zero and can be cyclic. Hence in practice integral mode is never used alone but combined with the proportional mode, to enjoy the advantages of both the modes.

16.20 Derivative Control Mode

In practice the error is function of time and at a particular instant it can be zero. But it may not remain zero forever after that instant. Hence some action is required corresponding to the rate at which the error is changing. Such a controller is called derivative controller.

The controller output is 50 % for the zero error. When error starts increasing, the controller output suddenly jumps to the higher value. It further jumps to a higher value for higher rate of increase of error. Then error becomes constant, the output returns to 50 %. When error is decreasing i.e. having negative slope, controller output decreases suddenly to a lower value.

The various characteristics of the derivative mode are,

1. For a given rate of change of error signal, there is a unique value of the controller output.
2. When the error is zero, the controller output is zero.
3. When the error is constant i.e. rate of change of error is zero, the controller output is zero.
4. When error is changing, the controller output changes by K_d % for even 1 % per second rate of change of error.

16.20.2 Applications

When the error is zero or a constant, the derivative controller output is zero. Hence it is never used alone. Its gain should be small because faster rate of change of error can cause very large sudden change of controller output. This may lead to the instability of the system.

16.21 Composite Control Modes

As mentioned earlier, due to offset error proportional mode is not used alone. Similarly integral and derivative modes are also not used individually in practice. Thus to take the advantages of various modes together, the composite control modes are used. The various composite control modes are,

1. Proportional + Integral mode (PI)
2. Proportional + Derivative mode (PD)
3. Proportional + Integral + Derivative mode (PID)

Let us see the characteristics of these three modes.

16.22 Proportional + Integral Mode (PI Control Mode)

This is a composite control mode obtained by combining the proportional mode and the integral mode.

The mathematical expression for such a composite control is,

$$p(t) = K_p\, e(t) + K_p K_i \int_0^t e(t)\, dt + p(0)$$

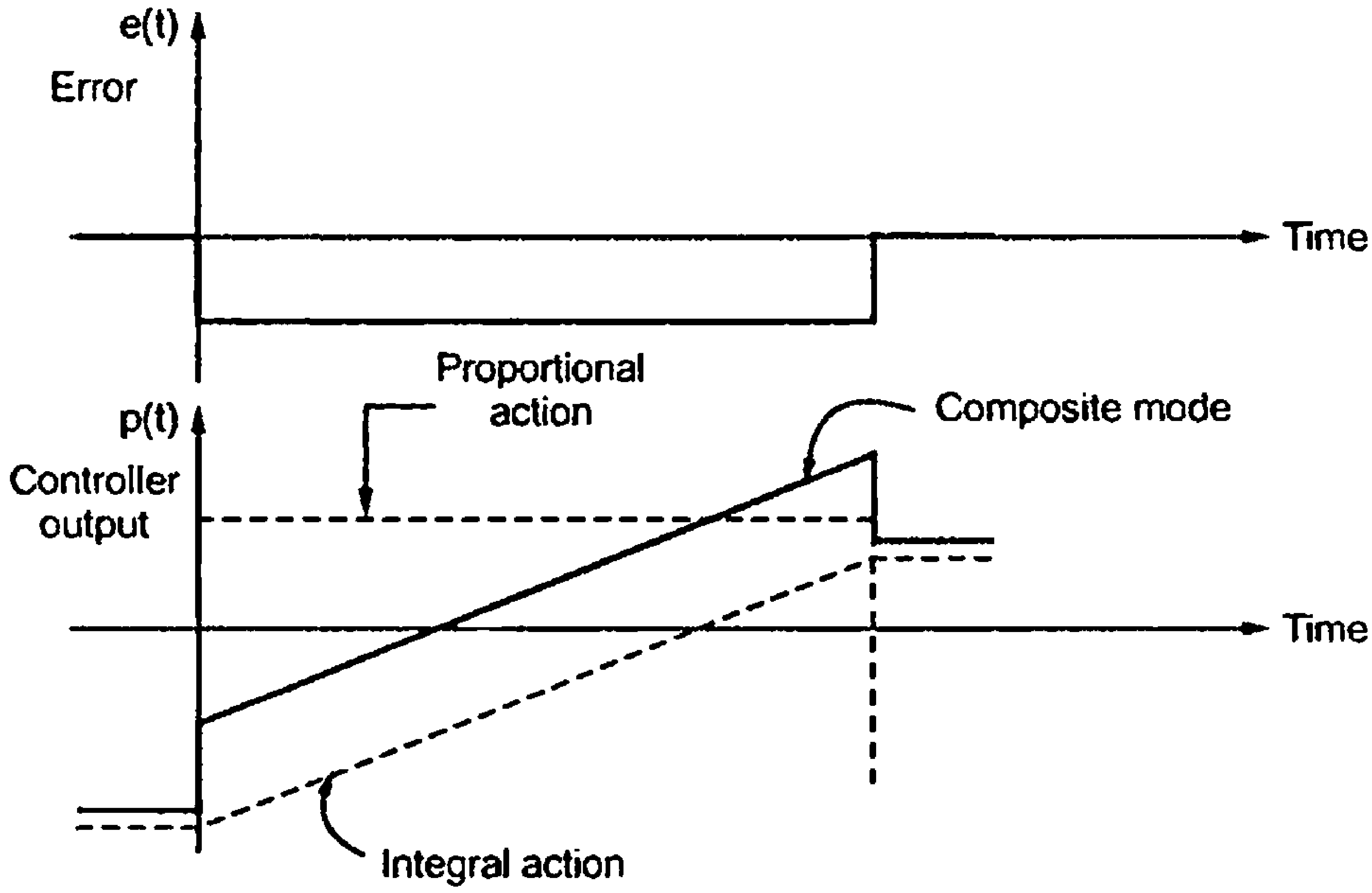


Fig. 16.39

16.22.1 Characteristics of PI Mode

The various characteristics of the composite PI mode are,

1. When the error is zero, the controller output is fixed at the value that integral mode had when the error went to zero. This is nothing but p(0).
2. When the error is not zero, proportional mode adds the correction while the integral term starts increasing or decreasing from its initial value depending upon reverse or direct action.
3. It improves the steady state accuracy.
4. It increases the rise time so response becomes slow.
5. It decreases bandwidth of the system.
6. It filters out the high frequency noise.
7. It makes the response more oscillatory.

16.22.2 Applications

The composite PI mode completely removes the offset problems of proportional mode. Such a mode can be used in the systems with the frequent or large load changes. But the process must have relatively slow changes in the load, to prevent the oscillations.

16.23 Proportional + Derivative Mode (PD Control Mode)

The series combination of proportional and derivative control modes gives proportional plus derivative control mode. The mathematical expression for the PD composite control is,

$$p(t) = K_p e(t) + K_p K_d \frac{de(t)}{dt} + p(0)$$

The behaviour of such a PD control to a ramp type of the input is shown in the Fig. 16.40.

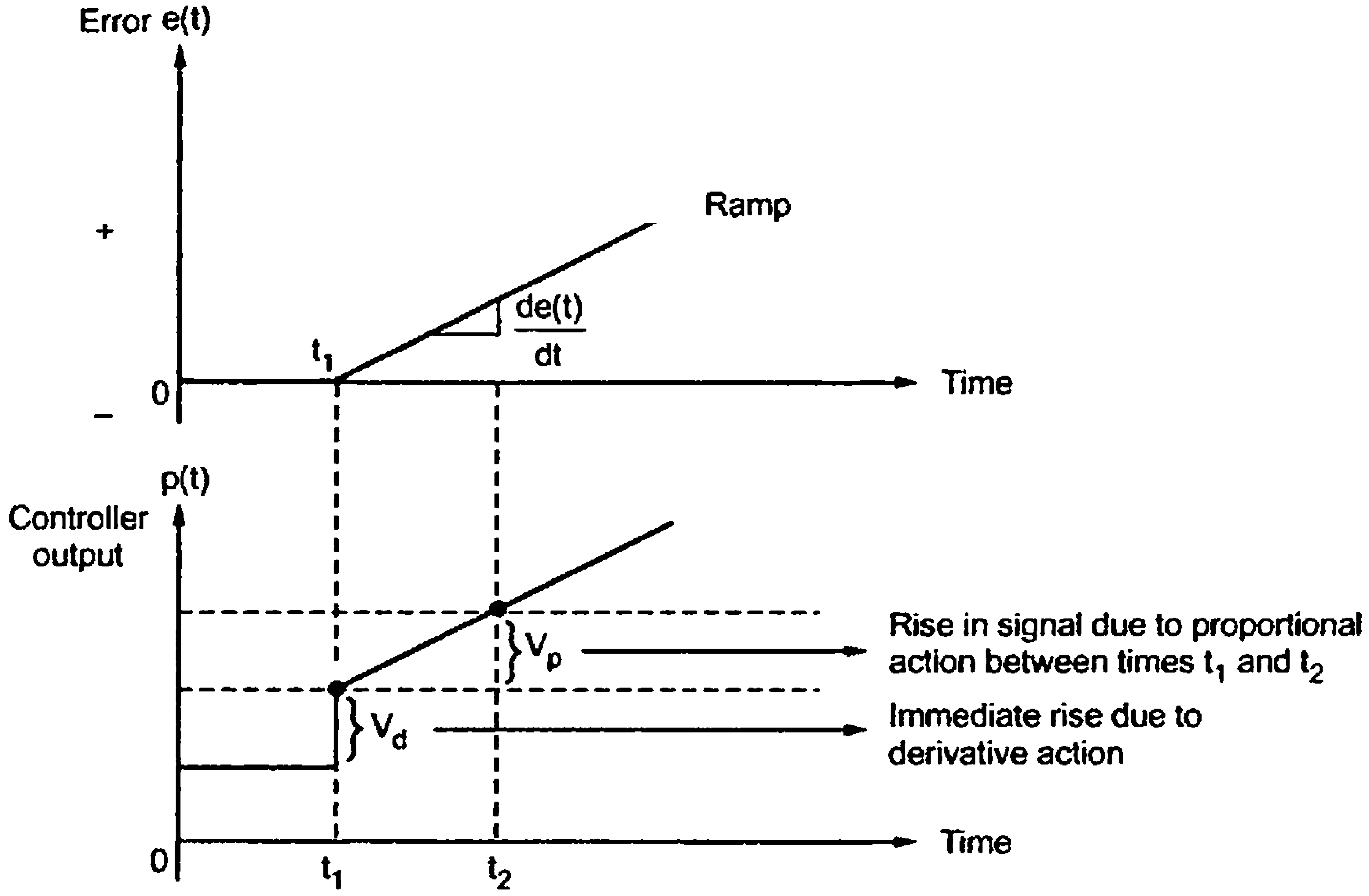


Fig. 16.40

The ramp function of error occurs at $t = t_1$. The derivative mode causes a step V_d at t_1 and proportional mode causes a rise of V_p equal to V_d at t_2. This is for direct action PD control.

The Fig. 16.41 shows the behaviour of PD control for the arbitrary load changes for reverse action. See Fig. 16.41 on next page.

For the reverse action, the controller output is image of the error for the proportional mode. While derivative mode causes sudden increase or decrease in the output corresponding to decrease or increase in the error. This mode can not eliminate the offset of the proportional mode.

16.23.1 Characteristics of PD Mode

The various characteristics of the PD mode are,

1. It improves the damping and reduces overshoot.
2. It reduces the rise time and makes response fast.
3. It makes the response stable very fast.

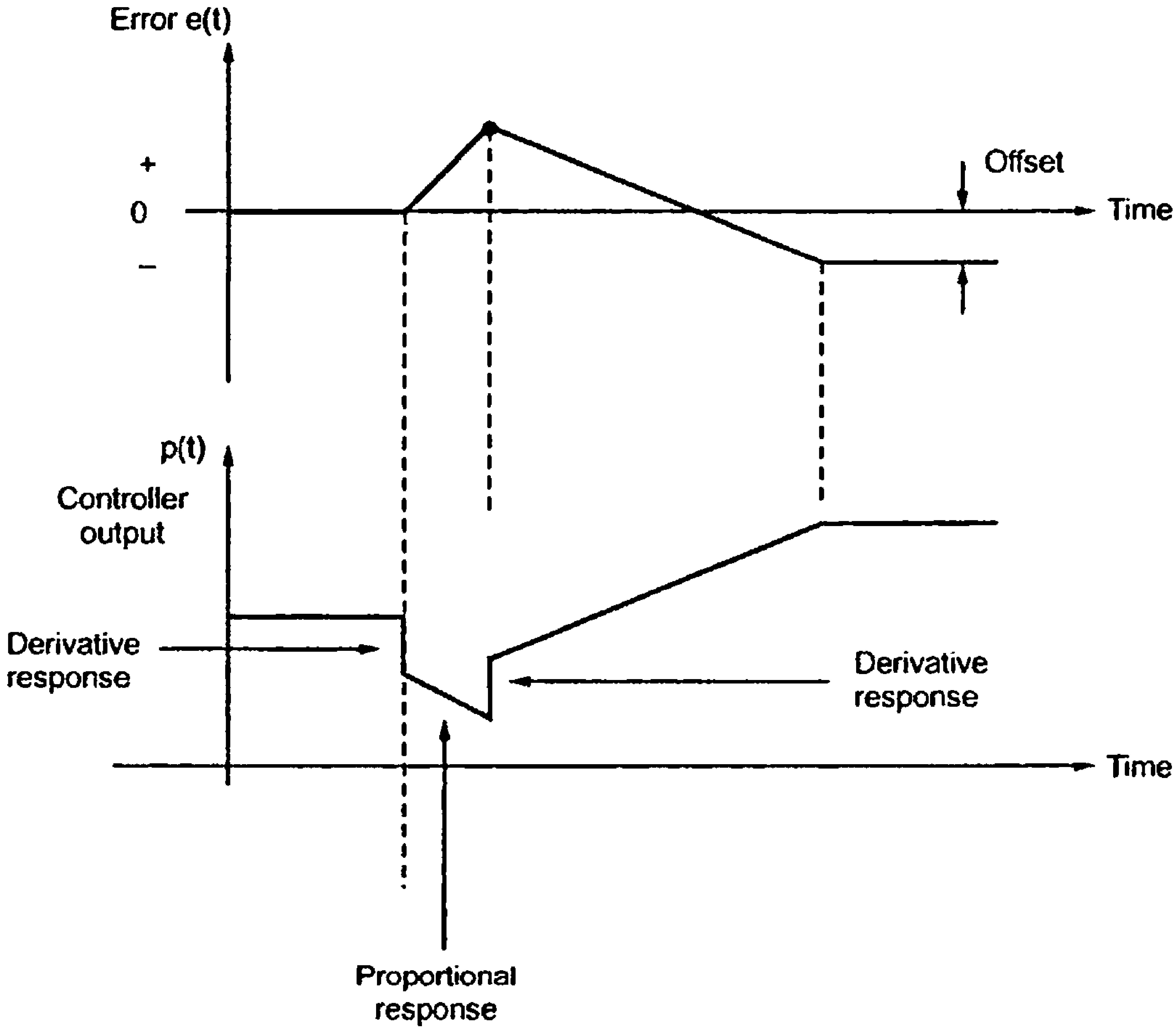


Fig. 16.41

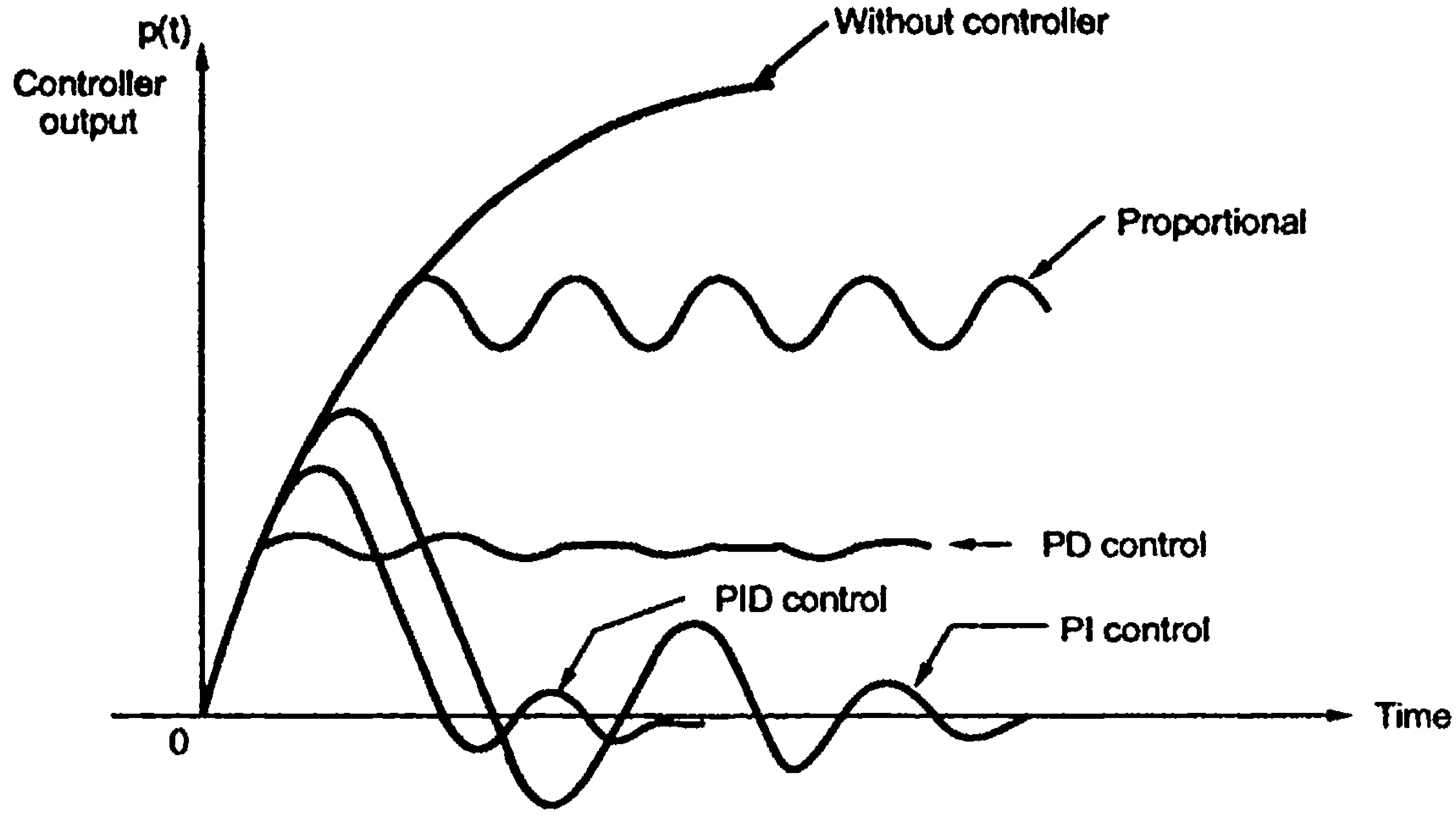
4. It improves the bandwidth of the system.
5. It cannot eliminate offset error.
6. It may make the noise dominant at high frequencies.
7. It is not very effective for lightly damped systems.
8. It may require a relatively large capacitor while the circuit implementation.

16.23.2 Applications

It cannot eliminate offset error of proportional mode. But it can be used to handle fast process load changes if offset error is tolerable. It is used in many industrial process control systems.

16.24 Three Mode Controller (PID Control Mode)

The composite controller including the combination of the proportional, integral and derivative control mode is called PID control mode and the controller is called three mode controller. It is very much complex to design but very powerful in action.



16.43

Example 16.8 : *A PI controller is used to control a certain process. The settings of the controller are K_p = 2 % and K_i = 4 % per min. While p(0) = 50 %. The error signal is found to be 3t + 7 where t is the time. Find the controller output in % after 1.5 minutes.*

Solution : For the PI controller,

$$p(t) = K_p\, e(t) + K_p K_i \int_0^t e(t)\,dt + p(0)$$

$$= 0.02\,[3t+7] + 0.02\times 0.04 \int_0^t (3t+7)\,dt + 0.5$$

$$= [0.06t] + 0.14 + 8\times 10^{-4}\left[\frac{3t^2}{t} + 7t\right]_0^t + 0.5$$

$$= [0.06\times 1.5] + [0.14] + 8\times 10^{-4}\left[\frac{3}{2} + (1.5)^2 + 7\times 1.5\right] + 0.5$$

$$= 0.09 + 0.14 + 0.0111 + 0.5 = 0.7411$$

i.e. $\quad p(t) =$ **74.11 % after t = 1.5 minutes**

And at t = 7, $e(t) = 0$

$$\therefore \quad m = \text{Slope} = \frac{0-4}{7-4} = -\frac{4}{3}$$

And substituting t = 4 and e(t) = 4 we get,

$$4 = -\frac{4}{3} \times 4 + c$$

$$\therefore \quad c = 4 + \frac{16}{3} = \frac{28}{3}$$

$$\therefore \quad e(t) = -\frac{4}{3}t + \frac{28}{3} \text{ in this interval}$$

$$\therefore \quad \frac{de(t)}{dt} = -\frac{4}{3}$$

$$\therefore \quad p_3(t) = 6\left[-\frac{4}{3}t + \frac{28}{3}\right] - \frac{4}{3} \times 6 \times 0.4 + 25$$

$$= -8t + 56 - 3.2 + 25 = -8t + 77.8$$

So at t = 4, $p_3(t) = -32 + 77.8 = 45.8\ \%$

So there is change from 49 % to 45.8 % at t = 4.

And at t = 7, $p_3(t) = -56 + 77.8 = 21.8\ \%$

So from t = 4 to 7, p(t) decreases linearily from 45.8 % to 21.8 %. Then it again becomes 25 %. The controller output is as shown in the Fig. 16.45.

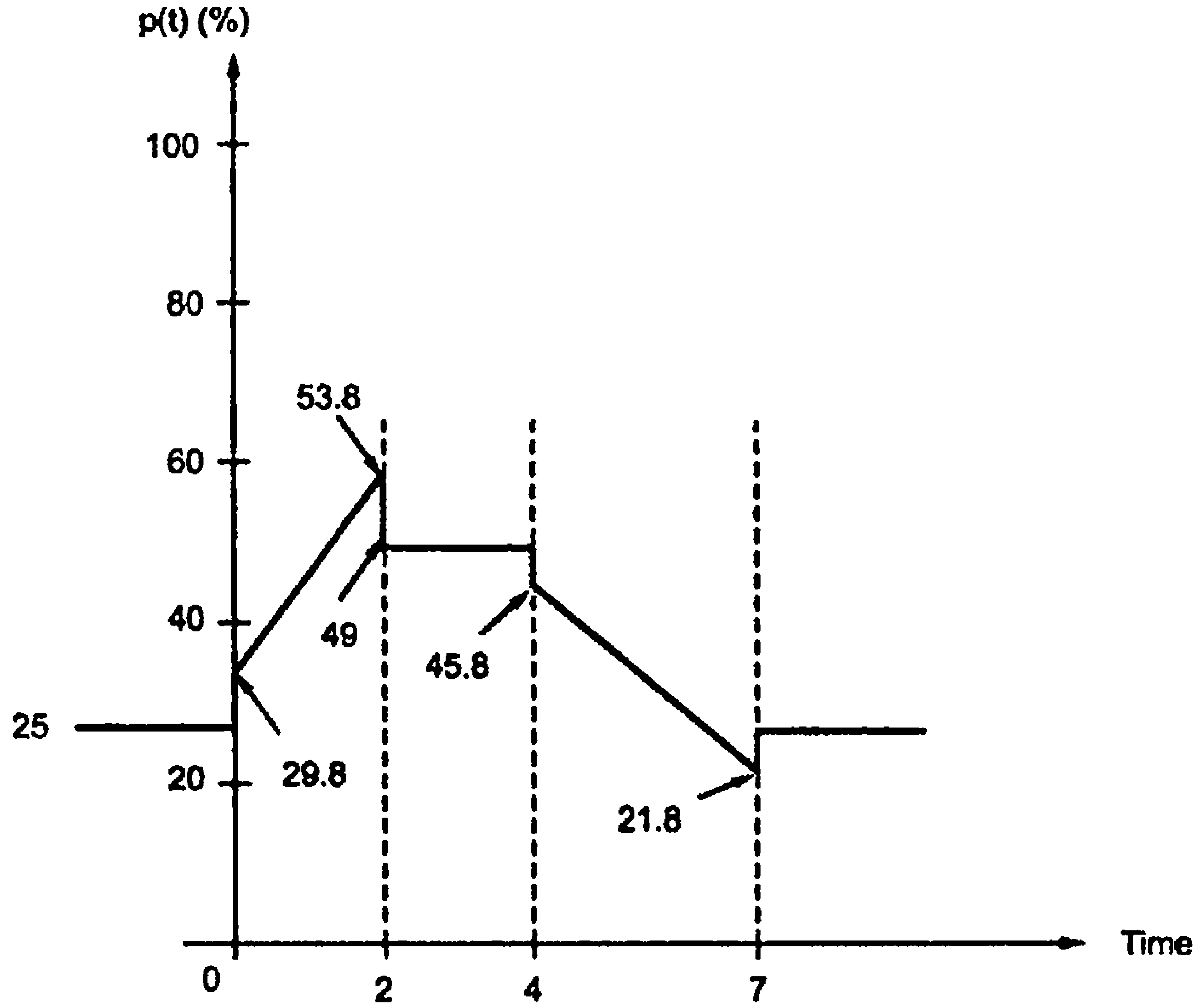


Fig. 16.45

For the system, $$G(s)H(s) = \frac{\omega_n^2}{s(s+2\xi\,\omega_n)}$$

And $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2}{s^2+2\xi\,\omega_n s+\omega_n^2}$$

It is standard second order system, having damping ratio of ξ and natural frequency of oscillations as ω_n.

For steady state error,

$$K_p = \lim_{s \to 0} s\, G(s)H(s) = \infty$$

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = 0 \text{ for the step input}$$

$$K_p = \lim_{s \to 0} s\, G(s)H(s) = \frac{\omega_n}{2\xi}$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{2\xi}{\omega_n}$$

So there is a finite error for the ramp input of magnitude A, which depends on ξ and ω_n.

For good time response, the system demands.

1. Less settling time.
2. Less overshoot.
3. Less rise time.
4. Smallest steady state error.

Increasing K_v, the steady state error for the ramp input can be reduced but it increases overshoot and settling time. This may lead to instability of the system. So to keep steady state error and overshoot well within the acceptable limits, the various composite controllers are used. Let us see the effect of PD, PI, PID and rate feedback controller on the time response of the second order system under consideration.

16.27 PD Type of Controller

A controller in the forward path, which changes the controller output corresponding to proportional plus derivative of error signal is called **PD controller.**

i.e. $$\boxed{\text{Output of controller} = K\,e(t) + T_d \frac{de(t)}{dt}}$$

Taking Laplace $= K\,E(s) + sT_d\,E(s) = E(s)\,[K + sT_d]$

While

$$K_p = \lim_{s \to 0} G(s)H(s) = \infty, \quad e_{ss} = 0$$

$$K_v = \lim_{s \to 0} s\,G(s)H(s) = \infty, \quad e_{ss} = 0$$

Key Point: *Hence as type is increased by one, error becomes zero for ramp type of inputs i.e. steady state of system gets improved and system becomes more accurate in nature*

Hence PI controller has following effects :

i) It increases order of the system.

ii) It increases TYPE of the system.

iii) Design of K_i must be proper to maintain stability of system. So it makes system relatively less stable.

iv) Steady state error reduces tremendously for same type of inputs.

Key Point: *In general this controller improves steady state part affecting the transient part.*

16.29 PID Type of Controller

As PD improves transient and PI improves steady state, combination of two may be used to improve overall time response of the system. This can be realized as shown in the Fig. 16.49 (a).

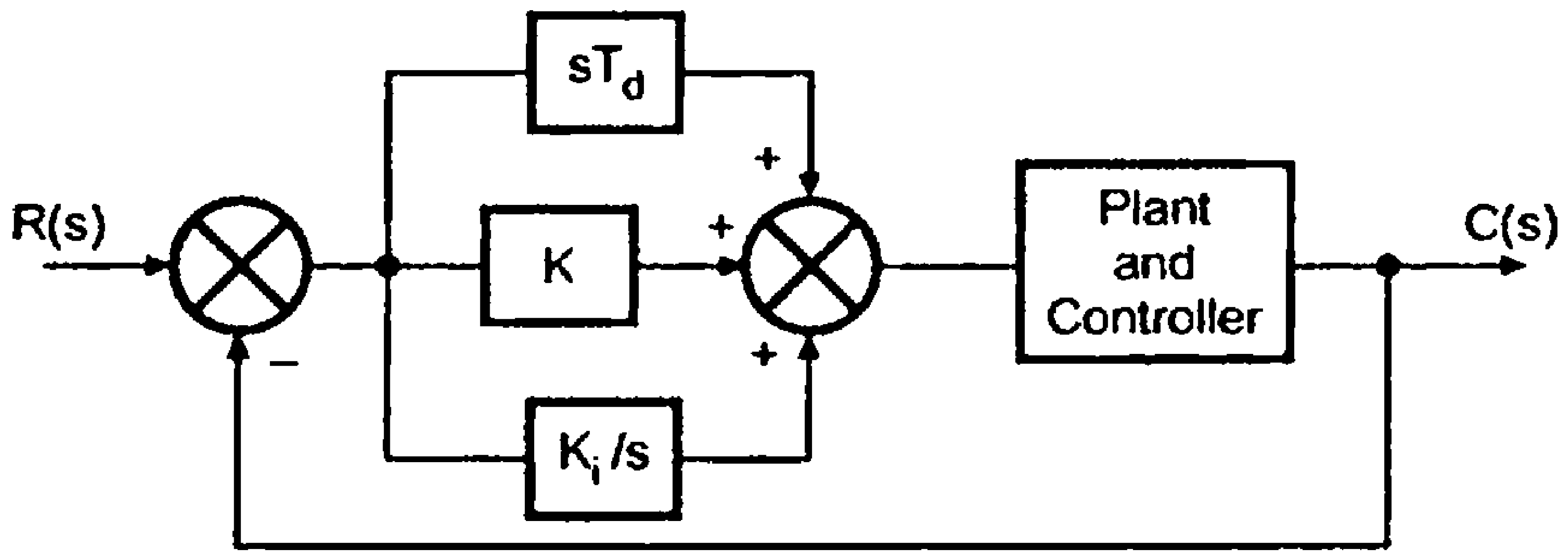


Fig. 16.49 (a)

The design of such controller is complicated in practice.

16.30 Rate Feedback Controller (Output Derivative Controller)

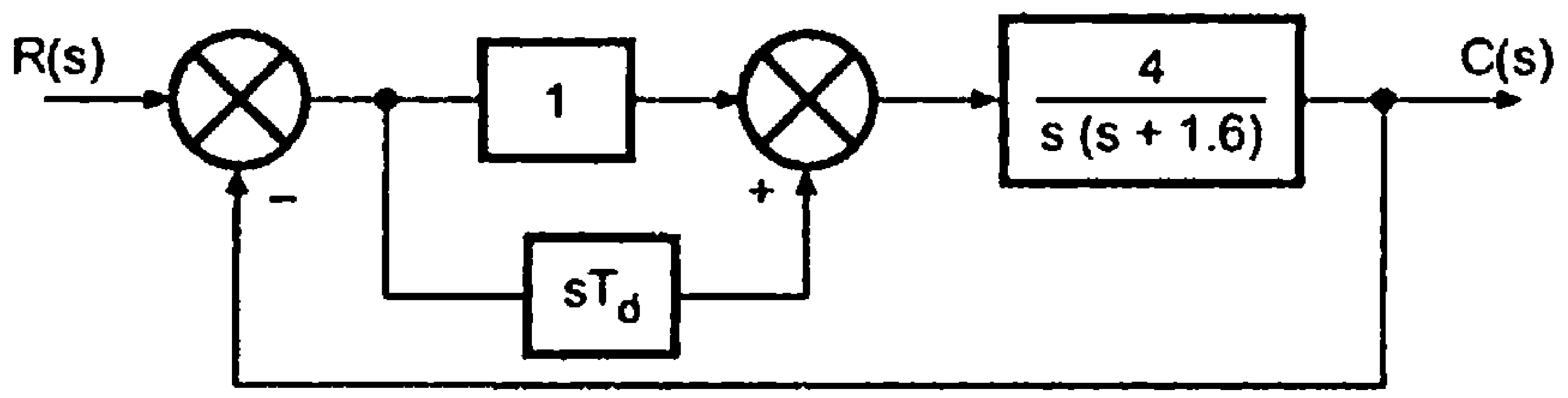
This is achieved by feeding back the derivative of output signal internally using a tachogenerator and comparing with signal proportional to error as shown. This is called minor loop feedback compensation.

$$\text{Output of controller} = K\,E(t) - K_t \frac{dc(t)}{dt}$$

But for step reference input, as TYPE of system is '1' before and after the implementation of the controller and hence error is always zero. So for step we can say, it behaves exactly similar to PD type of controller giving us improvement in transient response of the system. Though its behaviour is same for step, rise time for derivative controller is small i.e. response is fast in comparison with rate feedback controller because of added zero in the forward path.

Examples with Solutions

Example 16.10: *The figure shows PD controller used for the system. Determine the value of T_d so that system will be critically damped. Calculate its settling time.*



Solution :

$$G(s) = \frac{(1+sT_d)\,4}{s(s+1.6)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{(1+sT_d)\,4}{s(s+1.6)}}{1+\dfrac{(1+sT_d)4}{s(s+1.6)}} = \frac{(1+sT_d)\,4}{s^2+1.6s+4T_d\,s+4}$$

Comparing denominator with standard form,

$$\omega_n^2 = 4, \quad \omega_n = 2 \quad \text{and} \quad 2\xi\,\omega_n = 1.6 + 4\,T_d$$

$$\therefore \quad \xi = \frac{1.6+4T_d}{4}$$

Now system required is critically damped, i.e. $\xi = 1$

$$\therefore \quad 1 = \frac{1.6+4\,T_d}{4}$$

$$\therefore \quad 4 = 1.6+4T_d$$

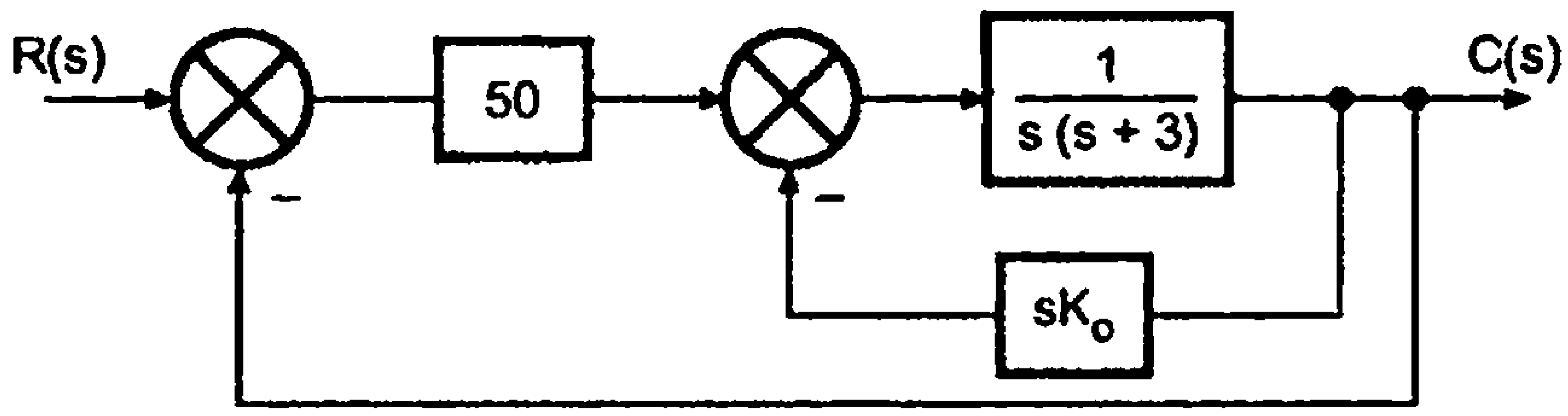
$$\therefore \quad T_d = 0.6 \quad \text{and settling time} = \frac{4}{\xi\,\omega_n}$$

$$T_s = \frac{4}{2\times 1} = 2 \text{ sec.}$$

Example 16.11 : *A unity feedback system is shown in following figure.*

i) In the absence of derivative feedback controller ($K_o = 0$).

Find ξ and ω_n *ii) Find K_o, if ξ is to be modified to 0.5 by use of controller.*



Solution :

$$G(s) = 50 \times \frac{1}{s(s+3)} \quad \text{for } K_o = 0$$

$$H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{50}{s^2+3s+50}$$

$$\therefore \quad \omega_n^2 = 50 \qquad \therefore \omega_n = \sqrt{50} = 7.071 \text{ rad/sec.}$$

$$2\xi\omega_n = 3 \qquad \therefore \quad \xi = \frac{3}{2\times\sqrt{50}} = 0.2121$$

$$\text{With controller,} G(s) = \frac{50 \times \frac{1}{s(s+3)}}{1+\frac{sK_o}{s(s+3)}} = \frac{50}{s^2+3s+sK_o} = \frac{50}{s[s+3+K_o]}$$

$$H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\frac{50}{s(s+3+K_o)}}{1+\frac{50}{s(s+3+K_o)}} = \frac{50}{s^2+(3+K_o)s+50}$$

$$\therefore \quad \omega_n^2 = 50 \qquad \text{so } \omega_n = 7.071$$

$$2\xi\omega_n = 3 + K_o$$

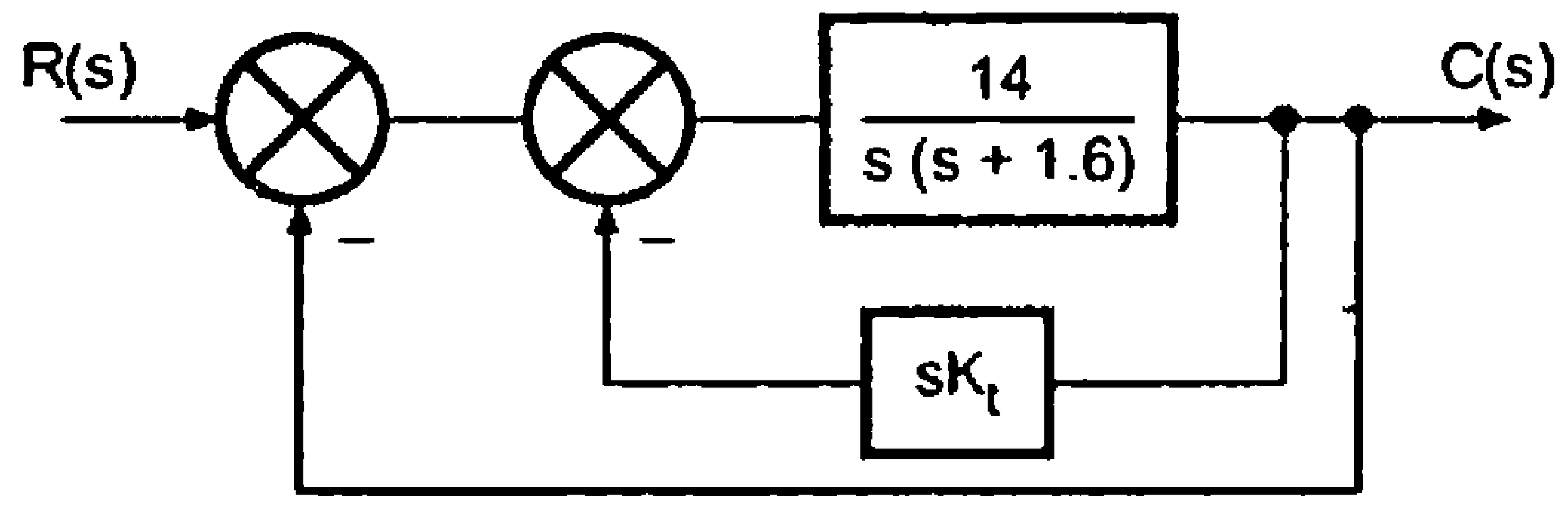
$$\therefore \quad \xi = \frac{3+K_o}{2\times 7.071}$$

Now ξ required is 0.5.

$$\therefore \quad 0.5 = \frac{3+K_o}{2\times 7.071}$$

$$\therefore \quad K_o = \mathbf{4.071}$$

Example 16.12 : *The system shown in figure uses a rate feedback controller. Deter aine the tachometer constant K_t so as to obtain the damping ratio as 0.5. Calculate corresponding T_p , M_p , ω_d* and *T_s.*



Solution :

$$G(s) = \frac{\frac{14}{s(s+1.6)}}{1+\frac{14}{s(s+1.6)}sK_t} = \frac{14}{s^2+1.6s+s\cdot 14\,K_t} = \frac{14}{s[s+14K_t+1.6]}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{14}{s(s+14\,K_t+1.6)}}{1+\frac{14}{s(s+14K_t+1.6)}} = \frac{14}{s^2+s(14\,K_t+1.6)+14}$$

$$\therefore \quad \omega_n^2 = 14$$

$$\therefore \quad \omega_n = \sqrt{14} = 3.7416 \text{ rad/sec.}$$

$$2\xi\,\omega_n = 14\,K_t+1.6$$

$$\therefore \quad \xi = \frac{14K_t+1.6}{2\times\sqrt{14}} = 0.5 \text{ given}$$

$$\therefore \quad K_t = \mathbf{0.1529}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.7416\sqrt{1-(0.5)^2} = 3.2403 \text{ rad/sec}$$

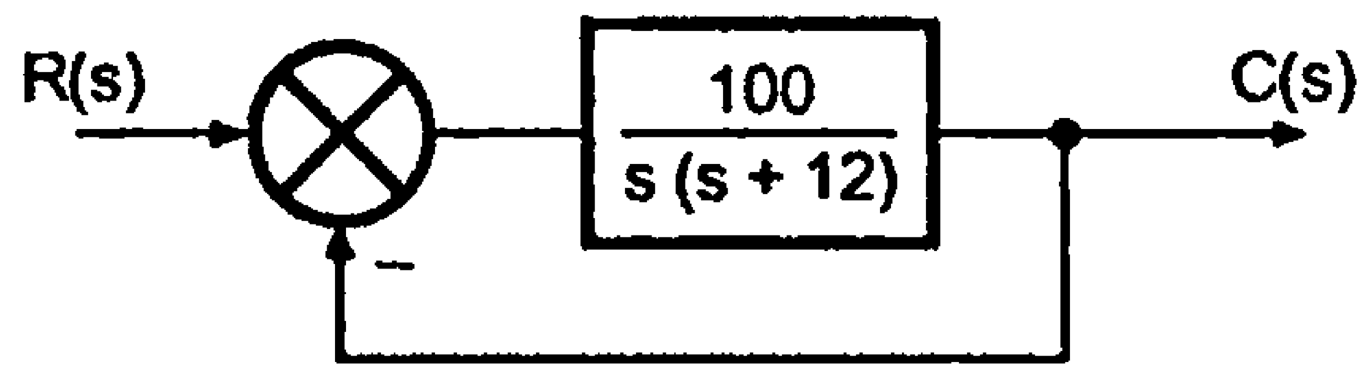
$$T_p = \frac{\pi}{\omega_d} = \mathbf{0.9695\ sec.}$$

$$\%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}\times 100 = \mathbf{16.3\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{2.1381\ sec.}$$

Example 16.13 : *For the system shown determine % M_p and T_s when it is excited by unit step input.*

If for the same system, PD controller having constant T_d = 1/30 is used in forward path, determine new values of damping ratio, M_p and T_s. Draw respective waveforms.



Solution : Without controller,

$$G(s) = \frac{100}{s(s+12)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{100}{s^2 + 12s + 100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore \ \omega_n = \mathbf{10\ rad/sec.}$$

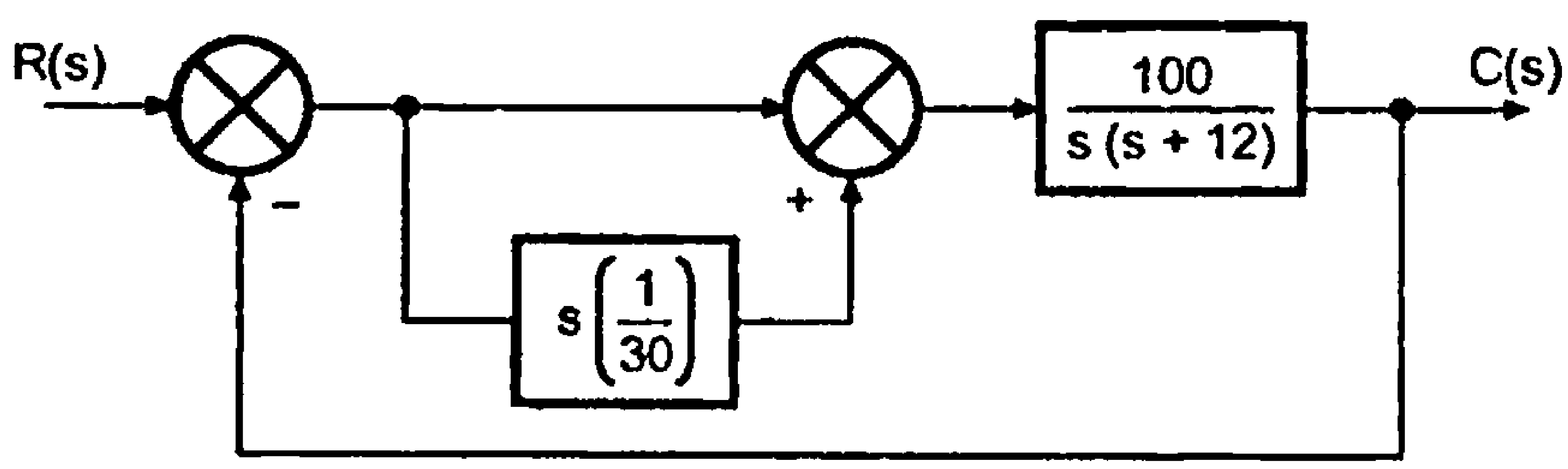
$$2\xi\,\omega_n = 12 \qquad \therefore \ \xi = \mathbf{0.6}$$

$$\therefore \quad \omega_d = \omega_n\sqrt{1-\xi^2} = 10 \times 0.8 = 8 \text{ rad/sec.}$$

$$\therefore \quad \%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = \mathbf{9.47\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{0.666\ sec.}$$

With controller



$$G(s) = \frac{\left(1+\frac{s}{30}\right)100}{s(s+12)} = \frac{(s+30)\times 3.33}{s(s+12)}, \quad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{\frac{(s+30)\,3.33}{s(s+12)}}{1+\frac{(s+30)\,3.33}{s\,(s+12)}} = \frac{3.33(s+30)}{s^2+12s+3.33s+100} = \frac{3.33\,(s+30)}{s^2+15.33s+100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore \ \omega_n = \mathbf{10\ rad/sec.}$$

$$2\xi\,\omega_n = 15.33$$

$$\therefore \quad \xi = \frac{15.33}{2\times 10} = \mathbf{0.7665}$$

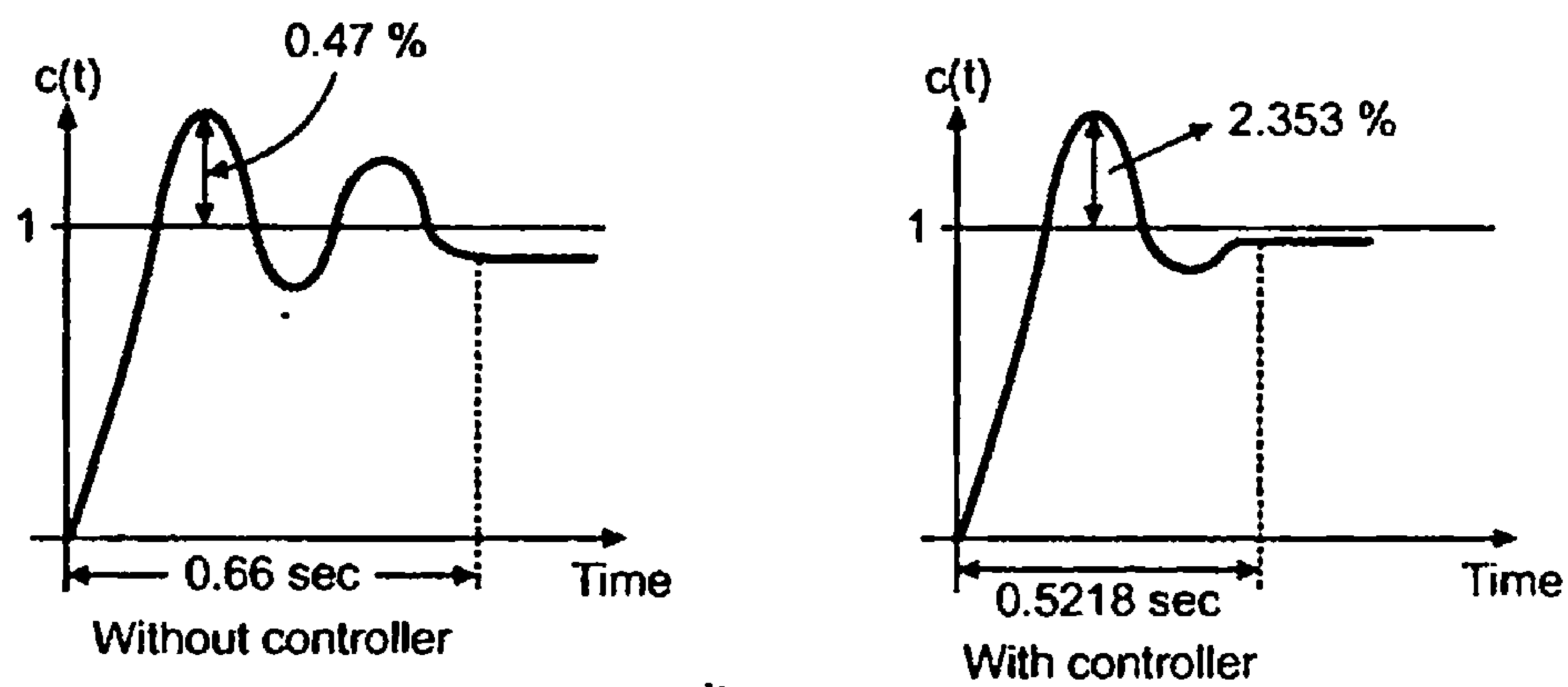
$$\therefore \ \xi \text{ is improved,} \quad \omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.7665)^2} = 6.4224 \text{ rad/sec.}$$

$$\%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}\times 100 = \mathbf{2.353\ \%}$$

Overshoot decreased to 2.3 % from 9.47 %.

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{0.5218\ sec.}$$

Comparison : Following Figure shows comparison between system with controller and system without controller.



Example 16.14 : *Block diagram model of a position control system is shown in the figure.*

a) In absence of derivative feedback $(K_t = 0)$, *determine the damping ratio of the system for amplifier gain* $K_A = 5$. *Also find the steady state error to unit ramp unit.*

b) Find suitable values of the parameters K_A and K_t *so that damping ratio of the system is increased to 0.7 without affecting the steady state error as obtained in part (a).*

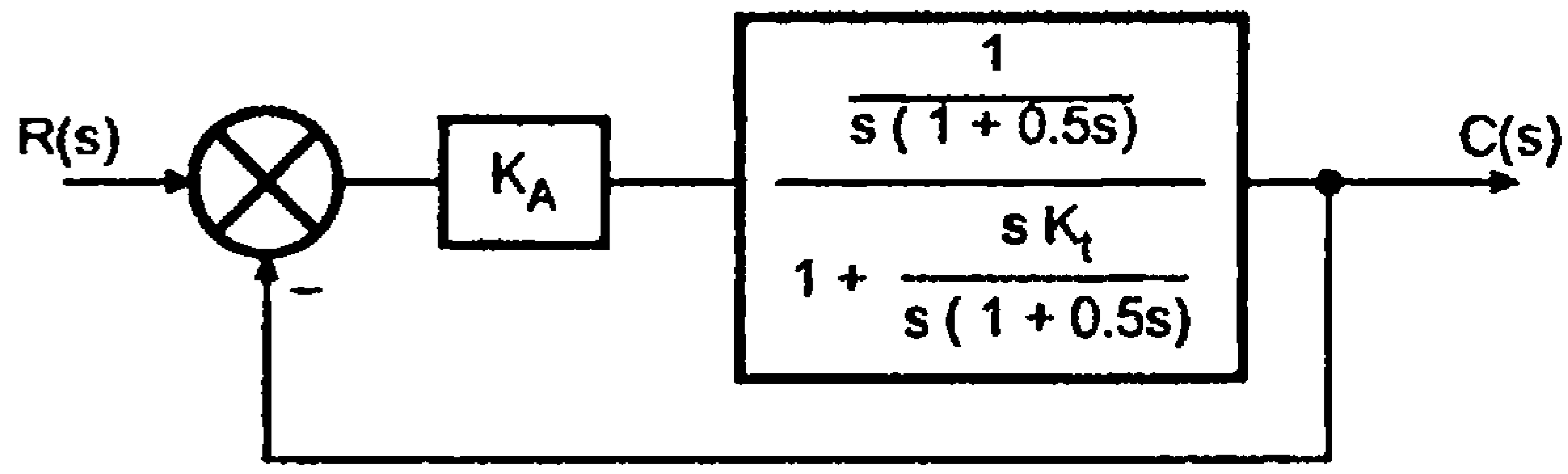
$$\therefore \quad K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \frac{s.5}{s\,(1+0.5s)} = 5$$

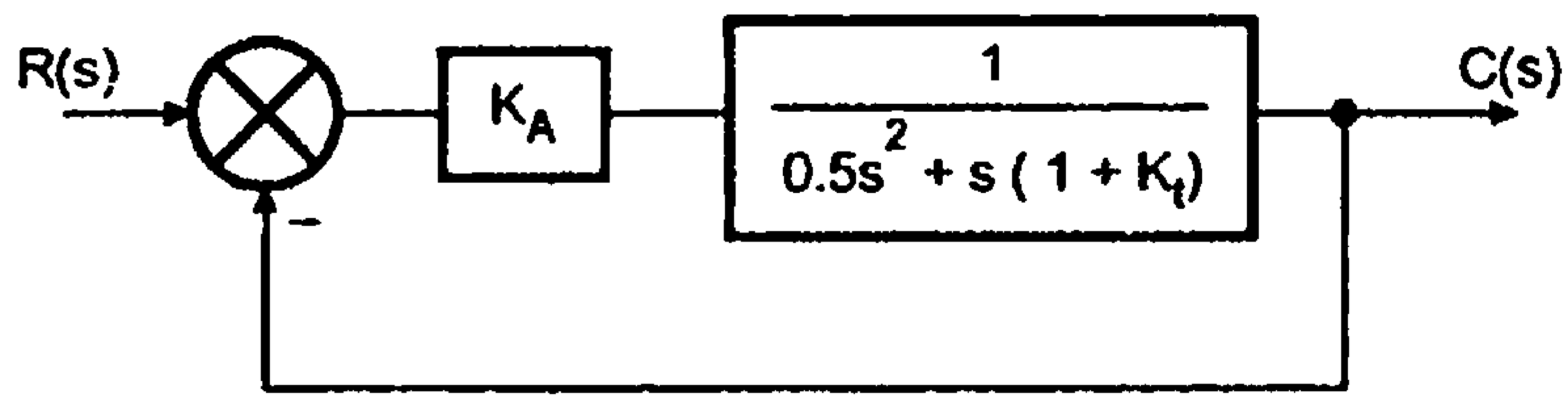
$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{1}{5} = 0.2$$

Case b) The derivative feedback is introduced in the system.

The system becomes,

$$\therefore \quad G(s) = \frac{K_A}{s\,[0.5s + 1 + K_t]} \quad \text{and} \quad H(s) = 1$$





$$\frac{C(s)}{R(s)} = \frac{\frac{K_A}{s\,[0.5s+1+K_t]}}{1+\frac{K_A}{s[0.5s+1+K_t]}} = \frac{K_A}{0.5s^2 + (1+K_t)\,s + K_A}$$

$$= \frac{2\,K_A}{s^2 + 2(1+K_t)\,s + 2\,K_A}$$

Comparing denominator to standard form,

$$\omega_n^2 = 2\,K_A$$

$$\therefore \quad \omega_n = \sqrt{2\,K_A} \qquad \text{... (1)}$$

$$\text{and} \quad 2\xi\,\omega_n = 2\,(1+K_t)$$

$$\therefore \quad \xi = \frac{(1+K_t)}{\sqrt{2\,K_A}} \qquad \text{... (2)}$$

The damping ratio required is 0.7. Substituting in equation (2),

$$\therefore \quad \frac{(1+K_t)}{\sqrt{2K_A}} = 0.7 \qquad \text{... (3)}$$

Now let us obtain the error interms of K_A and K_t, which must be same as obtained in case (a) i.e. 0.2.

$$K_v = \lim_{s \to 0} s\ G(s)\, H(s)$$

$$= \lim_{s \to 0} \frac{s \cdot K_A}{s[0.5s + 1 + K_t]} = \frac{K_A}{1+K_t} \quad \text{... (4)}$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{1}{\left(\frac{K_A}{1+K_t}\right)} = \frac{1+K_t}{K_A} \quad \text{... (5)}$$

But error e_{ss} required is 0.2.

$$\therefore \quad \frac{1+K_t}{K_A} = 0.2 \quad \text{... (6)}$$

$$\therefore \quad K_A = \frac{1+K_t}{0.2} \quad \text{... (7)}$$

Substituting value of $(1+K_t)$ from equation (3) in equation (7),

$$\therefore \quad K_A = \frac{0.7 \times \sqrt{2\,K_A}}{0.2}$$

$$\therefore \quad \sqrt{K_A} = 4.9497$$

$$\therefore \quad K_A = 24.5$$

$$\therefore \quad K_t = 3.9$$

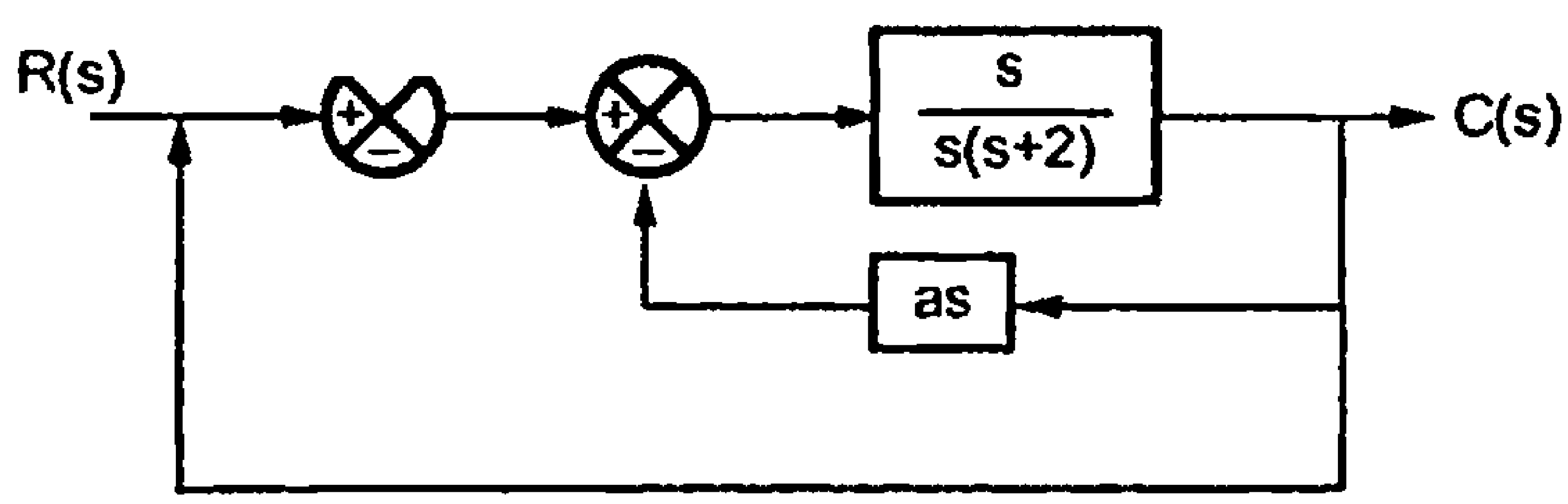
These are required values of the parameters to satisfy given specifications.

Example 16.15 : *The system given in figure is a unity feedback system with minor feedback loop.*

i) In the absence of derivative feedback (a = 0), determine the damping ratio and undamped natural frequency.

ii) Determine the constant 'a' which will increase damping ratio to 0.7.

iii) Find the overshoot in both the cases.



Review Questions

1. *Write short notes on*

   i) Synchro control transformer

   ii) Differential synchro transmitter and receiver

   iii) Servoamplifiers

   iv) Synchros

   v) Use of synchros as error detector in **(May-97)**

   a) A.C. system b) Hybrid systems

   vi) Two phase A.C. servomotors

   vii) Servomechanisms

   viii) Potentiometers and their use as error detectors

2. *Answer the following :*

   With respect to a precision potentiometer explain the following terms sensitivity, Linearity, Resolution, life.

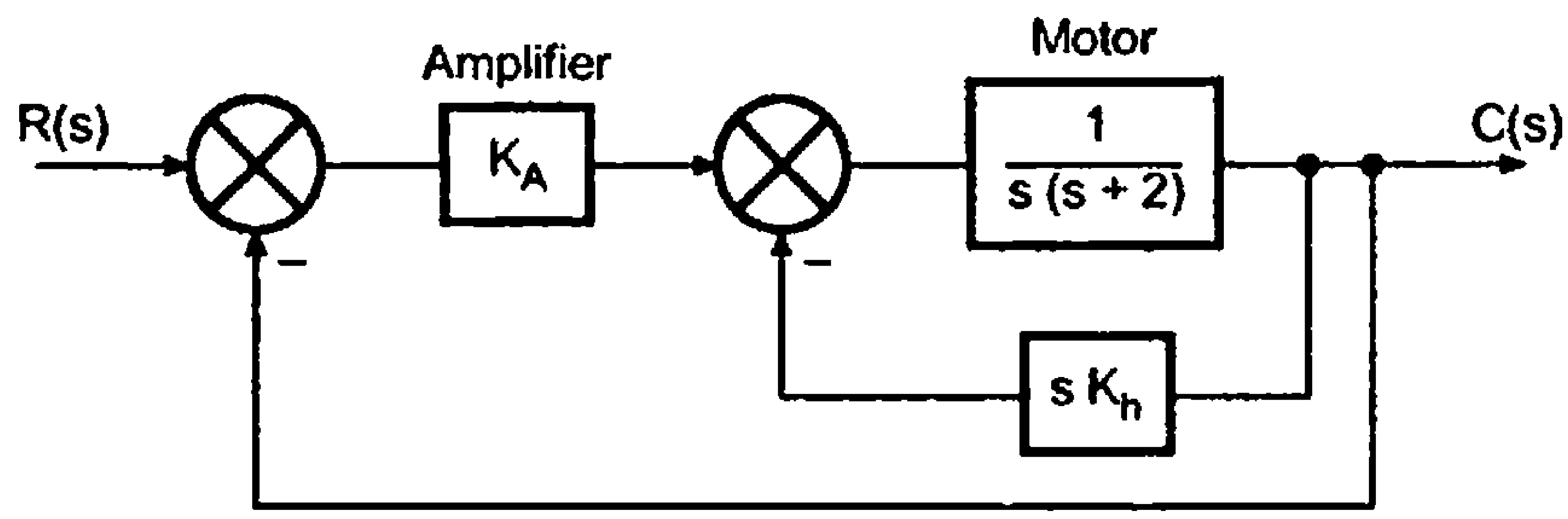
3. *Explain the working of the permanent magnet stepper motor.* **(Dec.-97)**
4. *What are the major applications of a stepper motor ?*
5. *Describe the construction and working of any one type of stepper motor.* **(Dec.-98)**
6. *How are synchros used as error detector ? Explain.* **(May-2003)**
7. *What is controller ? Explain its function in a system.*
8. *Give the classification of controllers.*
9. *Explain discontinuous mode of controllers.*
10. *Write a note on ON/OFF controller.*
11. *What is a major disadvantage of two position controller ?*
12. *State and explain the various properties of controller.*
13. *What is neutral zone in two position controller ?*
14. *What is multiposition mode of controller ?*
15. *State the various continuous controller modes.*
16. *Explain the proportional control mode. State its characteristics.*
17. *Explain the integral control mode. State its characteristics.*
18. *Explain the derivative control mode. State its characteristics.*
19. *Explain how constant K_i affect the output of PI control mode.*
20. *Why derivative mode is called anticipatory control mode ?*
21. *State the various composite control modes.*
22. *Explain the PI control mode, stating its characteristics.*
23. *Explain the PD control mode, stating its characteristics.*
24. *Write a note on three mode controller.*
25. *Discuss the effect of following controllers on the second order control system,*

    a) PI controller

b) PD controller

c) Rate feedback controller.

26. *A feedback system which uses a rate feedback controller is shown in the following figure.*



a) *For* $K_A = 10$, *in absence of derivative feedback* $(K_h = 0)$ *determine the damping ratio and natural frequency of oscillations. Also find the s.s error for unit ramp input.*

b) *Determine the constant* K_h *if the damping factor required is 0.6, with* $K_A = 10$. *With this value of* K_h, *determine s.s error for unit ramp input.*

(Ans. : 0.316, 3.16 rad/sec, 0.2, 1.8, 0.38)

❑❑❑

A Matrix Algebra

A.1 Definition of Matrix

A matrix is an ordered rectangular array of elements which may be real numbers, complex numbers, functions or mathematical operators. The order of matrix is always defined as $m \times n$

where,

m =	Number of rows
n =	Number of columns

For example, consider matrix A of order 2×4 of real numbers.

$$A = \begin{bmatrix} 4 & 3 & -2 & 1 \\ 6 & 1 & 0 & 8 \end{bmatrix} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \text{ 2 Rows}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$$

$$\text{4 Columns}$$

Any element of a matrix is denoted as a_{ij} where i indicates position of row and j indicates position of column. Thus for matrix A above $a_{11} = 4$, $a_{13} = -2$, $a_{22} = 1$ and so on.

Key Point: *Note that matrix does not have a value but its determinant has a value.*

A matrix with number of columns as 1 i.e. order $m \times 1$ is called **column matrix** while a matrix with number of rows as 1 i.e. order $1 \times n$ is called **row matrix**. It is seen that vector matrices $\dot{X}$, X, Y are the column matrices.

A.1.1 Types of Matrices

The various types of matrices are,

1. **Square Matrix :** If number of rows (m) is equal to number of columns (n) of a matrix, it is called a square matrix. The system matrix A in a state model is always a square matrix of order $n \times n$.

$$A = \begin{bmatrix} 1 & +j2 & 1+j1 \\ 2 & -j4 & 3+j2 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & -j2 & 1-j1 \\ 2 & +j4 & 3-j2 \end{bmatrix}$$

8. **Singular Matrix :** A square matrix having its determinant value zero is called singular matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0.5 & 2 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} 1 & 4 \\ 0.5 & 2 \end{vmatrix} = 2 - 2 = 0$$

Thus A is singular matrix.

9. **Nonsingular Matrix :** A square matrix whose determinant is nonzero is called nonsingular matrix.

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = 24 - 10 = 14 \neq 0$$

10. **Skew Symmetric Matrix :** A square matrix which is equal to its negative transpose is called a skew symmetric matrix.

$$A^T = -A$$

A.1.2 Important Terminologies

Let us study the important terms and their meanings related to the matrix.

1. **Minor of an element :** Consider a square (n × n) matrix A and the element a_{ij}. If now i^{th} row and j^{th} column are deleted then the remaining (n − 1) rows and columns form a determinant M_{ij}. The value of this determinant is called the minor of an element a_{ij}.

   Consider a square matrix A as,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 3 \\ 3 & 2 & 5 \end{bmatrix}$$

Let us obtain minor of element $a_{21} = 1$. Delete second row and first column to get determinant M_{21}.

$$\therefore \quad M_{21} = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 15 - 8 = 7$$

Thus minor of 'a_{21}' is 7.

2. **Cofactor of an element :** If M_{ij} is the minor of an element a_{ij} then cofactor C_{ij} is defined as,

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

Control System Engineering

✎ **Chapterwise University Questions with Answers**

May-98 to Dec.-2007

Q.3 *Obtain the transfer function* $\frac{C(s)}{R(s)}$ **(May-2004, 2 Marks)**

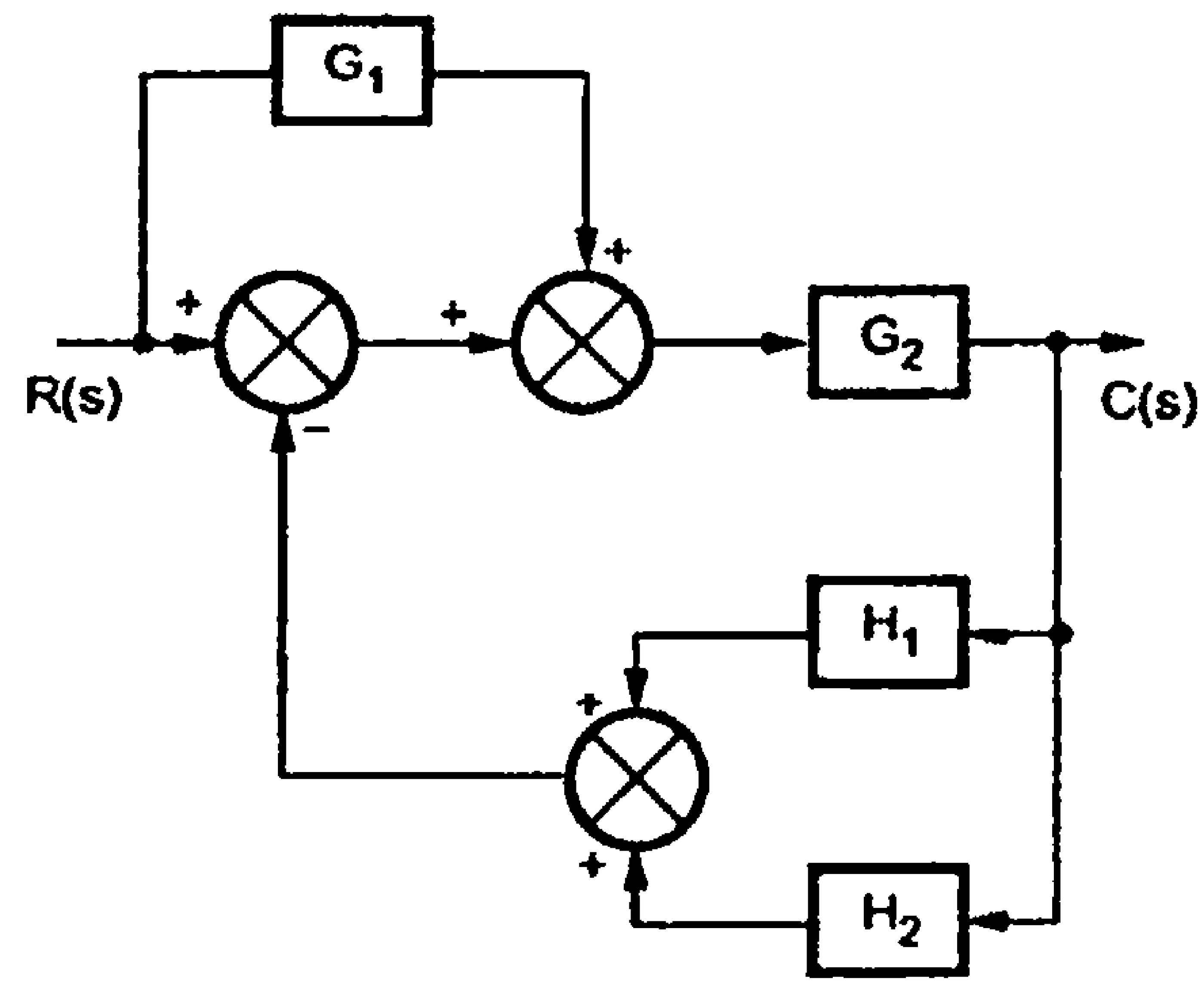


Fig. 3

Ans. : Refer example 5.25.

Q.4 *Obtain overall transfer function.* **(May-2005, May-2007, 5 Marks)**

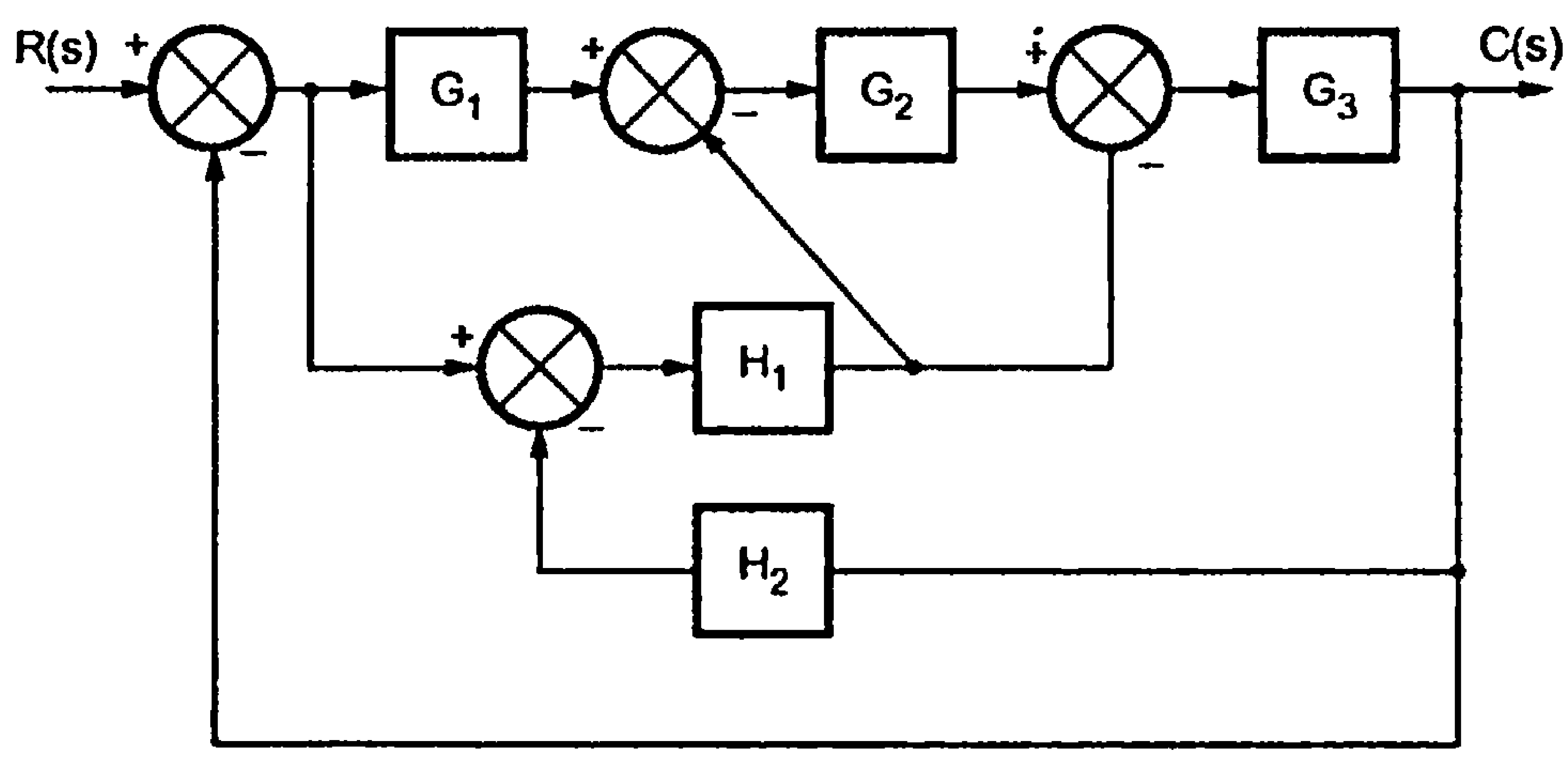


Fig. 4

Ans. : Refer example 5.26.

Q.5 *Obtain overall transfer function.* **(Dec.-2005, 10 Marks)**

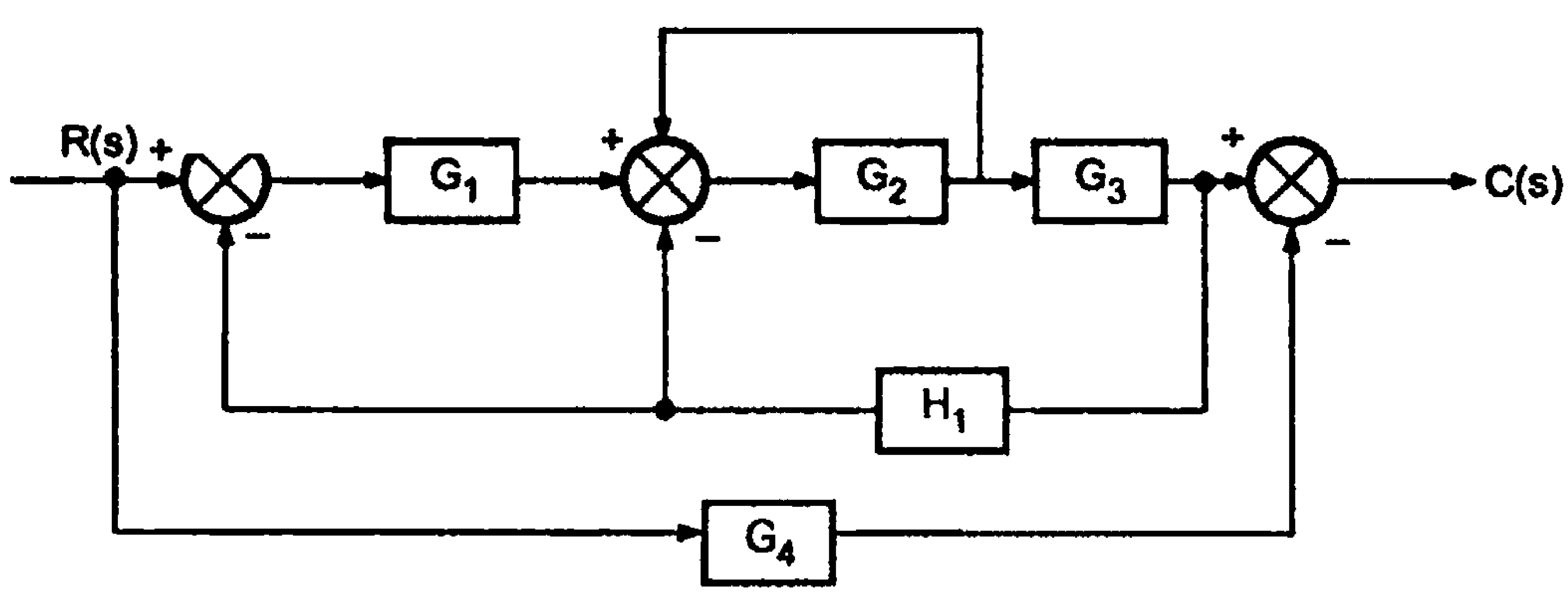


Fig. 5

Ans. : Refer example 5.12 for the procedure and verify the answer :

$$\frac{C(s)}{R(s)} = G_4 + \frac{G_1 G_2 G_3}{1+G_2+G_2 G_3 H_1 + G_1 G_2 G_3 H_1}$$

Q.6 *For the system represented by the block diagram shown, evaluate the closed loop transfer functions when the input R is*
i) at station I ii) at station II. **(May-2006, 10 Marks)**

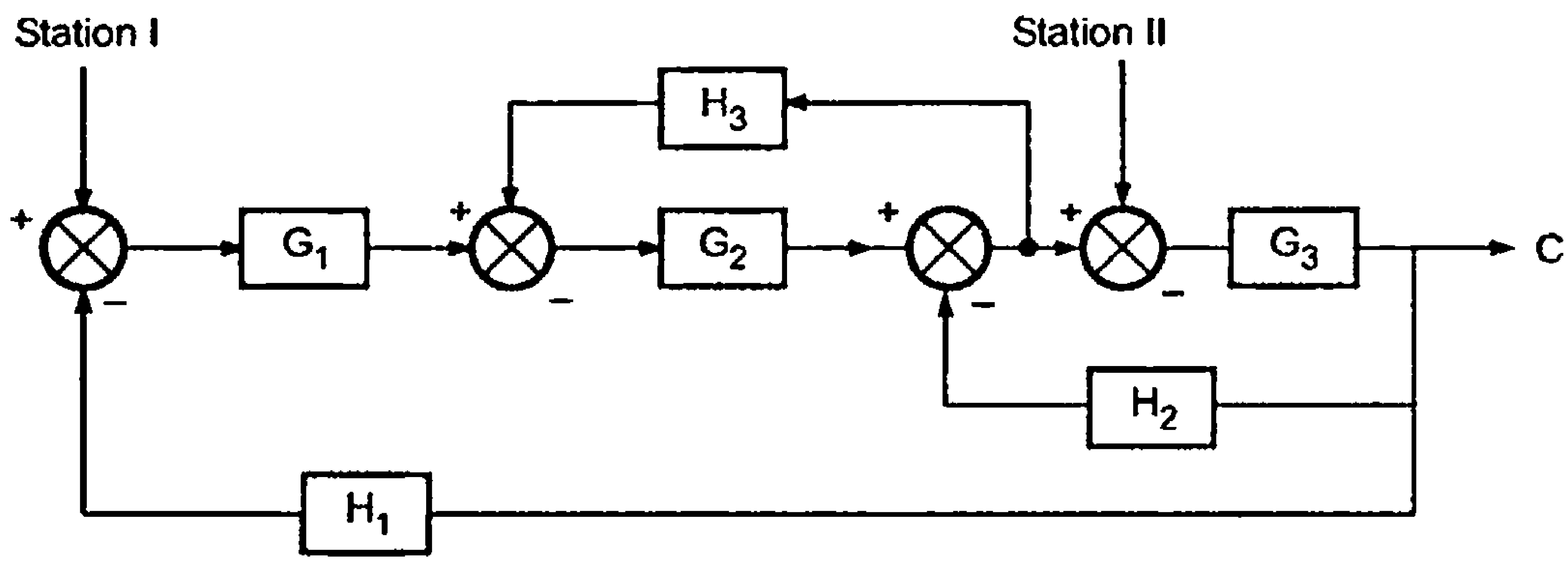


Fig. 6

Ans. : Refer example 5.27.

Q.7 *Find C using block diagram reduction techniques.* **(Dec.-2006, 10 Marks)**

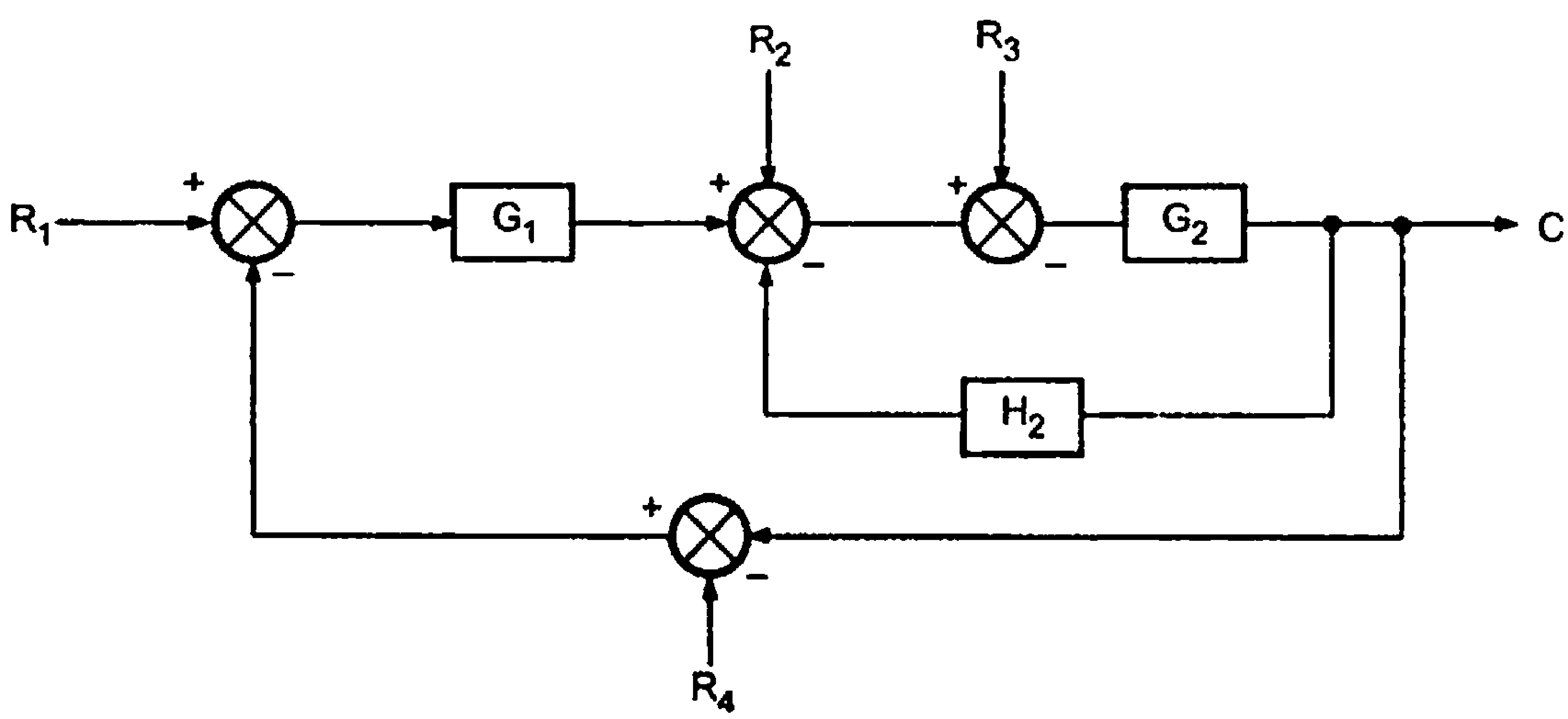


Fig. 7

Ans. : Refer example 5.22.

Q.8 *Find C/R using block diagram reduction technique.* **(Dec.- 2007, 10 Marks)**

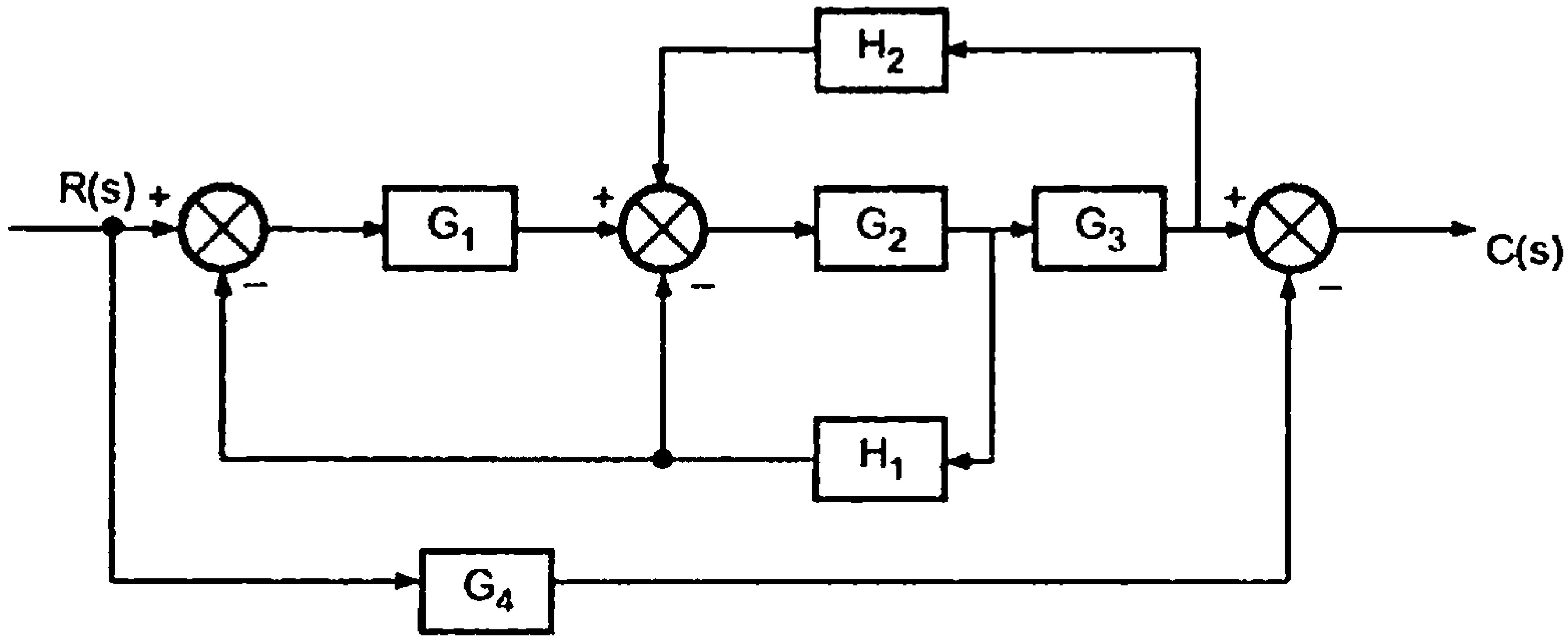


Fig. 8

Ans. : Refer example 5.12.

□□□

Q.8 *Construct the SFG of the following set of simultaneous equations :*

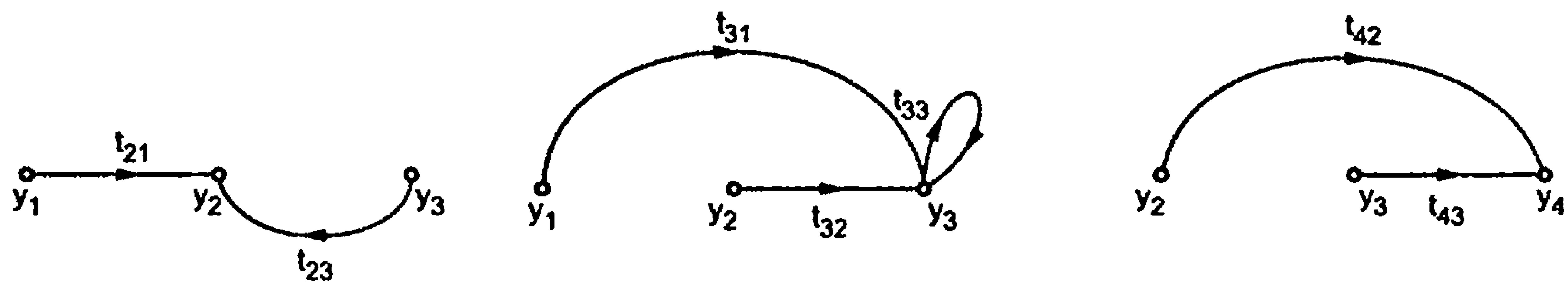
$$y_2 = t_{21} y_1 + t_{23} y_3$$

$$y_3 = t_{31} y_1 + t_{32} y_2 + t_{33} y_3$$

$$y_4 = t_{42} y_2 + t_{43} y_3$$

(Dec.-2007, 4 Marks)

Ans. :



Equation (1) Equation (2) Equation (3)

Fig. 6

The total signal flow graph is,

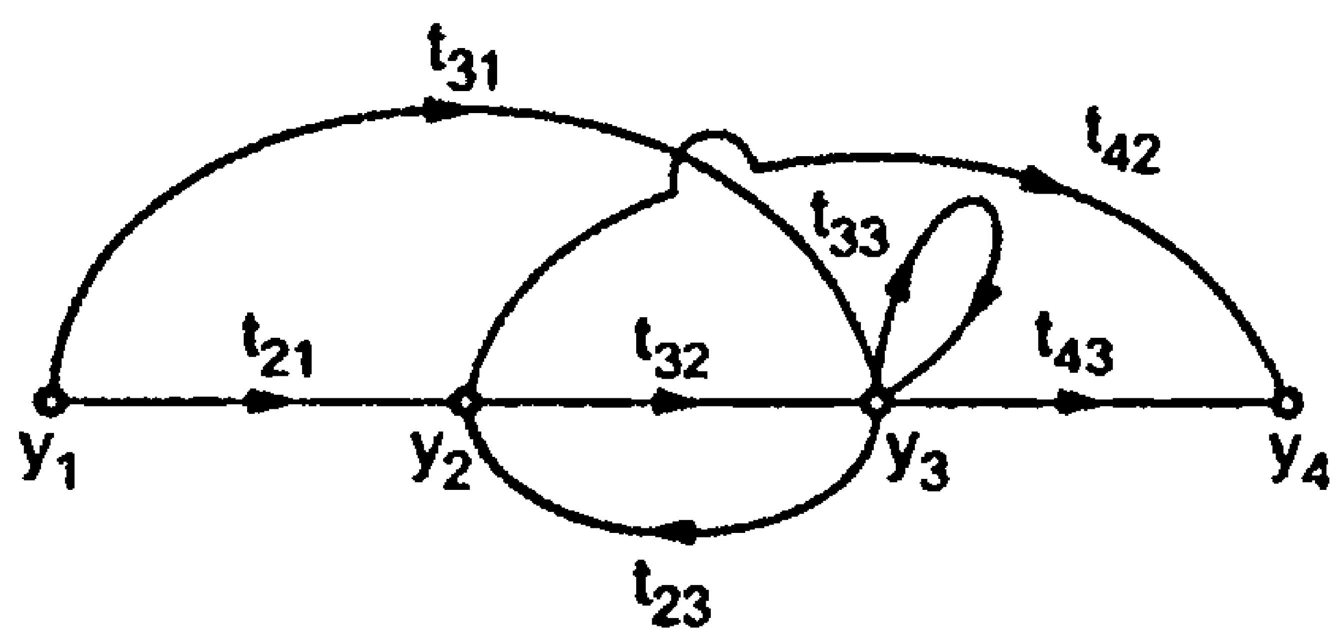


Fig. 7

7 Time Response Analysis of Control Systems

Q.1 *What will happen to damping factor ξ and natural frequency of oscillations ω_n if gain K of a second order system is increased ?* **(May-97, 2 Marks)**

Ans. : As the value of gain K increases the natural frequency of oscillations also increases and the value of damping ratio ξ decreases. Due to this system starts becoming more oscillatory in nature.

Q.2 *Pole locations in s-plane are given for two transfer functions $G_1(s)$ and $G_2(s)$ in Fig. 1. Show their corresponding time response.* **(May-97, 2 Marks)**

Ans. : For $G_1(s)$, real part is –1, while imaginary part is ± j. As the poles are complex conjugates with negative real part, time response will be damped oscillations with the term e^{-t} sin t. While in case of $G_2(s)$, the term will be e^{-2t} sin 0.5 t. The real part dominates exponential index while imaginary part controls the frequency. The response are shown in the Fig. 1 (a) and (b).

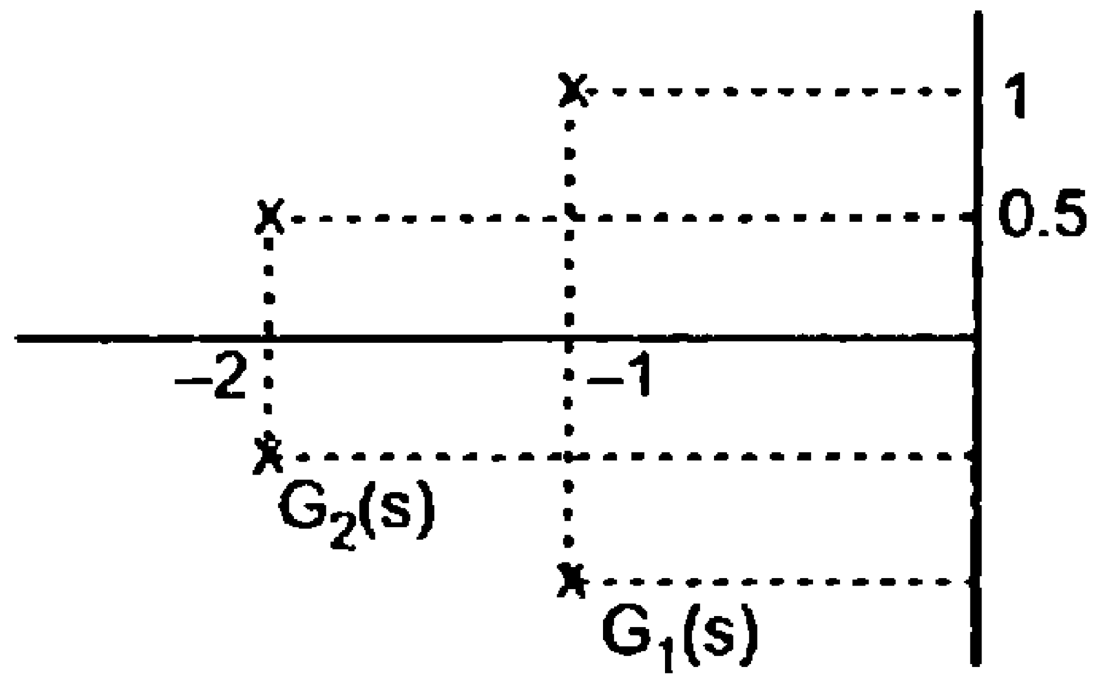


Fig. 1

(a) $G_1(s)$ **(b) $G_2(s)$**

Fig. 1

Q.3 *A system is critically damped. How will the system behave if the gain of the system in increased ?* **(Dec.-97, 2 Marks)**

Ans. : A system is critically damped means gain is at its marginal value and system closed loop poles are on the imaginary axis. If gain is increased beyond this marginal value, the closed loop poles on the imaginary axis gets shifted in the right half of s-plane making the system unstable in nature.

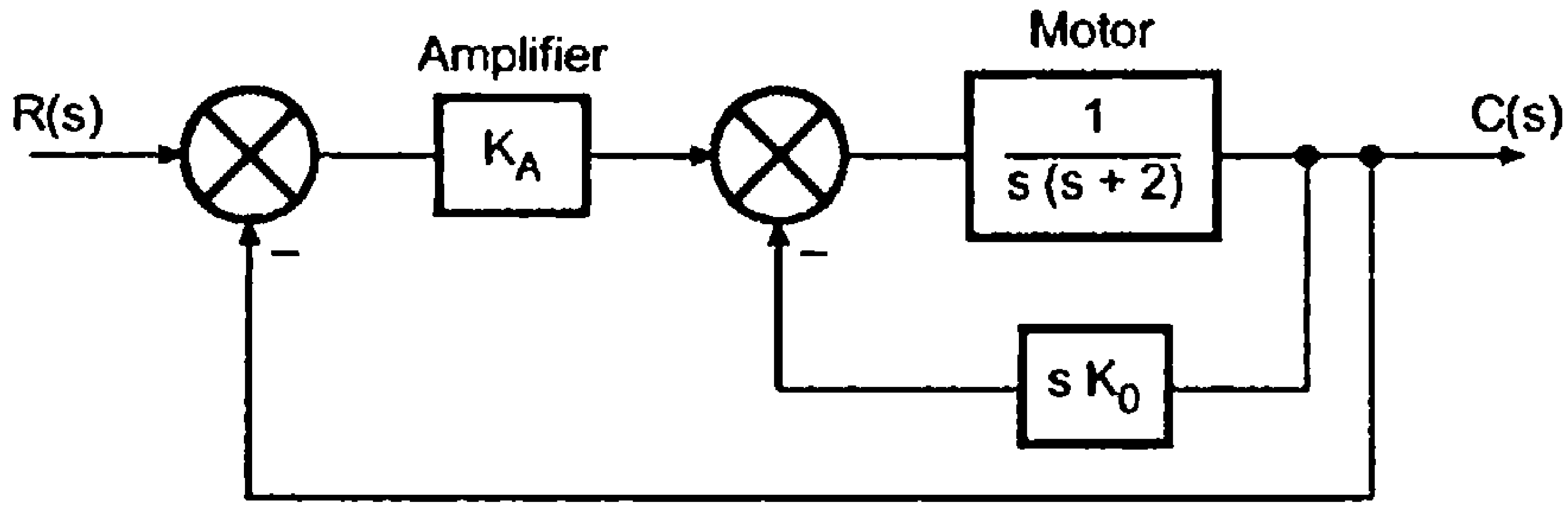


Fig. 13

Ans. : Refer example 7.62.

Q.45 *List the types of static error constants and comment of it's value as the type of system increases.* **(Dec.-2006; May-2007, 4 Marks)**

Ans. : Refer section 7.6.

Q.46 *For the system shown in Fig. 14. Find the values of K_1 and K_2 to yield a peak time of 1 second and a settling time of 2 seconds for the closed loop system's step response (for 2% tolerance bond).* **(May-2007, 10 Marks)**

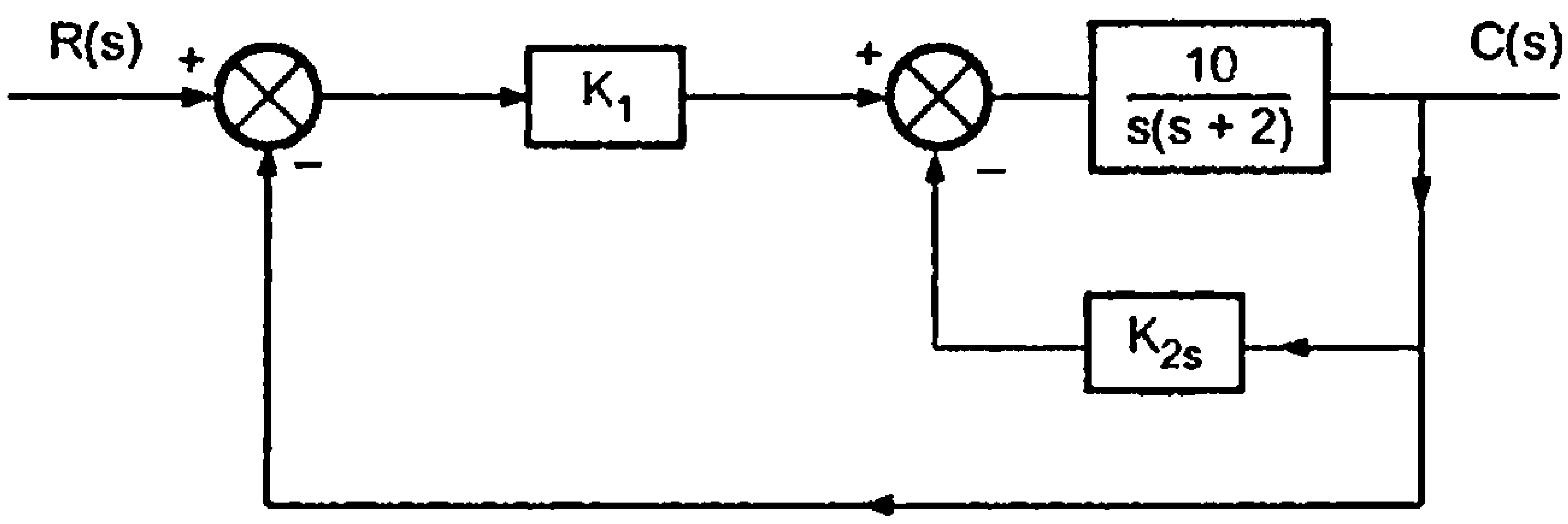


Fig. 14

Ans. : Refer example 7.52 for the procedure. Verify the values of K_1 and K_2 as K_1 = **1.3869,** K_2 = **0.2.**

Q.47 *Define the following terms :*

i) Delay time ii) Rise time iii) Settling time iv) Peak overshoot. **(Dec.-2007, 4 Marks)**

Ans. : Refer section 7.16.

Q.48 *Derive the expression for the response c(θ) of critically damped second order control system.* **(Dec.-2007, 10 Marks)**

Ans. : Refer section 7.14.

Q.49 *Write a short note on effect of ξ on second order system performance.* **(Dec.-2007, 5 Marks)**

Ans. : Refer section 7.14.

❑❑❑

(i) Find the range of values of $K_A > 0$ for which system is stable.

(ii) The acceptable value of steady state error is 1 °C for the step input of 10 °C. Find the value of K_A that meets this specification on static accuracy.

Ans. : Refer example 8.25.

Q.5 *The block diagram of a control system is shown in Fig. 3. Find the region in the K Vs α plane for the system to be asymptotically stable. (Use K as vertical and α as horizontal axis).*

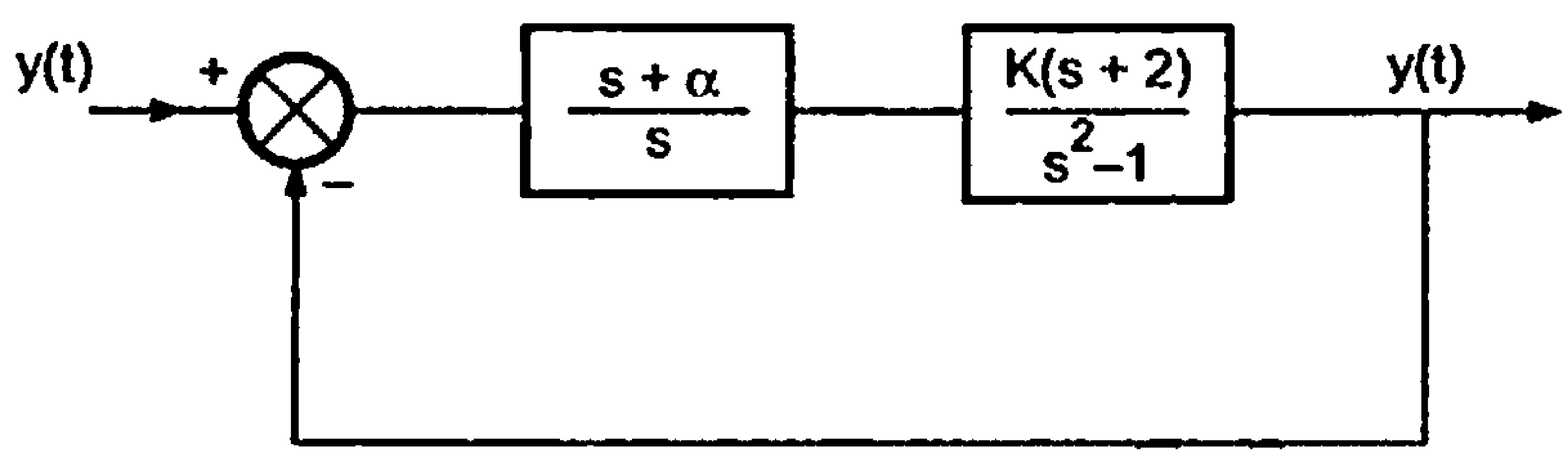


Fig. 3

(May-2003, 8 Marks)

Ans. : Refer example 8.20.

Q.6 *Determine whether the largest time constant of the roots of the characteristics equation given below is greater than, less than or equal to 1.0 sec.* **(May-2003, 8 Marks)**

$s^3 + 4s^2 + 6s + 4 = 0$

Ans. : Refer example 8.26.

Q.7 *The first two rows of Routh's tabulation of a fourth order are -* **(Dec.-2003, 2 Marks)**

s^4 1 10 5

s^3 2 10

Ans. :

s^4	1	10	5
s^2	2	10	0
s^2	5	5	
s^1	8	0	
s^0	5		

As there are no sign changes in the first column of the Routh's array, there are no **roots lying in the right -half of s-plane.**

9

Root Locus

Q.1 *Given* $G(s) = \frac{K(s+2)}{s(s+1)}$

Plot root locus for the above transfer function.

(i) Find breakaway and entry points on real axis.

(ii) Find gain and roots when the real part of complex roots is located at - 2.

(May-2003, 8 Marks)

Ans. : Refer example 9.35.

Q.2 *The root locus of a unity feedback system is shown in Fig. 1.* **(Dec.-2003, 2 Marks)**

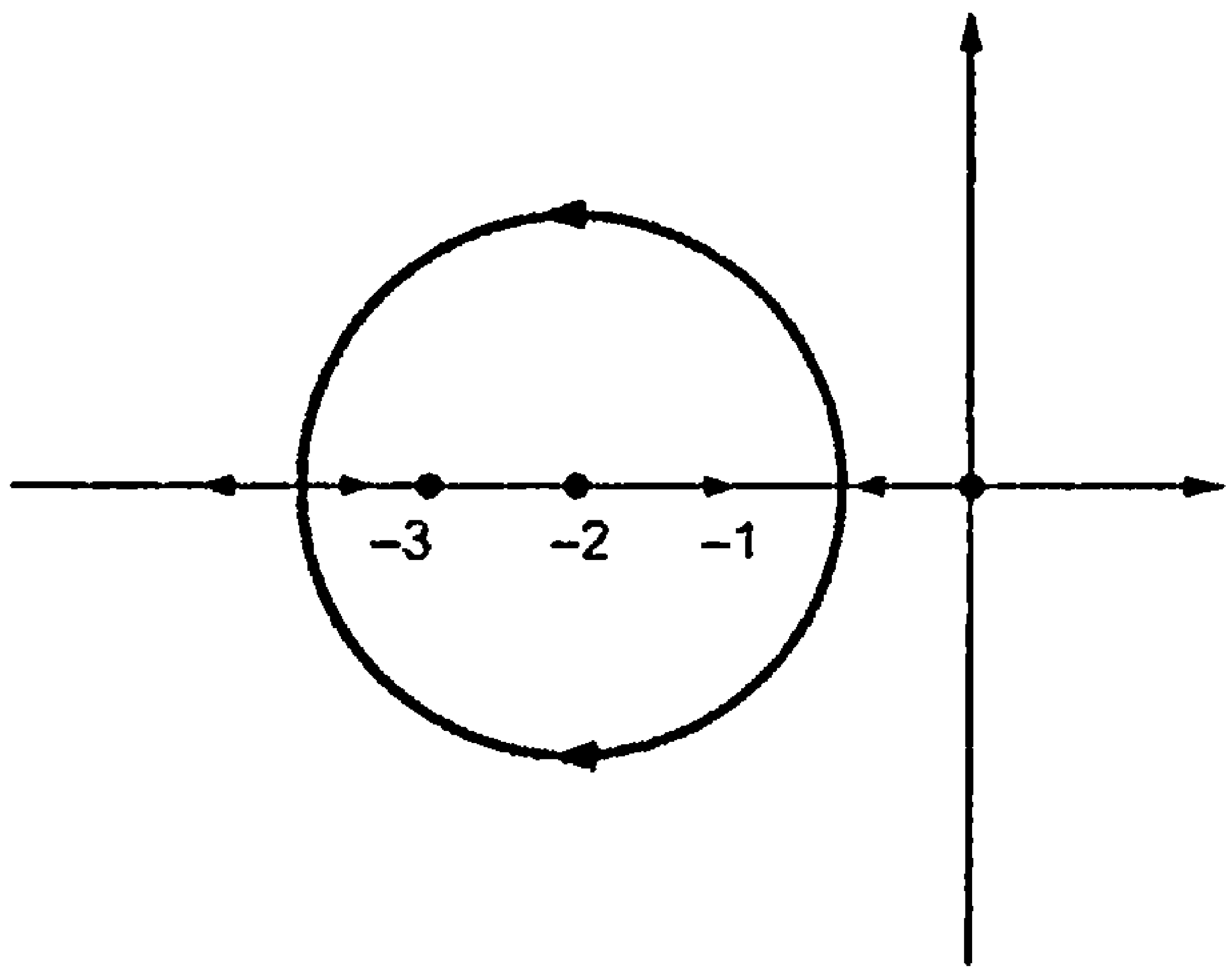


Fig. 1

Determine the open loop transfer function of the system.

Ans. : $G(s)H(s) = \frac{K(s+3)}{s(s+1)}$

Q.3 *Sketch the root locus plot for the transfer function.*

$$G(s) = \frac{K(s+2)}{s^2+2s+3}.$$

Determine the approximate damping ratio for a value of K = 1.33

(Dec.-2003, 8 Marks; Dec.-2005, 10 Marks)

Ans. : Refer example 9.39.

Q.4 *Plot root locus for the following open loop transfer function.*

(i) K/s (ii) K/s^2

(iii) K/s^3 (iv) K/s^4 **(May-2004, 20 Makrs)**

Ans. :

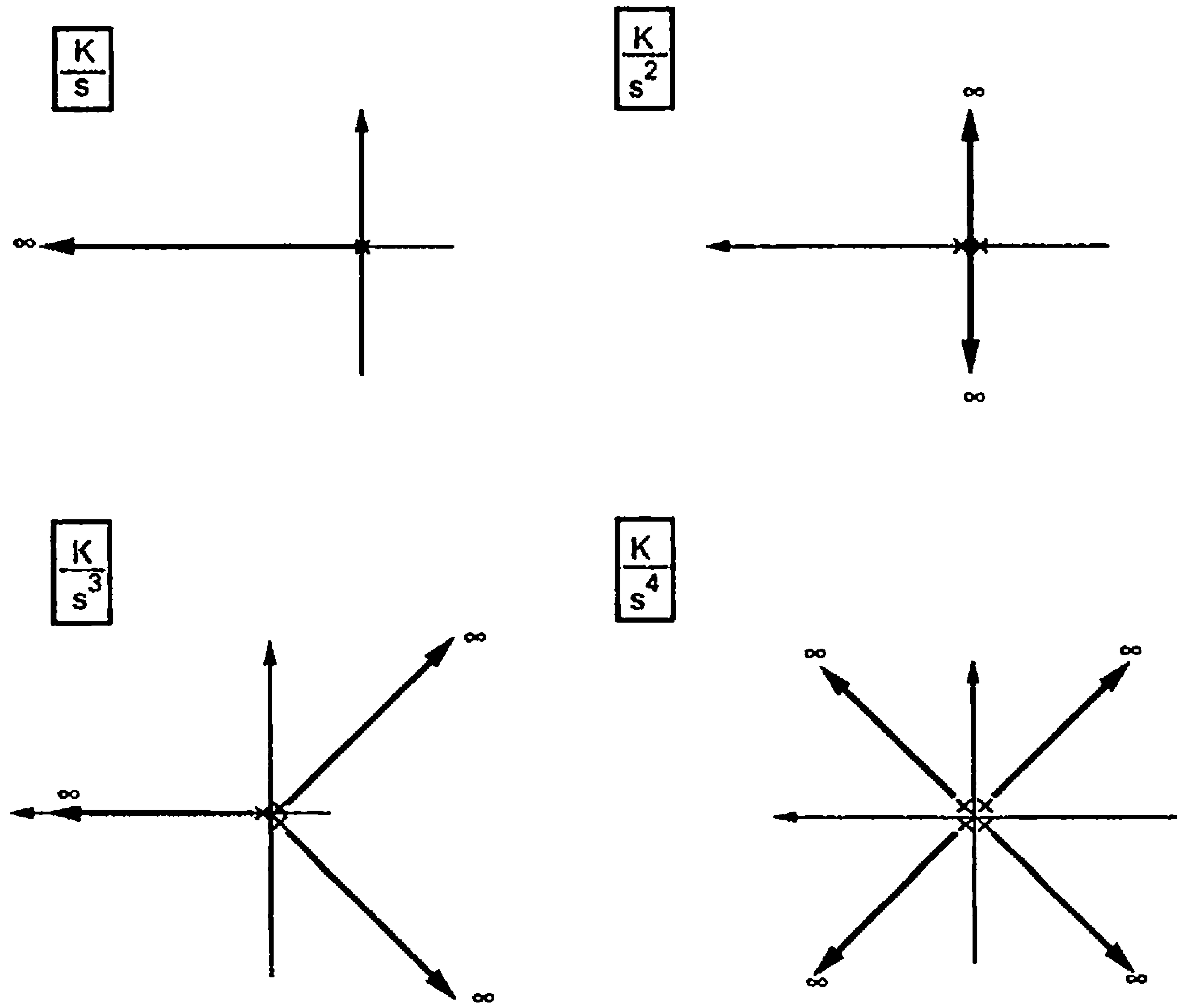


Fig. 2

Q.5 *Plot root locus of a unity feedback control sytem.*

$$G(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}.$$

Also find K marginal and K critical for damping $\xi = 1$ at breakway point. **(May-2004, 12 Marks)**

Ans. : Refer example 9.36.

Q.6 *List the two condition on which root locus are based.* **(Dec.-2004, 4 Marks)**

Ans. : Refer section 9.3.

Q.7 *List any two applications or use of root locus technique.* **(Dec.-2004, 4 Marks)**

Ans. : Refer section 9.10.

Q.8 *Sketch the root locus of a unity feedback system having -*

$$G(s) = \frac{K}{s(1+0.02s)(1+0.1s)}$$

Find K marginal, K critical and K for $\xi = 0.4$. **(Dec.-2004, 12 Marks)**

Ans. : Refer example 9.37.

Q.9 *Explain any four rules of root locus.* **(Dec.-2004, 4 Marks; Dec.-2004, 4 Marks)**

Ans. : Refer section 9.6.

Q.10 *Plot root locus :* **(May-2005, 12 Marks)**

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

Hence obtain :

(i) K for $\xi = 0.707$ *(ii) K for* $\xi = 0.866$ *(iii) K for* $\xi = 1$.

Ans. : Refer example 9.38.

Q.11 *What is root locus? How are the conditions on which root locus are based formulated?* **(May-2006, 4 Marks)**

Ans. : *Refer sections 9.2, 9.3.*

Q.12 *Sketch the root loci of the control system shown in Fig. 3. Determine the range of K for stability.* **(May-2006, 10 Marks)**

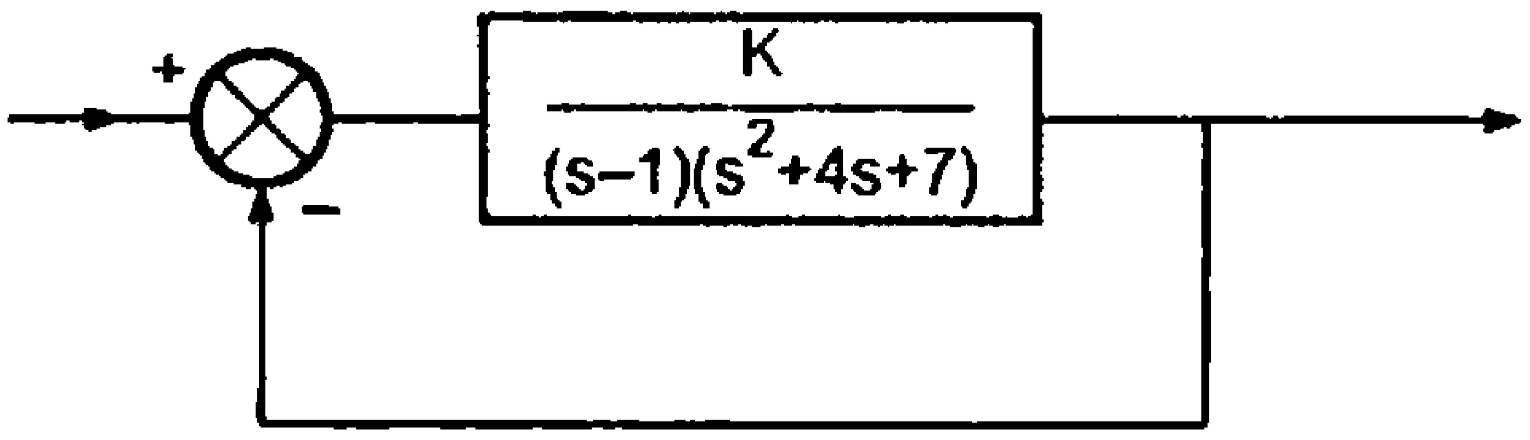


Fig. 3

Ans. : Refer section 9.40.

Q.13 *The root locus a unity feedback system is as shown. Determine the open loop transfer function of the system,* **(May-2007, 4 Marks)**

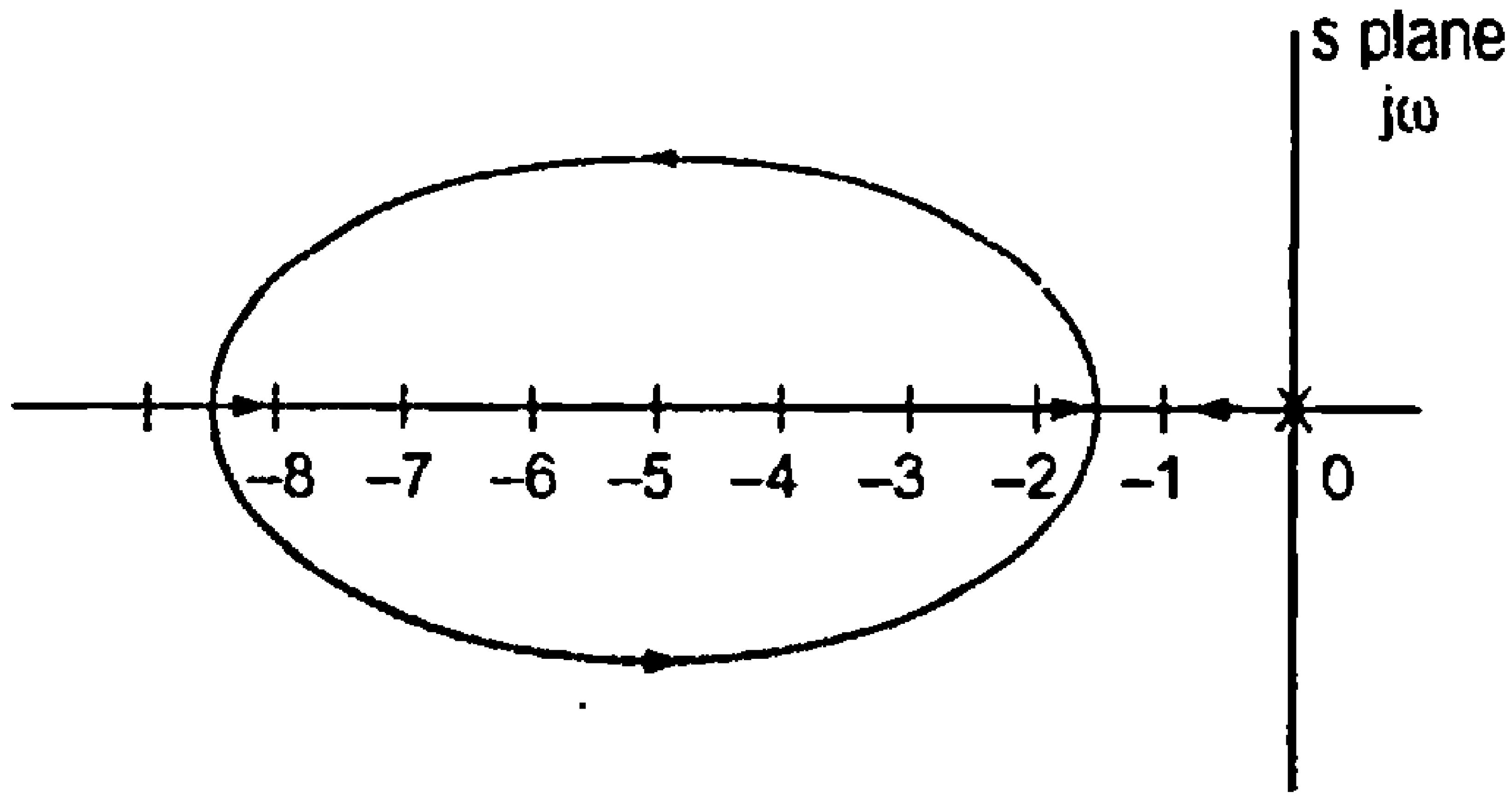


Fig. 4

Ans. : From the root locus there are two poles at s = 0 and – 2 and a zero at s = – 8.

$$\therefore \quad G(s)H(s) = \frac{K(s+8)}{s(s+2)}$$

Q.14 *Sketch the root locus :* **(May-2007, 10 Marks)**

$$G(s)H(s) = \frac{K}{s(s+6)(s^2+2s+2)}$$

Find K marginal, ω marginal and comment on stability.

Ans. : Refer example 9.41.

Q.15 *Sketch the complete root locus for the system having* $G(s)H(s) = \frac{K(s+5)}{s^2+4s+20}$ *and comment on stability.* **(Dec.-2007, 10 Marks)**

Ans. : Refer example 9.18.

□□□

10

Basics of Frequency Domain Analysis

Q.1 *Determine the magnitude and phase of the transfer function* $G(s) = \frac{1}{(s+1)}$ *at* $\omega = 1$. **(May-2003)**

Ans. :

$$G(j\omega) = \frac{1}{1+j\omega}$$

$$\therefore \quad |G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} \quad \text{and phase} = -\tan^{-1}\frac{\omega}{1}$$

At $\omega = 1$,

$$M = \frac{1}{\sqrt{2}} \quad \text{and} \quad \phi = -45^\circ$$

Q.2 *Consider the feedback system shown in Fig. 1.* **(May-2003, 8 Marks)**

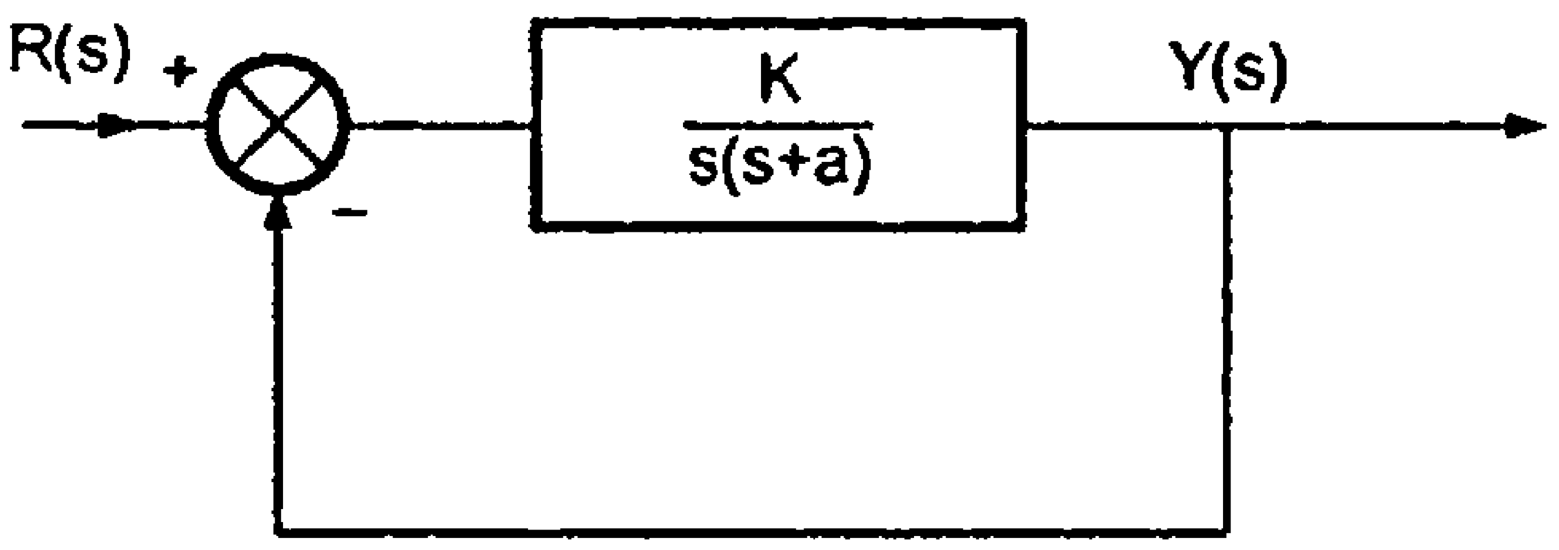


Fig. 1

(i) Find the value of. K and a to satisfy the following frequency domain specification.
Resonant peak $M_r = 1.04$

Resonant frequency $\omega_r = 11.55$ *rad/sec.*

(ii) For the values of K and Q determined in (i) Calculate the settling time and bandwidth of the system.

Ans. : Refer example 10.8.

Q.3 *Compare time and frequency domain specifications.* **(May-2003, 8 Marks)**

Ans. : Refer section 10.9.2.

Q.4 *If a system has an open loop transfer function* $\frac{1-s}{1+s}$, *determine the gain of the system at frequency 1 rad/sec.* **(Dec.-2003)**

11

Bode Plots

Q.1 *What are minimum and non-minimum phase systems?*

(May-98; May 99, 8 Marks)

Ans. : Transfer functions can be classified as minimum phase and non-minimum phase. The system having transfer function having neither poles nor zeros in the right half 's' plane is called as minimum phase system. The system having transfer function containing either a pole or a zero or both in the right half of 's' plane is called as non-minimum phase system.

For minimum phase systems, the magnitude and phase curves are uniquely related i.e. if we know the magnitude curve from this phase angle curve can be found and viceversa. This property does not hold good for non-minimum phase system.

For systems with the same magnitude characteristics, the range in phase angle of the minimum phase systems is minimum for all such systems, while the range in phase angle of non-minimum phase systems is greater than this minimum. When frequency is varied from zero to infinity, a minimum phase function goes through a minimum amount of phase change in negative direction. This minimum phase property does not hold good for non-minimum phase systems.

Non–minimum phase systems are slow in response. In designing a system, if fast speed of response is desired then non-minimum phase components should be avoided.

Consider following transfer functions.

$$G_1(s) = \frac{(s+2)}{s(s+4)\,(s+6)} \qquad G_2(s) = \frac{(s-2)}{s(s+4)\,(s+6)}$$

Minimum phase   Non–minimum phase

as $\omega \to 0$, $\angle G_1(j\omega) = -90°$ as $\omega \to 0$, $\angle G_2(j\omega) = +90°$

as $\omega \to \infty$, $\angle G_1(j\omega) = -180°$ as $\omega \to \infty$, $\angle G_2(j\omega) = +180°$

In general transfer functions can be considered as

Q.19 *Find transfer function in pole zero format from the given semi-log plot.*

(May-2005, Dec.-2006, 12 Marks)

(i) Without any correction at ω = 80 rad/sec.

(ii) With correction of + 8 dB at ω = 80 rad/sec.

(iii) With correction of + 4 dB at ω = 80 rad/sec.

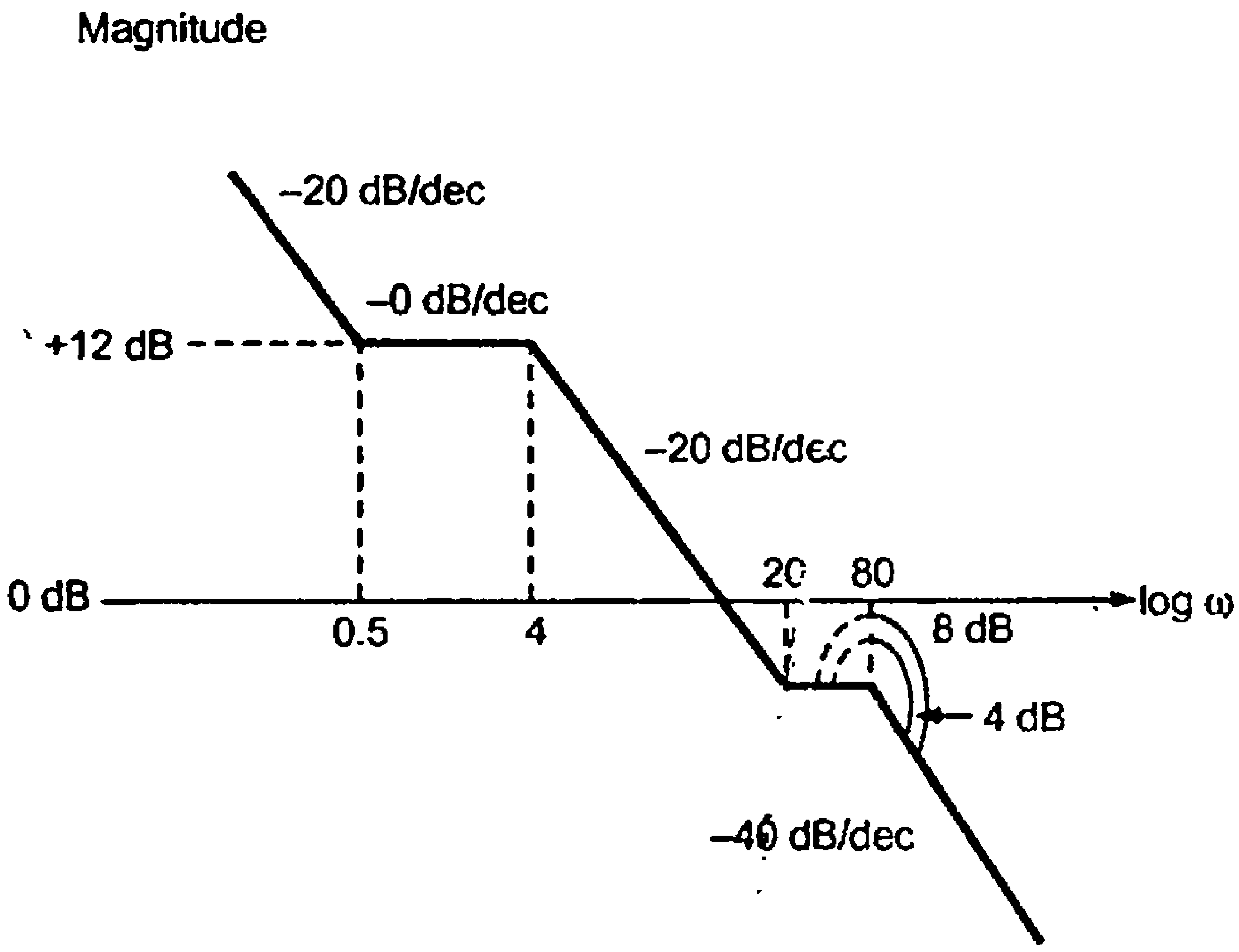


Fig. 8

Ans. : Refer example 11.20.

Q.20 *Obtain Bode plot.*

$$G(s)H(s) = \frac{10(1-s)}{s(s+2)(s^2+2s+25)}$$

Hence obtain gain margin and phase margin. **(May-2006, 10 Marks)**

Ans. : Refer example 11.23.

Q.21 *Draw the Bode for following functions :*

(i) $G(s)H(s) = 1/s$

(ii) $G(s)H(s) = s^2$

(iii) $G(s)H(s) = 1/s^3$

(iv) $G(s)H(s) = s$ **(Dec.-2007, 4 Marks)**

Q.15 *Obtain Nyquist plot -* **(May-2005, 12 Marks)**

(i) $G(s)H(s) = \frac{10}{s(s-6)}$ *(ii)* $G(s)H(s) = \frac{4(s-1)}{s(s-2)}$

Hence comment on stability and number of poles on R.H.S. of $j\omega$ axis.

Ans. : Refer example 12.30.

Q.16 *Explain :* **(May-2005, 10 Marks)**

(i) Principle of arguments

(ii) Selection of Nyquist path is s-plane

(iii) Mapping of Nyquist path in F-plane

(iv) Nyquist stability criteria.

Ans. : Refer sections 12.9, 12.10 and 12.11.

Q.17 *How is gain margin and phase margin found from magnitude-phase plot?*

(May-2006, 5 Marks)

Ans. : Refer section 12.15.

Q.18 *Draw the approximate polar plots of the following transfer functions :*

(i) $\frac{1}{1+j\omega T_1}$ *(ii)* $\frac{1}{j\omega(1+j\omega T_1)}$

(iii) $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$ *(iv)* $\frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$

(v) $\frac{1}{(j\omega)^3(1+j\omega T_1)}$. **(May-2006, 10 Marks)**

Ans. : Refer example 12.2, 12.3, 12.4 and 12.5 for procedure.

Q.19 *Discuss the stability of the system using Nyquist plot for* $G(s)H(s) = \frac{K(s-2)}{(s+1)^2}$

(May-2006, 10 Marks)

Ans. : Refer example 12.33.

Q.20 *Draw Nyquist plot for* $G(s)\,H(s) = \frac{20}{s(s+4)(s-2)}$ *and hence comment on stability.* **(Dec.-2006, Dec.-2007, 10 Marks)**

Ans. : Refer example 12.36.

Q.21 *If* $G(s)H(s) = \frac{K(s+1)}{s^2(s+2)(s+4)}$ *using polar plot determine the range of K for stability.*

Verify your results by Routh's criterion. **(Dec.-2006, 10 Marks)**

Ans. : Refer example 12.34.

Q.22 *If* $G(s)H(s) = \dfrac{K}{s(1+2s)(1+0.1\,s)}$ *, using polar plot determine the range of K for stability. Verify your results by Routh criterion.* **(May-2007, 10 Marks)**

Ans. : Refer example 12.35.

Q.23 *Draw the Nyquist plot for :* **(May-2007, 10 Marks)**

$$G(s)H(s) = \frac{(4s+1)}{s^2(s+1)(2s+1)}$$

Hence, comment on stability.

Ans. : Refer example 12.8 for the procedure and verify N = + 2, unstable.

Q.24 *Draw the polar plot for a system given by,*

$G(s)H(s) = \dfrac{100}{s(s+2)(s+4)(s+8)}$. *Find whether the system is stable and if so find G.M. and P.M. If 100 is replaced by K find critical value of K by Routh criteria and also verify G.M. and P.M.* **(Dec.-2007, 12 Marks)**

Ans. : Refer example 12.32 for the procedure.

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Contents

- **Control System Analysis**
  Examples of control systems, Open loop control systems, Closed loop control systems Transfer function Types of feedback and feedback control system characteristics Noise rejection, Gain, Sensitivity, Stability.

- **Mathematical Modeling of Systems**
  Importance of a mathematical model, Block diagrams, Signal flow graphs, Mason's gain formula and its application to block diagram reduction State space method, Solving time-invariant system, Transfer matrix

- **Transient and Steady State - Response Analysis**
  Impulse response function, First order system, Second order system, Time domain specifications of systems, Analysis of transient-response using second order model
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First Edition : 2008

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ISBN 978-81-8431-463-2

9788184314632

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